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COMMUTATOR OF TWO PROJECTIONS IN PREDICTION THEORY

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Let w be a nonnegative weight function in $L^1 = L^1(d\theta/2\pi)$. Let $\mathcal{Q}$ and $\mathcal{P}$ denote the orthogonal projections to the closed linear spans in $L^2(wd\theta/2\pi)$ of $\{e^{in\theta} : n \leqslant 0\}$ and $\{e^{in\theta} : n > 0\}$, respectively. The commutator of $\mathcal{Q}$ and $\mathcal{P}$ is studied. This has applications for prediction problems when such a weight arises as the spectral density of a discrete weakly stationary Gaussian stochastic process.

1. Introduction

Let w be a nonnegative weight function in $L^1 = L^1(d\theta/2\pi)$. Let Z^- denote the closed linear span in Z of $\{e^{in\theta} : n \leqslant 0\}$ and Z^+ denote the closed linear span in Z of $\{e^{in\theta} : n > 0\}$, where $Z = L^2(wd\theta/2\pi)$. $\mathcal{Q}$ denotes the orthogonal projection onto Z^- in Z and $\mathcal{P}$ denotes the orthogonal projection onto Z^+ in Z. In this paper we assume that $\log w \in L^1$ because $Z = Z^- = Z^+$ in the case that $\log w \notin L^1$. Hence we may assume $w = |h|^2$ for some outer function h in H^2, the Hardy space for the unit disc. Then $Z^- = h^{-1}\overline{H^2}$ and $Z^+ = h^{-1}zH^2$. Set

$$\phi = \bar{h}/h .$$

Let P be the orthogonal projection from L^2 onto H^2 and $Q = I - P$ where I is the identity operator on L^2. Let P_0 be the

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orthogonal projection from L^2 onto $H^2 \cap \bar{H}^2$. Put $M_k f = kf$, $k \in L^\infty$ and $f \in L^2$. Then $Q + P_0 = M_z Q M_{\bar{z}}$ and $P - P_0 = M_z P M_{\bar{z}}$. Let H_k denote the Hankel operator on H^2 defined by $H_k f = Q(M_k f)$, and T_k denote the Toeplitz operator on H^2 defined by $T_k f = P(M_k f)$.

In this paper it is shown that $\|QP - PQ\| \leqslant 1/4$. Let $H_\phi^* H_\phi = \int_0^1 \lambda dE_\lambda$. Then it is shown that $QP - PQ$ is compact if and only if $\int_0^\varepsilon \lambda dE_\lambda$ is compact for any ε with $0 < \varepsilon < 1$. Also $QP - PQ$ is compact and $Z^- + Z^+ = Z$ if and only if $H_{\bar{\phi}}$ is compact. It is shown that $H_{\bar{\phi}}$ is compact if and only if $\int_0^1 \lambda dE_\lambda$ is compact.

Helson and Sarason [2] showed that QP is compact if and only if H_ϕ is compact. Levinson and McKean ([4], pp. 103-105) studied the weight functions which satisfy $QP - PQ = 0$ and Hayashi [1] considered the operator $QP - Q \wedge P$ where $Q \wedge P$ denote the orthogonal projection onto $Z^- \cap Z^+$ in Z.

The problem of characterizing the weight functions w that satisfy some kind of interdependence of Z^- and Z^+ is of interest in the theories of weighted trigonometric approximations and discrete weakly stationary Gaussian stochastic process (such weights arise as the spectral densities of processes). We call Z^- the past of Z and Z^+ the future of Z. The results in this paper (see [1], [2], [3] and [5]) have applications for prediction problems when such a weight arises as the spectral density of a discrete weakly stationary Gaussian stochastic process.

2. Norm of $QP - PQ$

We give three lemmas which relate the commutator $QP - PQ$ and the Hankel operator of ϕ where $\phi = \bar{h}/h$ and $w = |h|^2$.

LEMMA 1. *For* $k \in L^2$, $Q\overline{h^{-1}k} = \overline{h^{-1}M_z Q M_{\bar{z}} k}$ and $Ph^{-1}k = h^{-1}M_z P M_{\bar{z}} k$.

