COMPOSITIO MATHEMATICA

Proper isometric actions of hyperbolic groups on L^p-spaces

Bogdan Nica

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Abstract

We show that every non-elementary hyperbolic group Γ admits a proper affine isometric action on $L^p(\partial\Gamma \times \partial\Gamma)$, where $\partial\Gamma$ denotes the boundary of Γ and p is large enough. Our construction involves a Γ-invariant measure on $\partial\Gamma \times \partial\Gamma$ analogous to the Bowen–Margulis measure from the CAT(-1) setting, as well as a geometric, Busemann-type cocycle. We also deduce that Γ admits a proper affine isometric action on the first ℓ^p-cohomology group $H^1_{(p)}(\Gamma)$ for large enough p.

1. Introduction

With respect to the geometry of L^2-spaces, the class of hyperbolic groups appears to be both 'soft' and 'rigid'. This ambivalence is vividly illustrated by cocompact lattices in isometry groups of rank-1 symmetric spaces. For real or complex hyperbolic spaces, cocompact lattices admit proper isometric actions on Hilbert spaces. For quaternionic hyperbolic spaces or the octonionic hyperbolic plane, every isometric action of a cocompact lattice on a Hilbert space is bounded. In other words, cocompact lattices enjoy the Haagerup property, or a-T-menability, in the real or complex case (see [CCJJV01]), respectively Kazhdan's property (T), or Serre's property FH, in the quaternionic or octonionic case (see [BlHV08]).

When tested against general L^p-spaces, hyperbolic groups reveal themselves to be 'soft': every hyperbolic group admits a proper isometric action on an L^p-space, where p depends on the group. This fact, due to Yu, is one of the most interesting results in the study of isometric group actions on uniformly convex Banach spaces. More precisely, the following is shown in [Yu05].

THEOREM 1 (Yu). *Let Γ be a hyperbolic group. Then, for large enough p, the linear isometric action of Γ on $\ell^p(\Gamma \times \Gamma)$ admits a proper cocycle.*

If we let a non-elementary hyperbolic group Γ act on its boundary $\partial\Gamma$ rather than on itself, then a number of finiteness properties emerge at infinity. An example is the fact that, although Γ is non-amenable, the action of Γ on $\partial\Gamma$ is amenable (Adams [Ada94]). In a recent joint work with Emerson [EN12], we construct Fredholm modules for the C*-crossed product $C(\partial\Gamma) \rtimes \Gamma$ which are p-summable for every $p \in (2, \infty)$ greater than the visual dimension of $\partial\Gamma$. Very informally, this means that the action of a hyperbolic group on its boundary is summable above the Hausdorff dimension of the boundary. This finiteness phenomenon is the inspiration for the following boundary analogue of Yu's theorem.

THEOREM 2. *Let Γ be a non-elementary hyperbolic group and let $\partial\Gamma$ denote its boundary. Then, for large enough p, the linear isometric action of Γ on $L^p(\partial\Gamma \times \partial\Gamma)$ admits a proper cocycle.*

On the surface, Theorem 2 conveys the same idea as Theorem 1, namely that hyperbolic groups admit proper isometric actions on L^p-spaces. The two theorems are nevertheless different both in statement and in proof. Our approach has the following novel features.

Framework. In the first part of the paper, we put forth the following principle: a Möbius action on an Ahlfors regular, compact metric space gives rise to an affine isometric action on an L^p-space for each p greater than the Hausdorff dimension of the metric space. See § 5.

Measure. For the action of a non-elementary hyperbolic group Γ on its boundary $\partial\Gamma$, the Möbius philosophy yields two ingredients. The first is an explicit Γ-invariant measure on $\partial\Gamma \times \partial\Gamma$, twin to the Bowen–Margulis measure encountered in the CAT(-1) setting. Our generalized Bowen–Margulis measure significantly improves a previous construction by Furman [Fur02, Proposition 1]. See § 7.1.

Cocycle. The second ingredient is a beautiful geometric cocycle for the action of Γ on $\partial\Gamma \times \partial\Gamma$. Its memorable form suggests that it could be interpreted as the *other* Busemann cocycle. The properness of this cocycle depends on the hyperbolicity of Γ. See § 7.2.

Exponent. The final conceptual advantage of the Möbius philosophy is that it provides an integrability exponent p which is related to a suitable interpretation of the Hausdorff dimension of the boundary $\partial\Gamma$, the *hyperbolic dimension* introduced by Mineyev [Min07]. Alternatively, and somewhat less sharply, p is related to a modified growth exponent of Γ. See § 7.3.

ℓ^p-cohomology. Our construction of a proper isometric action of Γ on $L^p(\partial\Gamma \times \partial\Gamma)$ has the following ℓ^p-cohomological interpretation:

THEOREM 3. *Let Γ be a non-elementary hyperbolic group. Then, for large enough p, the linear isometric action of Γ on the first ℓ^p-cohomology group $H^1_{(p)}(\Gamma)$ admits a proper cocycle.*

Again, we provide an appealing cocycle for the action, see § 8. Typical results on the first ℓ^p-cohomology of finitely generated groups are concerned with the vanishing–non-vanishing dichotomy, and Theorem 3 is entirely new in that respect. Its proof uses a result of Bourdon and Pajot [BP03].

Differences aside, the proofs of Theorems 2 and 1 share a common technical point, and that is Mineyev's powerful re-metrization procedure. For our purposes, the relevant upshot of this procedure is that it leads to visual metrics on the boundary with much better properties than the visual metrics coming from the word metric. These new visual metrics were constructed by Mineyev in [Min07].

We would like to mention another recent result concerning proper isometric actions of hyperbolic groups on L^p-spaces. In [Bou], Bourdon shows the following: for every non-elementary hyperbolic group Γ, there is a positive integer n such that the linear isometric action of Γ on $\ell^p(\sqcup_1^n \Gamma)$ admits a proper cocycle for every $p > \text{Confdim } \partial\Gamma$. Here, too, the integrability exponent p is related to a suitable interpretation of the Hausdorff dimension of the boundary: the Ahlfors regular, conformal dimension Confdim, which is easily seen to be no larger than the hyperbolic dimension mentioned above. Thus, Bourdon's exponent bound is in principle better than ours, though discriminating examples are probably very hard to construct (if there are any at all). Note however that the linear part of the action requires several copies of the regular representation, and there seems to be no explicit formula for the number of copies one needs to consider. Bourdon's

construction of proper isometric actions is achieved via ℓ^p-cohomology; in particular, it has the advantage of dispensing with Mineyev's technical procedure.

2. Preliminaries

2.1 Notation

We write $\preccurlyeq$ to mean inequality up to a positive multiplicative constant, and the corresponding equivalence is denoted $\asymp$. The constants involved in these relations often depend on some parameter, and we record this dependence as a subscript (e.g., $\preccurlyeq_\varepsilon$).

