A THEOREM ON PARTIALLY ORDERED SETS AND ITS APPLICATION

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Let $(S, \leqq)$ be a (non-void) partially ordered set with the property that for every (non-void) chain C (i.e., every totally ordered subset) of S, there exists in S the element $sup\ C$. Let S_M be the set of all maximal elements s of S. [1] Let $f : S \setminus S_M \to S$ be a *slowly increasing* mapping in the sense that

$$(\forall s \in S \setminus S_M)[s < fs \ \& \ non\ (\exists z \in S)\ s < z < fs].$$

Let s_0 be a fixed element of S.

A subset Z of S will be called *closed* if

(i) $s_0 \in Z$,
(ii) for every (non-void) chain C in Z, $sup\ C$ is in Z,
(iii) if $z \in Z \setminus S_M$, then $fz \in Z$.

There exist closed subsets of S; for example, S itself. Let T be the intersection of all closed subsets of S; T itself is closed.

THEOREM. *T is well-ordered.*

We prove first that T is a chain (i.e. totally ordered). Let us suppose the contrary, that is, let us suppose that in T there are incomparable elements and let W be the set of all elements of T which are less than or equal to each element t of T for which there is in T an element incomparable to t.

For all $t \in T$, $s_0 \leqq t$ (otherwise, the proper subset of T consisting of all elements t of T for which $s_0 \leqq t$ would be closed, contrary to the definition of T). Hence W is non-void. By definition of W, the elements of W are comparable with every $t \in T$, W is a chain and $sup\ W \in W$. Let X be the subset of T defined by

$$X \underset{\text{Df}}{=} \{t | t \in T \ \& \ (t \leqq sup\ W \vee f\ sup\ W \leqq t)\}.$$

X is closed (since for $t < sup\ W$, by definition of W ft must be comparable to $sup\ W$ and by the assumption on f it cannot be $sup\ W < ft$, hence

1 We are not interested here that, assuming the axiom of choice, according to Zorn's lemma S_M is necessarily non-void.

$ft \leqq sup\ W$). Hence, by definition of X, $X = T$. But then $f\,sup\ W$, too, would be an element of W, contrary to the definition of W. This proves that T cannot contain incomparable elements i.e. $W = T$.

To prove that the ordering of T is a well-ordering, consider a non-void subset Y of T and let T_0 be the set of all elements of T which are less than or equal to each element of Y. Let $sup\ T_0 = t_0$; we have to prove that $t_0 \in Y$. But this is immediate, since otherwise ft_0 would contradict either the definition of t_0 or the assumption that f is slowly increasing.

COROLLARY. *Assuming the axiom of choice, every set can be well-ordered.*

PROOF. Let R be a non-void set and $S = \mathscr{P}R$ its power-set. We define in S an order relation $\leqq$ by

$$(\forall s_1, s_2 \in S) \qquad s_1 \leqq s_2 \underset{\mathrm{Df}}{\Leftrightarrow} s_1 \supset s_2.$$

Obviously, for every chain C in S, $sup\ C \in S$. $S_M = \{\emptyset\}$. Let γ be a choice function for S, i.e. $(\forall s \in S\diagdown\{\emptyset\})\ \gamma s \in s$. Then $f : S\diagdown\{\emptyset\} \to S$ defined by

$$(\forall s \in S\diagdown\{\emptyset\}) \qquad fs \underset{\mathrm{Df}}{=} s\diagdown\{\gamma s\}$$

is slowly increasing. Let furthermore $s_0 = R$.

By the Theorem, the corresponding subset T of S is well-ordered. To prove that R can be well-ordered, it suffices to define a $1-1$ mapping φ of R into T. This can be done by putting

$$(\forall r \in R) \qquad \varphi r = \sup_{r \in t \in T} t,$$

since then $\gamma\varphi r = r$ (otherwise, it would be $r \in \varphi r\diagdown\{\gamma\varphi r\}$, contrary to the definition of φr).

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