Proof. Since $Z = \overline{h^{-1}L^2} = \overline{h^{-1}\bar{H}^2} \oplus \overline{h^{-1}zH^2}$, $\overline{h^{-1}k} = \overline{h^{-1}}(Q + P_0)k =$

$\overline{h^{-1}M_z QM_{\bar z}k}$, $k \in L^2$. The statement for P follows similarly.

LEMMA 2. *Let* $k \in L^2$. *Then the following hold :*

(1) $QPh^{-1}k = \overline{h^{-1}M_z QM_\phi PM_{\bar z}k}$,

(2) $(QP - PQ)h^{-1}k = h^{-1}M_z(M_{\bar\phi}QM_\phi P - PM_{\bar\phi}QM_\phi)M_{\bar z}k$.

Proof. This is clear.

LEMMA 3. *Let* $A = M_{\bar\phi}QM_\phi P - PM_{\bar\phi}QM_\phi$. *Then* $A|H^2 = -H_{\bar\phi}T_\phi$ *and* $A|\bar z\bar H^2 = T_{\bar\phi}H^*_{\bar\phi}$.

Proof. We shall prove only $A|H^2 = -H_{\bar\phi}T_\phi$, since $A|\bar z\bar H^2 = T_{\bar\phi}H^*_{\bar\phi}$ follows similarly.

$$\begin{aligned} A|H^2 &= (M_{\bar\phi}QM_\phi P - PM_{\bar\phi}QM_\phi P)|H^2 \\ &= QM_{\bar\phi}QM_\phi P|H^2 \\ &= QM_{\bar\phi}(I-P)M_\phi P|H^2 \\ &= -H_{\bar\phi}T_\phi . \end{aligned}$$

The map $S : f \to hf$ is an isometry of Z onto L^2 . Then

$$M_{\bar z}S(QP - PQ) = (M_{\bar\phi}QM_\phi P - PM_{\bar\phi}QM_\phi)M_{\bar z}S$$

THEOREM 1. $\|QP - PQ\| \leqslant 1/4$.

Proof. By the remark above, $\|QP - PQ\| = \|H_{\bar\phi}T_\phi\|$.

$$\begin{aligned} (H_{\bar\phi}T_\phi)^*H_{\bar\phi}T_\phi &= T_{\bar\phi}(I - T_\phi T_{\bar\phi})T_\phi = T_{\bar\phi}T_\phi(I - T_{\bar\phi}T_\phi) \\ &= (I - H^*_\phi H_\phi)H^*_\phi H_\phi . \end{aligned}$$

If $H^*_\phi H_\phi = \int_0^1 \lambda dE_\lambda$ then

$$(H_{\bar\phi}T_\phi)^*H_{\bar\phi}T_\phi = \int_0^1 \lambda(1-\lambda)dE_\lambda$$

and this implies $\|QP - PQ\| \leqslant 1/4$.

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A theorem of Helson and Szegö [3] shows that $\|QP\| < 1$ if and only if $\sigma(H_\phi^* H_\phi) \subset [0,1-\varepsilon]$ for some $\varepsilon > 0$ where $\sigma(H_\phi^* H_\phi)$ is the spectrum of $H_\phi^* H_\phi$. The proof of Theorem 1 shows that $\|QP - PQ\| < 1/4$ if and only if $\sigma(H_\phi^* H_\phi) \subset [0,1/2-\varepsilon] \cup [1/2+\varepsilon,1]$.

3. Compactness of $QP - PQ$

$w = |h|^2$ for some outer function h in H^2 and $\phi = \bar{h}/h$. Let

$$H_\phi^* H_\phi = \int_0^1 \lambda dE_\lambda .$$

THEOREM 2. *The following three properties are equivalent.*

(1) $QP - PQ$ *is compact ;*

(2) $H_{\bar\phi} T_\phi$ *is compact ;*

(3) For any ε *with* $0 < \varepsilon < 1$, $\int_0^\varepsilon \lambda dE_\lambda$ *and* $\int_\varepsilon^1 (1-\lambda) dE_\lambda$ *are compact.*

Proof. Lemmas 2 and 3 imply (1) $\Leftrightarrow$ (2).