2.2 Isometric actions on L^p-spaces

By the Mazur–Ulam theorem (see [Nic12] for a short proof), isometric group actions on real Banach spaces are affine. An isometric action of a discrete group Γ on a real L^p-space is obtained from two ingredients. The first ingredient is a measured space X on which Γ acts in a measure-preserving way; then $(g, F) \mapsto g \cdot F$ is a linear isometric action of Γ on $L^p(X)$. The second ingredient is a cocycle for this linear isometric action, that is a map $c \colon \Gamma \to L^p(X)$ satisfying $c_{gh} = g \cdot c_h + c_g$ for all $g, h \in \Gamma$; then $(g, F) \mapsto g \cdot F + c_g$ is an affine isometric action of Γ on $L^p(X)$. The latter isometric action is proper if $\|c_g\|_{L^p(X)} \to \infty$ as $g \to \infty$ in Γ.

Throughout the paper, it is understood that $p \in [1, \infty)$.

3. Proper isometric actions of free groups on L^p-spaces

As a warm-up, we start by giving a simple and self-contained proof of Theorem 2 in the case of free groups. The purpose of this discussion is to foreshadow two key points, developed later: the construction of affine isometric actions on L^p-spaces from Möbius actions (§5), and the properness of the cocycle in the case of hyperbolic groups (§7).

3.1 The boundary Ω

Let $\mathbb{F}_n$ be the free group on $n \geqslant 2$ generators, and put $q = 2n - 1$. The Cayley graph of $\mathbb{F}_n$ with respect to the standard generators is the $2n$-valent tree, rooted at the identity element. The boundary of this tree, customarily denoted by Ω, is the set of all infinite rooted paths without backtracks. We endow Ω with a probability measure μ defined by the requirement that

$$\mu(\Omega_x) = \frac{q}{q+1} q^{-|x|}$$

for all non-identity elements $x \in \mathbb{F}_n$. Here Ω_x denotes the 'boundary under x', that is, the boundary subset consisting of all those $\omega \in \Omega$ which start with x, and $|x|$ is the length of x.

The Poisson kernel is given by

$$P_g(\omega) = q^{-|g|+2(g,\omega)} \quad (g \in \mathbb{F}_n, \omega \in \Omega)$$

where $(\cdot, \cdot)$, the Gromov product based at the identity, measures the length of the longest shared path. The Poisson kernel P_g represents the Radon–Nikodym derivative $d(g_*\mu)/d\mu$, and it satisfies the cocycle relation $P_{gh} = P_g \, g \cdot P_h$ for $g, h \in \mathbb{F}_n$. (For more details see, for instance, Figà-Talamanca and Picardello [FP82].)

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3.2 An invariant measure on $\Omega \times \Omega$

The following key relation is easy to check:

$$q^{-2(g\xi, g\omega)} = P_{g^{-1}}(\xi) P_{g^{-1}}(\omega) q^{-2(\xi, \omega)} \tag{†}$$

for all $g \in \mathbb{F}_n$ and $\xi, \omega \in \Omega$. The main consequence of this relation is that a suitably weighted product measure on $\Omega \times \Omega$ is $\mathbb{F}_n$-invariant. Namely, let ν be the measure on $\Omega \times \Omega$ given by

$$d\nu = q^{2(\xi, \omega)} \, d\mu(\xi) \, d\mu(\omega).$$

The diagonal of $\Omega \times \Omega$ is $(\mu \times \mu)$-negligible, since points of Ω are μ-negligible. Therefore ν is well defined, and the diagonal of $\Omega \times \Omega$ is ν-negligible as well.

Note also that ν is infinite and σ-finite. Indeed, consider the countable partition of $\Omega \times \Omega - \mathrm{diag}$ given by the sets $K_n = \{(\xi, \omega) \in \Omega \times \Omega : (\xi, \omega) = n\}$, where $n \geqslant 0$. We claim that each K_n has finite ν-measure, and $\sum \nu(K_n) = \infty$. When $n \geqslant 1$, we have $\mu(\{\xi \in \Omega : (\xi, \omega) = n\}) = \frac{q-1}{q+1} q^{-n}$ for each $\omega \in \Omega$. Then $(\mu \times \mu)(K_n) = \frac{q-1}{q+1} q^{-n}$, and hence $\nu(K_n) = \frac{q-1}{q+1} q^n$. A similar argument shows that $\nu(K_0) = \frac{q}{q+1}$.

LEMMA 4. *The measure ν is invariant for the diagonal action of $\mathbb{F}_n$.*

Proof. Let $F \in L^1(\Omega \times \Omega, \nu)$ and $g \in \mathbb{F}_n$. Then

$$\int g \cdot F \, d\nu = \iint F(g^{-1}\xi, g^{-1}\omega) q^{2(\xi, \omega)} \, d\mu(\xi) \, d\mu(\omega)$$

$$= \iint F(\xi, \omega) q^{2(g\xi, g\omega)} \, dg^*\mu(\xi) \, dg^*\mu(\omega)$$

$$= \iint F(\xi, \omega) q^{2(g\xi, g\omega)} P_{g^{-1}}(\xi) P_{g^{-1}}(\omega) \, d\mu(\xi) \, d\mu(\omega)$$

$$= \iint F(\xi, \omega) q^{2(\xi, \omega)} \, d\mu(\xi) \, d\mu(\omega) = \int F \, d\nu.$$

The above computation involves a change of variables, followed by an application of (†). □

3.3 A cocycle for the action on $\Omega \times \Omega$

A natural cocycle for the action of $\mathbb{F}_n$ on Ω is the logarithm of the Poisson kernel $g \mapsto \log P_g$, and a cocycle in two variables can be obtained by taking the difference of two such cocycles. More precisely, if

$$c_g(\xi, \omega) := \tfrac{1}{2}(\log P_g(\xi) - \log P_g(\omega)) = (g, \xi) - (g, \omega)$$

then $g \mapsto c_g$ a cocycle for the diagonal action of $\mathbb{F}_n$ on $\Omega \times \Omega$. The next proposition shows that, for each p, c is a proper cocycle for the linear isometric representation of $\mathbb{F}_n$ on $L^p(\Omega \times \Omega, \nu)$.

PROPOSITION 5. *We have $\|c_g\|_{L^p(\nu)} \asymp_p |g|^{1/p}$.*

Proof. Let $g \in \mathbb{F}_n$ and write

$$\|c_g\|_{L^p(\nu)}^p = \iint |(g, \xi) - (g, \omega)|^p \, q^{2(\xi, \omega)} \, d\mu(\xi) \, d\mu(\omega).$$

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As ξ runs over Ω, the Gromov product (g, ξ) takes on the values $0, 1, \ldots, |g|$. Thus we have

$$\|c_g\|_{L^p(\nu)}^p = \sum_{i,j=0}^{|g|} \iint_{\substack{(g,\xi)=i \\ (g,\omega)=j}} |(g,\xi) - (g,\omega)|^p q^{2(\xi,\omega)} \, d\mu(\xi) \, d\mu(\omega)$$

$$= \sum_{i,j=0}^{|g|} |i-j|^p q^{2\min\{i,j\}} \mu(\{\xi : (g,\xi) = i\}) \mu(\{\omega : (g,\omega) = j\}),$$

using the fact that $(\xi, \omega) = \min\{(g,\xi), (g,\omega)\}$ whenever $(g,\xi) \neq (g,\omega)$. For $i \in \{0, 1, \ldots, |g|\}$, we have $\mu(\{\xi : (g,\xi) = i\}) \asymp q^{-i}$. In fact,

$$\frac{q-1}{q+1} q^{-i} \leqslant \mu(\{\xi : (g,\xi) = i\}) \leqslant \frac{q}{q+1} q^{-i}$$

with equality on the left for all $i \neq 0, |g|$, and equality on the right at the endpoints $0, |g|$. Hence, letting

$$S_N := \sum_{i,j=0}^{N} |i-j|^p q^{2\min\{i,j\}} q^{-i} q^{-j} = \sum_{i,j=0}^{N} |i-j|^p q^{-|i-j|},$$

we have $\|c_g\|_{L^p(\nu)}^p \asymp S_{|g|}$. Now $S_N \asymp_p N$: the recurrence

$$S_{N+1} = \sum_{i,j=0}^{N+1} |i-j|^p q^{-|i-j|} = S_N + 2 \sum_{i=0}^{N} (N+1-i)^p q^{-(N+1-i)} = S_N + 2 \sum_{i=1}^{N+1} i^p q^{-i}$$

gives $S_N + 2q^{-1} \leqslant S_{N+1} < S_N + 2(\sum_{i \geqslant 1} i^p q^{-i})$. We conclude that $\|c_g\|_{L^p(\nu)}^p \asymp_p |g|$. $\qquad\square$

There are, certainly, easier ways to produce proper isometric actions of free groups on L^p-spaces. Here is one such action. Let $\vec{X}$ be the directed Cayley graph of $\mathbb{F}_n$ with respect to the standard generators. Let $\{g \to h\}$ denote the shortest oriented path in $\vec{X}$ from $g \in \mathbb{F}_n$ to $h \in \mathbb{F}_n$. Then $g \mapsto c_g := (\sum_{e \in \{1 \to g\}} e - \sum_{e \in \{g \to 1\}} e)$ defines a proper cocycle for the linear isometric action of $\mathbb{F}_n$ on the ℓ^p-space of the edge-set of $\vec{X}$.

Our aim in what follows is to promote the boundary-based method presented in this section to general non-elementary hyperbolic groups.

4. Interlude: Möbius calculus

In this section we discuss 'derivatives' of Möbius maps. The facts established herein will be used in the next section to construct affine isometric actions of Möbius groups on L^p-spaces.

Throughout, we let (X, d) be a compact metric space without isolated points and we consider Möbius self-homeomorphisms of X. Recall that the *cross-ratio* of a quadruple of distinct points in X is defined by the formula

$$(z_1, z_2; z_3, z_4) = \frac{d(z_1, z_3) d(z_2, z_4)}{d(z_1, z_4) d(z_2, z_3)}.$$

A homeomorphism $g : X \to X$ is called a *Möbius homeomorphism* if g preserves the cross-ratios, i.e., $(gz_1, gz_2; gz_3, gz_4) = (z_1, z_2; z_3, z_4)$ for all quadruples of distinct points $z_1, z_2, z_3, z_4 \in X$.

LEMMA 6. *Let g be a self-homeomorphism of X. Then g is Möbius if and only if there exists a positive continuous function on X, denoted $|g'|$, with the property that for all $x, y \in X$*

we have

$$d^2(gx, gy) = |g'|(x)|g'|(y)d^2(x, y). \tag{$*$}$$

Before we prove the lemma, let us observe that a continuous function $|g'|$ satisfying $(*)$ has, in particular, the property that

$$\lim_{y \to x} \frac{d(gx, gy)}{d(x, y)} = |g'|(x)$$

for all $x \in X$. This property justifies the notation, as well as the interpretation of $|g'|$ as the *metric derivative* of g. Following Sullivan [Sul79, §4], we also interpret the relation $(*)$ as a geometric mean-value property.

Proof of Lemma 6. We prove the forward implication. The converse is a trivial verification.

Assume that g is a Möbius homeomorphism. Let x, u, v be a triple of distinct points in X. For any fourth distinct point y we have

$$\frac{d(gx, gy)}{d(x, y)} \frac{d(gu, gv)}{d(u, v)} = \frac{d(gx, gu)}{d(x, u)} \frac{d(gy, gv)}{d(y, v)}$$

since g preserves the cross-ratios. When $y \to x$, we obtain

$$\lim_{y \to x} \frac{d(gx, gy)}{d(x, y)} = \frac{d(gx, gu)}{d(x, u)} \frac{d(gx, gv)}{d(x, v)} \frac{d(u, v)}{d(gu, gv)}. \tag{1}$$

Let $|g'|_{u,v}$ denote the expression on the right-hand side of (1), viewed as a function of x. Then $|g'|_{u,v}$ is a positive continuous function on $X - \{u, v\}$. However, the left-hand side of (1) is independent of the choice of u, v. Thus, picking $\tilde{u}, \tilde{v}$ distinct points in $X - \{u, v\}$, we have that $|g'|_{u,v} = |g'|_{\tilde{u},\tilde{v}}$ on $X - \{u, \tilde{u}, v, \tilde{v}\}$. Defining $|g'|$ on X as $|g'|_{u,v}$ on $X - \{u, v\}$, and $|g'|_{\tilde{u},\tilde{v}}$ on $X - \{\tilde{u}, \tilde{v}\}$, we obtain a positive continuous function.

Now let us prove $(*)$ for distinct $x, y \in X$. Let u, v be distinct points in $X - \{x, y\}$, so that we can use the local formula $|g'|_{u,v}$ for $|g'|$. The equality $d^2(gx, gy) = |g'|_{u,v}(x)|g'|_{u,v}(y)d^2(x, y)$ can be readily checked by rearranging factors and using the g-invariance of the cross-ratios. $\square$

The next lemma shows that metric derivatives are more than just continuous.

LEMMA 7. *Let g be a Möbius self-homeomorphism of X. Then $|g'|$ is Lipschitz.*

Proof. We show that $\sqrt{|g'|}$ is Lipschitz. This is equivalent to $|g'|$ being Lipschitz, since

$$2\sqrt{\min |g'|} \left| \sqrt{|g'|}(x) - \sqrt{|g'|}(y) \right| \leqslant ||g'|(x) - |g'|(y)| \leqslant 2\sqrt{\max |g'|} \left| \sqrt{|g'|}(x) - \sqrt{|g'|}(y) \right|.$$

Let $x, y \in X$. There exists $z \in X - \{y\}$ such that $d(x, z) \geqslant (\operatorname{diam} X)/2$. By the geometric mean-value property $(*)$, we have

$$\sqrt{|g'|}(x) - \sqrt{|g'|}(y) = \frac{1}{\sqrt{|g'|}(z)} \left(\frac{d(gx, gz)}{d(x, z)} - \frac{d(gy, gz)}{d(y, z)} \right).$$

Using the fact that g is $(\max|g'|)$-Lipschitz, we estimate

$$\frac{d(gx,gz)}{d(x,z)} - \frac{d(gy,gz)}{d(y,z)} \leqslant \frac{d(gx,gy)}{d(x,z)} + \frac{d(gy,gz)}{d(x,z)} - \frac{d(gy,gz)}{d(y,z)}$$

$$\leqslant \frac{d(gx,gy)}{d(x,z)} + \frac{d(gy,gz)}{d(y,z)}\frac{d(x,y)}{d(x,z)}$$

$$\leqslant 2\,(\max|g'|)\frac{d(x,y)}{d(x,z)} \leqslant \frac{4\,(\max|g'|)}{\operatorname{diam} X}\,d(x,y).$$

Thus

$$\sqrt{|g'|}(x) - \sqrt{|g'|}(y) \leqslant \frac{4}{\operatorname{diam} X}\,\frac{\max|g'|}{\sqrt{\min|g'|}}\,d(x,y) \tag{2}$$

so $\sqrt{|g'|}$ is, indeed, Lipschitz. $\qquad\square$

We now take a measure-theoretic turn. Recall that the Hausdorff measure of dimension $D \geqslant 0$ on X is defined by the formula

$$\mu_D(A) = \lim_{\delta \to 0}\left(\inf\left\{\sum(\operatorname{diam} U_i)^D\,:\,(U_i)\;\delta\text{-cover of }A\right\}\right) \quad (A \subseteq X).$$

This is a Borel measure which is interesting for a single D only, the Hausdorff dimension of X.