(2) $\Leftrightarrow$ (3). $(H_{\bar\phi} T_\phi)^* H_{\bar\phi} T_\phi = \int_0^1 \lambda(1-\lambda) dE_\lambda$ by the proof of Theorem 1.

Hence $H_{\bar\phi} T_\phi$ is compact if and only if for any ε, with $0 < \varepsilon < 1$, $\int_0^\varepsilon \lambda(1-\lambda) dE_\lambda$ and $\int_\varepsilon^1 \lambda(1-\lambda) dE_\lambda$ are compact, that is if and only if (3) holds.

THEOREM 3. *The following three properties are equivalent.*

(1) $QP - PQ$ *is compact and* $Z^- + Z^+ = Z$ *;*

(2) $H_{\bar\phi}$ *is compact ;*

(3) For any ε, *with* $0 < \varepsilon < 1$, $\int_0^\varepsilon \lambda dE_\lambda$ *is compact and* $\int_\varepsilon^1 (1-\lambda) dE_\lambda$ *has finite rank .*

Proof. (1) $\Rightarrow$ (2). Since $\overline{h^{-1}\bar{H}^2} + h^{-1} z H^2 = h^{-1} L^2$, $\bar{z}\bar{H}^2 + \phi H^2 = L^2$ and T_ϕ is right invertible. By Theorem 2, $H_{\bar\phi}$ is compact.

(2) $\Rightarrow$ (3). If $H_{\bar\phi}$ is compact and $\|H_{\bar\phi}\| = 1$ then there exists f

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in H^2 with $\|f\|_2 = 1$ such that $H^*_{\bar\phi}H_{\bar\phi}f = f$ and so $T_{\bar\phi}f = 0$. This contradicts $\ker T_{\bar\phi} = \{0\}$. Hence if $H_{\bar\phi}$ is compact then $\|H_{\bar\phi}\| < 1$. Thus $T_\phi H^2 = H^2$ and so $\|T_\phi f\|_2 \geq \delta\|f\|_2$, $f \in M$ for some $\delta > 0$ where $M = (\ker T_\phi)^\perp$. Let P_M be the orthogonal projection from L^2 onto M. Then there exists a bounded linear operator K such that $KT_{\bar\phi}T_\phi = P_M$. $P_M H^*_\phi H_\phi$ is compact because $H_{\bar\phi}$ is compact and $(H_{\bar\phi}T_\phi)^* H_{\bar\phi}T_\phi = T_{\bar\phi}T_\phi H^*_\phi H_\phi$. Since $\ker T_\phi = (E_1 - E_{1-0})H^2$, $P_M H^*_\phi H_\phi = H^*_\phi H_\phi P_M$. Thus $H^*_\phi H_\phi P_M$ is compact and so $\int_0^{1-} \lambda dE_\lambda$ is compact. This implies (3).

(3) $\Rightarrow$ (2). By Theorem 2 $H_{\bar\phi}T_\phi$ is compact. Since $\int_0^{1-} \lambda dE_\lambda$ is compact, $\|T_\phi f\|_2 \geq \delta\|f\|_2$, $f \in M$ for some $\delta > 0$ because $H^*_\phi H_\phi | M = (I - T^*_\phi T_\phi) | M$ is compact. This implies that T_ϕ is right invertible and so $H_{\bar\phi}$ is compact.

(2) $\Rightarrow$ (1). As in the proof of (2) $\Rightarrow$ (3) if $H_{\bar\phi}$ is compact then T_ϕ is left invertible. T_ϕ is left invertible if and only if $\bar z\bar H^2 + \phi H^2 = L^2$. This and Theorem 2 imply (1).

THEOREM 4. *The following three conditions are equivalent.*

(1) $QP - PQ$ *has finite rank* $2n$ *;*

(2) $H_{\bar\phi}$ *has finite rank* n *;*

(3) for any ε *, with* $0 < \varepsilon < 1$ *,* $\int_0^\varepsilon \lambda dE_\lambda$ *has finite rank* m *and* $\int_\varepsilon^1 (1-\lambda) dE_\lambda$ *has finite rank* ℓ *with* $m + \ell = n$ *.*

Proof. By Lemma 3 rank $(QP - PQ) = 2$ rank $H_{\bar\phi}T_\phi$. Since $\ker T_{\bar\phi} = \{0\}$,

$$\text{rank } H_{\bar\phi}T_\phi = \text{rank } T_{\bar\phi}H^*_{\bar\phi} = \text{rank } H^*_{\bar\phi} = \text{rank } H_{\bar\phi} .$$

These imply (1) $\Leftrightarrow$ (2). (2) $\Leftrightarrow$ (3) can be shown in a similar manner to (2) $\Leftrightarrow$ (3) in Theorem 3.