The following lemma is a variation on a basic observation of Sullivan (cf. [Sul79, p. 174]).

LEMMA 8. *Assume that X has finite, non-zero Hausdorff dimension D, and let g be a Möbius self-homeomorphism of X. Then $|g'|^D$ represents the Radon–Nikodym derivative $dg^*\mu_D/d\mu_D$.*

Proof. Fix a measurable subset $S \subseteq X$. We want to show that

$$\mu_D(gS) = \int_S |g'|^D\,d\mu_D. \tag{3}$$

From the geometric mean-value property $(*)$, we get that

$$\left(\inf_U |g'|\right)\operatorname{diam} U \leqslant \operatorname{diam} gU \leqslant \left(\sup_U |g'|\right)\operatorname{diam} U$$

for every $U \subseteq X$. It follows that, for every measurable $T \subseteq X$, we have

$$\left(\inf_T |g'|\right)^D \mu_D(T) \leqslant \mu_D(gT) \leqslant \left(\sup_T |g'|\right)^D \mu_D(T). \tag{4}$$

Now let $\varepsilon > 0$. Let also $\eta > 0$ be such that $\big||g'|(x) - |g'|(y)\big| < \varepsilon \min|g'|$ whenever $d(x,y) < \eta$. Hence, if T is a measurable η-set, in the sense that $\operatorname{diam} T < \eta$, then $\sup_T |g'| \leqslant (1+\varepsilon)\inf_T |g'|$, which in turn yields

$$(1+\varepsilon)^{-D}\left(\sup_T |g'|\right)^D \mu_D(T) \leqslant \int_T |g'|^D\,d\mu_D \leqslant (1+\varepsilon)^D\left(\inf_T |g'|\right)^D \mu_D(T).$$

Thus, in light of (4), we get that for every measurable η-set T the following holds:

$$(1+\varepsilon)^{-D}\,\mu_D(gT) \leqslant \int_T |g'|^D\,d\mu_D \leqslant (1+\varepsilon)^D\,\mu_D(gT). \tag{5}$$

The measurable set S we started with may be partitioned into a finite number of measurable η-subsets. (Indeed, pick a finite cover of X by open η-subsets. This cover gives rise to a finite partition of X into Borel η-subsets, which partition can be used on any measurable subset of X.)

Applying (5) to these η-pieces of S, and then adding up, we get

$$(1+\varepsilon)^{-D}\, \mu_D(gS) \leqslant \int_S |g'|^D \, d\mu_D \leqslant (1+\varepsilon)^D\, \mu_D(gS).$$

Since ε is arbitrary, we conclude that (3) holds. $\qquad\square$

5. Affine isometric actions of Möbius groups on L^p-spaces

As before, (X, d) is a compact metric space without isolated points. Throughout this section we also assume that the Hausdorff dimension of X, denoted by D, is finite and non-zero. The goal is to construct an affine isometric action of $\mathrm{M\ddot{o}b}(X)$, the group of Möbius self-homeomorphisms of X, on $L^p(X \times X)$.

5.1 A Möbius-invariant measure on $X \times X$

The relevant measure on $X \times X$ is given by a suitable weighting of the product measure $\mu_D \times \mu_D$, where μ_D is the D-dimensional Hausdorff measure on X.

LEMMA 9. *Assume $\mu_D(X) < \infty$. Then*

$$d\nu(x, y) = d^{-2D}(x, y)\, d\mu_D(x)\, d\mu_D(y)$$

defines a σ-finite Borel measure on $X \times X$. The diagonal of $X \times X$ is ν-negligible, and, on the locally compact and σ-compact space $X \times X - \mathrm{diag}$, the measure ν is a Radon measure.

Proof. By the separability of X, the product measure $\mu_D \times \mu_D$ is a Borel measure on $X \times X$. Points are negligible for a Hausdorff measure of positive dimension, so the diagonal of $X \times X$ is $(\mu_D \times \mu_D)$-negligible. Thus ν is well defined, and the diagonal of $X \times X$ is ν-negligible. Since ν is obtained by weighting a Borel measure, namely $\mu_D \times \mu_D$, by a Borel map, namely $d^{-2D}(\cdot, \cdot)$, it follows that ν is Borel on $X \times X$. The σ-finiteness of ν, as well as the σ-compactness of the 'slashed square' $X \times X - \mathrm{diag}$, follow by writing $X \times X - \mathrm{diag} = \bigcup_{n \geqslant 1} \{(x, y) : d(x, y) \geqslant 1/n\}$. When restricted to $X \times X - \mathrm{diag}$, ν is a Borel measure which is finite on compact subsets. The regularity of ν is automatic: it follows from [Rud87, Theorem 2.18] that, on a locally compact and σ-compact metric space, a Borel measure which is finite on compact sets is a Radon measure. $\qquad\square$

The key property of ν is its Möbius invariance.

LEMMA 10. *Assume $\mu_D(X) < \infty$. Then ν is invariant for the diagonal action of $\mathrm{M\ddot{o}b}(X)$.*

Proof. By the σ-finiteness of ν, it suffices to show that sets of finite measure are invariant. This is shown as in the proof of Lemma 4. For $F \in L^1(X \times X, \nu)$ and $g \in \mathrm{M\ddot{o}b}(X)$ we have

$$\int g \cdot F \, d\nu = \iint F(g^{-1}x, g^{-1}y)d^{-2D}(x, y)\, d\mu_D(x)\, d\mu_D(y)$$

$$= \iint F(x, y)d^{-2D}(gx, gy)\, dg^*\mu_D(x)\, dg^*\mu_D(y)$$

$$= \iint F(x, y)d^{-2D}(gx, gy)|g'|^D(x)|g'|^D(y)\, d\mu_D(x)\, d\mu_D(y)$$

$$= \iint F(x, y)d^{-2D}(x, y)\, d\mu_D(x)\, d\mu_D(y) = \int F \, d\nu.$$

The second equality is a change of variables, the third relies on Lemma 8, and the fourth is due to the geometric mean-value property ($*$). □

5.2 A cocycle for the Möbius action on $X \times X$

At this point, we have a linear isometric action of $\text{Möb}(X)$ on $L^p(X \times X, \nu)$ for each p. We need a cocycle in order to get an affine isometric action, and this arises as follows. The metric derivatives satisfy the chain rule

$$|(gh)'|(x) = |g'|(hx)|h'|(x) \tag{6}$$

for all $g, h \in \text{Möb}(X)$ and $x \in X$. In other words, $g \mapsto \log |(g^{-1})'|$ is a cocycle for the action of $\text{Möb}(X)$ on X. Hence

$$g \mapsto c_g(x, y) := \log |(g^{-1})'|(x) - \log |(g^{-1})'|(y)$$

defines a cocycle for the diagonal action of $\text{Möb}(X)$ on $X \times X$. Now $g \mapsto c_g$ is a cocycle for the linear action of $\text{Möb}(X)$ on $L^p(X \times X, \nu)$ if and only if each c_g is in $L^p(X \times X, \nu)$. This turns out to be the case for all p greater than D, the Hausdorff dimension of X, as soon as we require μ_D to be *Ahlfors regular*. This means that the measure of balls, viewed as a function of the radius $r \in [0, \text{diam}\, X]$, satisfies

$$\mu_D(r-\text{ball}) \asymp r^D.$$

In particular, if μ_D is Ahlfors regular, then $\mu_D(X) < \infty$.

LEMMA 11. *Assume that μ_D is Ahlfors regular. Then the metric d is not in $L^D(X \times X, \nu)$, but it does belong to the weak L^D-space $L^{D,\infty}(X \times X, \nu)$. Consequently, $d \in L^p(X \times X, \nu)$ for each $p > D$, and ν is an infinite measure on $X \times X$.*

Proof. To show that d is in the weak L^D-space $L^{D,\infty}(X \times X, \nu)$, we have to check that

$$\nu(\{(x, y) : d(x, y) > t\}) \preccurlyeq t^{-D}$$

as t runs over positive reals. To that end, it suffices to show that for each fixed $y \in X$ we have

$$\int_{d(x,y)>t} d^{-2D}(x, y)\, d\mu_D(x) \preccurlyeq t^{-D}$$

independent of y; and indeed

$$\int_{d(x,y)>t} d^{-2D}(x, y)\, d\mu_D(x) = \int_0^{t^{-2D}} \mu_D(\{x : d^{-2D}(x, y) > s\})\, ds$$

$$= \int_{(\text{diam}\, X)^{-2D}}^{t^{-2D}} \mu_D(\{x : d(x, y) < s^{-1/2D}\})\, ds$$

$$\leqslant \int_{(\text{diam}\, X)^{-2D}}^{t^{-2D}} C s^{-1/2}\, ds \leqslant (2C) t^{-D}.$$

The inequality in the last line uses the upper polynomial bound on the measure of balls. Next, we show that d is not in the subspace $L^D(X \times X, \nu) \subseteq L^{D,\infty}(X \times X, \nu)$, and here we use the

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lower polynomial bound on the measure of balls. For each $y \in X$ we have

$$
\int d^{-D}(x, y)\, d\mu_D(x) = \int_0^\infty \mu_D(\{x : d^{-D}(x, y) > s\})\, ds
$$
$$
= \int_{(\mathrm{diam}\, X)^{-D}}^\infty \mu_D(\{x : d(x, y) < s^{-1/D}\})\, ds
$$
$$
\geqslant \int_{(\mathrm{diam}\, X)^{-D}}^\infty c s^{-1}\, ds = \infty.
$$

Integrating with respect to y, we get $\|d\|_{L^D(\nu)} = \infty$.

The second part of the lemma follows by using the boundedness of d. As d is in weak $L^D(\nu)$ and in $L^\infty(\nu)$, interpolation yields that d is in $L^p(\nu)$ for each $p > D$. Finally, d not in $L^D(\nu)$ implies in particular that ν is infinite. $\qquad\square$

On the other hand, we may bound the cocycle by the metric as follows.

LEMMA 12. *We have*

$$
|c_g| \preccurlyeq \frac{\max |g'|}{\min |g'|}\, d.
$$

Proof. Let $g \in \mathrm{M\ddot{o}b}(X)$. Using the fact that $|\log a - \log b| \leqslant |a - b|/m$ whenever $a, b \geqslant m > 0$, together with the Lipschitz estimate (2), we get

$$
\left| \log |g'|(x) - \log |g'|(y) \right| = 2 \left| \log \sqrt{|g'|}(x) - \log \sqrt{|g'|}(y) \right|
$$
$$
\leqslant \frac{2}{\sqrt{\min |g'|}} \left| \sqrt{|g'|}(x) - \sqrt{|g'|}(y) \right| \leqslant \frac{8}{\mathrm{diam}\, X} \frac{\max |g'|}{\min |g'|}\, d(x, y).
$$

The chain rule (6) implies that $|(g^{-1})'| = 1/g \cdot |g'|$, so $\max |(g^{-1})'|/\min |(g^{-1})'| = \max |g'|/\min |g'|$. Thus, the Lipschitz estimate for $\log |g'|$ becomes

$$
|c_g(x, y)| \leqslant \frac{8}{\mathrm{diam}\, X} \frac{\max |g'|}{\min |g'|}\, d(x, y),
$$

as desired. $\qquad\square$

Lemmas 11 and 12 imply that our cocycle c takes values in $L^p(X \times X, \nu)$ for each $p > D$. Summarizing, we have proved the following.

PROPOSITION 13. *Assume that μ_D is Ahlfors regular. Then, for each $p > D$,*

$$
g \mapsto c_g(x, y) = \log |(g^{-1})'|(x) - \log |(g^{-1})'|(y)
$$

is a cocycle for the linear isometric action of $\mathrm{M\ddot{o}b}(X)$ on $L^p(X \times X, \nu)$.

5.3 A topological perspective

The groups we are ultimately interested in, namely hyperbolic groups, are discrete. Proposition 13, in which we are treating the Möbius group $\mathrm{M\ddot{o}b}(X)$ as a discrete group, suffices for our purposes. However, the Möbius context we have developed so far has a topological layer as well, and we will look at it before moving on to the case of hyperbolic groups.

Recall that the natural topology on the space $C(Z, Y)$ of continuous maps between a compact space Z and a metric space Y is the compact-open topology or, equivalently, the topology of

uniform convergence. This topology is induced by the metric

$$\mathrm{dist}(f_1, f_2) = \sup_{z \in Z} d_Y(f_1 z, f_2 z).$$

For a compact metric space Z, the group of self-homeomorphisms $\mathrm{Homeo}(Z)$ is a topological group under the topology of uniform convergence. In general, $\mathrm{Homeo}(Z)$ is not locally compact.

PROPOSITION 14. *Endow the Möbius group of X with the topology of uniform convergence. Then we have the following.*

(i) *The metric differentiation map $\mathrm{M\ddot{o}b}(X) \to C(X, \mathbb{R})$, given by $g \mapsto |g'|$, is continuous.*

(ii) *The topological group $\mathrm{M\ddot{o}b}(X)$ is a locally compact and σ-compact. It contains the isometry group $\mathrm{Isom}(X)$ as a compact subgroup.*

(iii) *For each $p > D$, the affine isometric action of $\mathrm{M\ddot{o}b}(X)$ on $L^p(X \times X, \nu)$, given by $(g, F) \mapsto g \cdot F + c_g$, is continuous.*

Proof. We start by showing that

$$\mathrm{dist}(|g'|, 1) \leqslant C^{-1} \mathrm{dist}(g, \mathrm{id}), \quad \text{provided that } \mathrm{dist}(g, \mathrm{id}) \leqslant C \tag{7}$$

for some $C > 0$ depending on X only.