4. Weight functions

$w = |h|^2$ for some outer function h in H^2 and $\phi = \bar{h}/h$. Let C denote the set of all continuous functions on the unit circle. Using a theorem of Wolff [6] we can show that $w = |h_0|^2 e^{u+\tilde{v}}$ where h_0 is an outer function in $H^{2-} = \bigcap_{p<2} H^p$, $\bar{h}_0/h_0 = \bar{F}G$ for some inner functions F and G and where u and v are real functions in C ($\tilde{v}$ denotes the harmonic conjugate of v).

PROPOSITION 5. *$QP - PQ$ is compact and $Z^- + Z^+ = Z$ if and only if*

$$w = |h_0|^2 e^{u+\tilde{v}}$$

where h_0 is an outer function in H^{2-}, $\bar{h}_0/h_0 = \bar{F}$ for some inner function F and where u and v are real functions in C.

Proof. If $QP - PQ$ is compact and $Z^- + Z^+ = Z$ then by Theorem 3 $H_{\bar{\phi}}$ is compact. Hence $\bar{\phi} \in H^\infty + C$. By a theorem of Wolff [6] $\phi = \bar{F}e^{i(v-\tilde{u})}$ where F is an inner function and where u and v are real functions in C. Put $g = e^{-\tilde{v}-u+i(v-\tilde{u})}$ then

$$|g||h|^2 = \bar{F}gh^2 \geq 0 \qquad \text{a.e.}\ .$$

If $h_0^2 = gh^2$ then $h_0^2 \in H^{2-}$ by a theorem of Zygmund (see [4], p. 140). Since $|h_0|^2 = \bar{F}h_0^2$, $\bar{h}_0/h_0 = \bar{F}$ and $w = |h_0|^2 e^{u+\tilde{v}}$. Conversely if $|h|^2 = |h_0|^2 e^{u+\tilde{v}}$ then $\bar{\phi} \in H^\infty + C$. Since $H_{\bar{\phi}}$ is compact, Theorem 3 implies the proposition.

Let $Z^{+/-}$ be the closure of the projection of Z^+ on Z^- then $Z^- \supset Z^{+/-}$. Levinson and McKean ([5], pp. 103-105) showed that $Z^- \neq Z^{+/-}$ if and only if $\phi = \bar{F}G$ for some inner functions F and G.

PROPOSITION 6. *$QP - PQ$ has finite rank $2n$ if and only if $\phi = \bar{F}G$ where F is an inner function and G is a finite Blashke product of degree n.*

Proof. If $QP - PQ$ has finite rank $2n$ then by Theorem 4 $H_{\bar{\phi}}$ has finite rank n. Then $\ker H_{\bar{\phi}} \neq \{0\}$ and so $\ker H_{\bar{\phi}} = GH^2$ for some inner function G by Beurling's theorem. Hence $\phi = \bar{F}G$ for some inner

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function F . Since rank $H_{\bar{\phi}}$ = codim ker $H_{\bar{\phi}} = \dim(H^2 \ominus GH^2)$, G is a finite Blashke product of degree n . Conversely if $\phi = \bar{F}G$ and G is a finite Blashke product of degree n then $H_{\bar{\phi}}$ has finite rank n . Theorem 4 implies $QP - PQ$ has finite rank $2n$.

Hayashi ([1], Theorem 2) showed that $QP - Q \wedge P$ is compact if and only if $H_{\bar{\phi}}$ is compact. The proof shows that $QP - Q \wedge P$ has finite rank n if and only if $H_{\bar{\phi}}$ has finite rank n . Thus

rank $(QP - PQ) = 2$ rank $(QP - Q \wedge P)$.

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