There is $\kappa > 0$ such that no two open balls of radius κ cover X; the easy proof, by contradiction, is left to the reader. Now assume that $D(g) := \mathrm{dist}(g, \mathrm{id}) \leqslant \kappa/10$. Fix $x \in X$. By the defining property of κ, there are $u, v \in X$ such that $d(x, u) \geqslant \kappa$, $d(u, v) \geqslant \kappa$, $d(x, v) \geqslant \kappa$. Recall from the proof of Lemma 6 the following local formula:

$$|g'|(x) = \frac{d(gx, gu)}{d(x, u)} \frac{d(gx, gv)}{d(x, v)} \frac{d(u, v)}{d(gu, gv)}.$$

We have $d(x, u) - 2D(g) \leqslant d(gx, gu) \leqslant d(x, u) + 2D(g)$. As $d(x, u) \geqslant \kappa$, we obtain

$$1 - 2\kappa^{-1}D(g) \leqslant \frac{d(gx, gu)}{d(x, u)} \leqslant 1 + 2\kappa^{-1}D(g).$$

The same bounds are valid for x and v, and for u and v, instead of x and u. Therefore

$$\frac{(1 - 2\kappa^{-1}D(g))^2}{1 + 2\kappa^{-1}D(g)} - 1 \leqslant |g'|(x) - 1 \leqslant \frac{(1 + 2\kappa^{-1}D(g))^2}{1 - 2\kappa^{-1}D(g)} - 1.$$

The lower bound is greater than $-6\kappa^{-1}D(g)$, whereas the upper bound is at most $8\kappa^{-1}D(g)$. The claim (7) is thus proved, with $C := \kappa/10$.

(i) By (7), the metric differentiation map is continuous at the identity element of $\mathrm{M\ddot{o}b}(X)$. The continuity on $\mathrm{M\ddot{o}b}(X)$ follows, since $\mathrm{dist}(|g'|, |h'|) \leqslant (\max |h'|)\mathrm{dist}(|(gh^{-1})'|, 1)$ for all g, h in $\mathrm{M\ddot{o}b}(X)$ by using the chain rule (6).

(ii) It is clear that $\mathrm{M\ddot{o}b}(X)$ is a topological group, and a closed subgroup of $\mathrm{Homeo}(X)$. For each $R \geqslant 1$ the subset $\{g \in \mathrm{M\ddot{o}b}(X) : \max |g'| \leqslant R\}$ is closed and equicontinuous, and hence compact by the Arzelà–Ascoli theorem. On the one hand, it follows that $\mathrm{M\ddot{o}b}(X)$ is σ-compact. On the other hand, (7) implies that the closed ball of radius C around the identity is contained in $\{g \in \mathrm{M\ddot{o}b}(X) : \max |g'| \leqslant 2\}$. Since the latter is compact, the former is a compact neighborhood of the identity, and we conclude that $\mathrm{M\ddot{o}b}(X)$ is locally compact. In what concerns the compactness of the isometry group, note that $\mathrm{Isom}(X) = \{g \in \mathrm{M\ddot{o}b}(X) : \max |g'| \leqslant 1\}$.

(iii) By definition, an affine isometric action of a topological group on a Banach space is continuous if the linear part of the action is strongly (that is, pointwise) continuous, and the cocycle is (norm) continuous.

Fix $p > D$. First, we show that the linear isometric action $(g, F) \mapsto g \cdot F$ of Möb(X) on $L^p(X \times X, \nu)$ is strongly continuous. To that end, it suffices to check that for a dense set of functions F in $L^p(X \times X, \nu)$ we have $g \cdot F \to F$ in $L^p(X \times X, \nu)$ whenever $g \to$ id in Möb(X). Recall from Lemma 9 that ν is a Radon measure on the locally compact space $X^{2-} := X \times X - \mathrm{diag}$. Hence $C_c(X^{2-})$, the subspace of compactly-supported continuous functions on X^{2-}, is dense in $L^p(X^{2-}, \nu) = L^p(X \times X, \nu)$. Let $F \in C_c(X^{2-})$. If $g \to$ id in Möb(X), then $g \cdot F \to F$ uniformly. Since $\|g \cdot F - F\|_{L^p(X^{2-}, \nu)} \leqslant \|g \cdot F - F\|_\infty \, \nu(\mathrm{supp}\, F)$ and $\nu(\mathrm{supp}\, F) < \infty$, we conclude that $g \cdot F \to F$ in $L^p(X \times X, \nu)$.

To show that the cocycle $g \mapsto c_g$ is continuous, it suffices to check continuity at the identity element of Möb(X); the cocycle rule will then imply continuity at every element of Möb(X). Let $r \in (D, p)$. Then

$$\|c_g\|_{L^p(\nu)}^p \leqslant \|c_g\|_\infty^{p-r} \|c_g\|_{L^r(\nu)}^r.$$

We have

$$\|c_g\|_\infty = \log \frac{\max |g'|}{\min |g'|}, \quad \|c_g\|_{L^r(\nu)} \preccurlyeq_r \frac{\max |g'|}{\min |g'|},$$

the latter by Lemma 12. Therefore

$$\|c_g\|_{L^p(\nu)}^p \preccurlyeq_r \left(\log \frac{\max |g'|}{\min |g'|} \right)^{p-r} \left(\frac{\max |g'|}{\min |g'|} \right)^r,$$

from which it follows that $\|c_g\|_{L^p(\nu)} \to 0$ as $g \to$ id in Möb(X). □

Without further assumptions, the properness of the affine isometric action of Möb(X) on $L^p(X \times X, \nu)$ seems elusive.

6. Interlude: visual metrics on boundaries of hyperbolic groups

The next goal is to apply the construction of § 5 to the action of a non-elementary hyperbolic group on its boundary. In order to do so, we need a metric on the boundary such that the group action is by Möbius maps, and such that the corresponding Hausdorff measure is Ahlfors regular.

6.1 Visual metrics induced by the word metric

Let Γ be a non-elementary hyperbolic group, and consider the Cayley graph of Γ with respect to a finite generating set. Throughout, we choose the identity element as the basepoint.

In the beginning there is the word metric, and the word metric is geodesic. Let $\partial\Gamma$ denote the Gromov boundary of the Cayley graph. Topologically, $\partial\Gamma$ is a canonical compact space without isolated points on which Γ acts by homeomorphisms. The metric structure on $\partial\Gamma$ is, however, more subtle. The classical approach runs as follows. First, the Gromov product $(\cdot, \cdot)$ on $\Gamma \times \Gamma$ is extended to $\overline{\Gamma} \times \overline{\Gamma}$, where $\overline{\Gamma} = \Gamma \cup \partial\Gamma$ is the boundary compactification. Such an extension involves a somewhat ad hoc choice and, a priori, it is neither unique nor continuous. Next, the expression $\exp(-\varepsilon(\cdot, \cdot))$ is turned into a compatible metric on the boundary. More precisely, for each sufficiently small parameter $\varepsilon > 0$ there is a *visual metric* $d_\varepsilon \asymp \exp(-\varepsilon(\cdot, \cdot))$ which agrees with the canonical topology on $\partial\Gamma$. (See, for instance, Bourdon [Bou95] and Kapovich and Benakli [KB02] for more details.) For a visual metric d_ε, the geometric mean-value property $(*)$

holds up to multiplicative constants depending on the visual parameter ε and the hyperbolicity constant δ. Clearly, starting with a 'quasi-definition' of the extended Gromov product snowballs into a quasified metric structure on $\partial\Gamma$.

The Patterson–Sullivan theory for non-elementary hyperbolic groups, due to Coornaert [Coo93], describes the Hausdorff dimensions and the Hausdorff measures associated to visual metrics on the boundary. Let

$$e(\Gamma) = \limsup_{n\to\infty} \frac{\log \#\{g \in \Gamma : l(g) \leqslant n\}}{n}$$

be the growth exponent of Γ, where l denotes the word length. Then $0 < e(\Gamma) < \infty$, and the following holds.

PROPOSITION 15 [Coo93]. *Equip $\partial\Gamma$ with a visual metric d_ε. If D_ε denotes the Hausdorff dimension of $\partial\Gamma$, then $D_\varepsilon = e(\Gamma)/\varepsilon$ and the D_ε-dimensional Hausdorff measure is Ahlfors regular.*

6.2 Visual metrics induced by Mineyev's hat metric

In the $\mathrm{CAT}(-1)$ setting, the metric structure on the boundary is better behaved. Namely, let X be a proper $\mathrm{CAT}(-1)$ space with a fixed basepoint $o \in X$. Then the Gromov product $(\cdot, \cdot)_o$ on $X \times X$ extends continuously to $\overline{X} \times \overline{X}$, and for each $\varepsilon \in (0, 1]$ the expression $\exp(-\varepsilon(\cdot, \cdot)_o)$ is a compatible metric on the boundary ∂X (see Bourdon [Bou95]). If we equip ∂X with a visual metric $\exp(-\varepsilon(\cdot, \cdot)_o)$, then the action of $\mathrm{Isom}(X)$ on ∂X is Möbius. Furthermore, the Möbius group of ∂X does not depend on the visual parameter ε.

Mineyev showed in [Min07] that properties similar to the ones in the $\mathrm{CAT}(-1)$ setting can be achieved on boundaries of hyperbolic groups. The main technical point is the replacement of the word metric on the group by a new metric, herein called the *hat metric*. First introduced by Mineyev and Yu in [MY02], the hat metric is the key geometric ingredient in their proof of the Baum–Connes conjecture for hyperbolic groups.

As before, let Γ be a non-elementary hyperbolic group. Let X be the Cayley graph of Γ with respect to a finite generating set, and endow X with the path metric d. The following result collects the properties of the hat metric which are relevant for this paper.

PROPOSITION 16 [Min05, Min07]. *There is a metric $\hat{d}$ on X having the following properties.*

(i) *The metric $\hat{d}$ is Γ-invariant, quasi-isometric to d, and roughly geodesic.*

(ii) *For each $o \in X$, the Gromov product $\langle\cdot, \cdot\rangle_o$ with respect to $\hat{d}$ on X extends to a continuous map $\langle\cdot, \cdot\rangle_o : \overline{X} \times \overline{X} \to [0, \infty]$, where $\langle\xi, \omega\rangle_o = \infty$ if and only if $\xi = \omega \in \partial X$.*

(iii) *(normalization) For each $\varepsilon \in (0, 1]$, $\exp(-\varepsilon\langle\cdot, \cdot\rangle_o)$ is a metric on ∂X for all $o \in X$.*

Thus, by passing from the path metric d to the hat metric $\hat{d}$, we get a Gromov product which behaves just like the one in the $\mathrm{CAT}(-1)$ setting. The metrics on the boundary $\partial X = \partial\Gamma$ are then simple and explicit, and they also have sharp properties. In the direction that concerns us, we have that the action on the boundary is Möbius, whereas classically the action is quasi-Möbius only. Recall, the identity element of Γ is our chosen basepoint.

COROLLARY 17 [Min07]. *Equip $\partial\Gamma$ with a visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon\langle\cdot, \cdot\rangle)$ defined by the Gromov product with respect to $\hat{d}$. Then Γ acts on $\partial\Gamma$ by Möbius homeomorphisms.*

Note also that the Möbius group of $\partial\Gamma$ is independent of the choice of visual parameter ε.

Proof. We verify that the geometric mean-value property ($*$) holds for each $g \in \Gamma$. A direct calculation shows that

$$2\langle gx, gw \rangle - 2\langle x, w \rangle = (\hat{l}(g^{-1}) - 2\langle g^{-1}, x \rangle) + (\hat{l}(g^{-1}) - 2\langle g^{-1}, w \rangle) \quad (x, w \in \Gamma),$$

where $\hat{l}(g) := \hat{d}(1, g)$ is the hat length of g. Letting $x \to \xi$ and $w \to \omega$, we get

$$\hat{d}_\varepsilon^2(g\xi, g\omega) = \exp\Big(\varepsilon(2\langle g^{-1}, \xi \rangle - \hat{l}(g^{-1}))\Big) \exp\Big(\varepsilon(2\langle g^{-1}, \omega \rangle - \hat{l}(g^{-1}))\Big)\hat{d}_\varepsilon^2(\xi, \omega) \qquad (8)$$

for all $\xi, \omega \in \partial\Gamma$. $\qquad\qquad\square$

Mineyev's hat metric straightens the outside while wrinkling the inside. Indeed, the previous corollary witnesses the fact that exchanging the path metric d for the hat metric $\hat{d}$ improves the metric structure on the boundary. However, there is a price to be paid within the space X: while the path metric d is geodesic, the hat metric $\hat{d}$ is only roughly geodesic. Recall, to say that $\hat{d}$ is *roughly geodesic* ($^+$geodesic, in the language of [Min07]) is to mean the following: there is a spatial constant $C \geqslant 0$ with the property that, for any points $x, y \in X$, there is a (not necessarily continuous) map $\gamma : [a, b] \to \mathbb{R}$ such that $\gamma(a) = x$, $\gamma(b) = y$, and γ is a C-rough isometry, i.e.,

$$|s - t| - C \leqslant \hat{d}(\gamma(s), \gamma(t)) \leqslant |s - t| + C$$

for all $s, t \in [a, b]$. Hyperbolicity is a quasi-isometry invariant for roughly geodesic spaces, so $\hat{d}$ is a hyperbolic metric on X.

Recent work of Blachère, Haïssinsky and Mathieu shows that we still have Ahlfors regularity for the visual metrics coming from the hat metric. More precisely, [BHM11, Theorem 2.3] leads to the following.

PROPOSITION 18. *Equip $\partial\Gamma$ with a visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon\langle \cdot, \cdot \rangle)$ defined by the Gromov product with respect to $\hat{d}$. If $\hat{D}_\varepsilon$ denotes the Hausdorff dimension of $\partial\Gamma$, then $\hat{D}_\varepsilon = \hat{e}(\Gamma)/\varepsilon$ and the $\hat{D}_\varepsilon$-dimensional Hausdorff measure is Ahlfors regular.*

Here

$$\hat{e}(\Gamma) = \limsup_{n\to\infty} \frac{\log \#\{g \in \Gamma : \hat{l}(g) \leqslant n\}}{n}$$

is the growth exponent of Γ with respect to the hat length $\hat{l}$.

To conclude, Corollary 17 and Proposition 18 fulfill the two desiderata announced at the beginning of the section.

Remark 19. In [BHM11], the authors are concerned with another interesting metric on a hyperbolic group, namely the Green metric coming from a random walk. The Green metric is a more natural, and much easier to construct, replacement of the word metric than the hat metric. It would be interesting to know whether the Green metric leads to a metric structure on the boundary as nice as the one coming from the hat metric.

7. Proper isometric actions of hyperbolic groups on L^p-spaces

Again, Γ is a non-elementary hyperbolic group. Equip $\partial\Gamma$ with a visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon\langle \cdot, \cdot \rangle)$ defined by the Gromov product with respect to the hat metric $\hat{d}$. Then, as explained in the previous section, the action of Γ on $\partial\Gamma$ falls under the framework developed in §5. Let us see what we obtain.

7.1 A Γ-invariant measure on $\partial\Gamma \times \partial\Gamma$

While the Hausdorff dimension of $\partial\Gamma$ depends on ε, the Hausdorff measure on $\partial\Gamma$ does not, and we simply denote it by μ. The Möbius-invariant measure ν on $\partial\Gamma \times \partial\Gamma$ is also independent of the visual parameter, and it takes the form

$$d\nu(\xi, \omega) = \exp(2\hat{e}(\Gamma)\langle \xi, \omega \rangle)\, d\mu(\xi)\, d\mu(\omega). \tag{9}$$

In particular, ν is Γ-invariant. Since the diagonal of $\partial\Gamma \times \partial\Gamma$ is ν-negligible, we may also view ν as a Γ-invariant Radon measure on $\partial^2\Gamma := \partial\Gamma \times \partial\Gamma - \mathrm{diag}$.

The measure ν is a generalization of the Bowen–Margulis measure from the CAT(-1) setting. Namely, consider the special case when Γ acts geometrically on a proper CAT(-1) space X. If we forgo the visual metrics coming from the hat metric on the Cayley graph of Γ, and we use instead the visual metrics coming from within X, then the Möbius-invariant measure ν we obtain is the so-called *Bowen–Margulis measure*. This has the form

$$d\nu_{\mathrm{BM}}(\xi, \omega) = \frac{d\mu_o(\xi)\, d\mu_o(\omega)}{d_{\varepsilon,o}(\xi,\omega)^{2D_\varepsilon}} = \exp(2e(\Gamma)(\xi,\omega)_o)\, d\mu_o(\xi)\, d\mu_o(\omega),$$

independent of the basepoint $o \in X$. Here D_ε and μ_o denote the Hausdorff dimension, respectively the Hausdorff measure, with respect to the visual metric $d_{\varepsilon,o} = \exp(-\varepsilon(\cdot, \cdot)_o)$.

The existence of a Γ-invariant measure on $\partial\Gamma \times \partial\Gamma$ analogous to the Bowen–Margulis measure was first established by Furman in [Fur02, Proposition 1] using a cohomological argument. Roughly speaking, Furman constructs the measure in the loose measurable sense whereas our ν is constructed in the sharp metric category. This grants ν some advantages: it is more explicit, it is much closer to the Bowen–Margulis measure, and it has the sharp properties needed for the construction of a proper isometric action on $L^p(\partial\Gamma \times \partial\Gamma)$.

7.2 A cocycle for the Γ-action on $\partial\Gamma \times \partial\Gamma$

By (8), the metric derivative of $g \in \Gamma$ is

$$|g'|^{(\varepsilon)}(\xi) = \exp\Big(\varepsilon(2\langle g^{-1}, \xi \rangle - \hat{l}(g^{-1}))\Big), \tag{10}$$

and the cocycle c becomes, up to a factor of 2ε,

$$g \mapsto c_g(\xi, \omega) = \langle g, \xi \rangle - \langle g, \omega \rangle. \tag{11}$$

We may interpret this attractive cocycle as follows. For $g \in \Gamma$ and $\xi \in \partial\Gamma$, let $\hat{\beta}(g, \xi) := 2\langle g, \xi \rangle - \hat{l}(g)$. If we fix a boundary point ξ and we view $\hat{\beta}$ as a function on the group, then $\hat{\beta}(\cdot, \xi)$ is the *Busemann function* corresponding to ξ, and the difference map $(g, h) \mapsto \hat{\beta}(g, \xi) - \hat{\beta}(h, \xi)$ is the *Busemann cocycle* with respect to ξ. If we fix a group element g and we view $\hat{\beta}$ as a function on the boundary, then the difference map $(\xi, \omega) \mapsto \hat{\beta}(g, \xi) - \hat{\beta}(g, \omega)$ is, up to a factor of 2, our cocycle c_g. We thus think of c as the *other Busemann cocycle*.

Let $p > \hat{D}_\varepsilon$; we recall that $\hat{D}_\varepsilon$ denotes the Hausdorff dimension of $(\partial\Gamma, \hat{d}_\varepsilon)$. By Proposition 13, c is a cocycle for the linear isometric action of Γ on $L^p(\partial\Gamma \times \partial\Gamma)$. The last ingredient is the following.

PROPOSITION 20. *The cocycle c is proper. In fact, the growth of $g \mapsto \|c_g\|_{L^p(\nu)}^p$ is at least linear with respect to the (hat or word) length.*

Proof. Let $\hat{\delta}$ denote the hyperbolicity constant of $\hat{d}$. Put $K := e^{\hat{e}(\Gamma)}$. Then

$$\|c_g\|_{L^p(\nu)}^p = \iint |\langle g, \xi \rangle - \langle g, \omega \rangle|^p \, K^{\,2\langle \xi, \omega \rangle} \, d\mu(\xi) \, d\mu(\omega)$$

$$\succcurlyeq \iint |\langle g, \xi \rangle - \langle g, \omega \rangle|^p \, K^{\,2\min\{\langle g, \xi\rangle, \langle g, \omega\rangle\}} \, d\mu(\xi) \, d\mu(\omega)$$

by using the hyperbolic inequality $\langle \xi, \omega \rangle \geqslant \min\{\langle g, \xi \rangle, \langle g, \omega \rangle\} - \hat{\delta}$.

Let $M(g) := \max\{\langle g, \xi \rangle : \xi \in \partial\Gamma\}$. First, we claim that $\mu(\{\xi \in \partial\Gamma : \langle g, \xi \rangle \geqslant R\}) \asymp_\varepsilon K^{-R}$ for $0 \leqslant R \leqslant M(g) - \hat{\delta}$. To prove the claim, pick $\omega \in \partial\Gamma$ such that $\langle g, \omega \rangle = M(g) \geqslant R + \hat{\delta}$. Then

$$B(\omega, e^{-\varepsilon(R+\hat{\delta})}) \subseteq \{\xi \in \partial\Gamma : \langle g, \xi \rangle \geqslant R\} \subseteq B(\omega, e^{-\varepsilon(R-\hat{\delta})}) \tag{12}$$

where the balls are closed, and taken with respect to the $\hat{d}_\varepsilon$ metric. Indeed, let $\xi \in B(\omega, e^{-\varepsilon(R+\hat{\delta})})$; then $\langle \xi, \omega \rangle \geqslant R + \hat{\delta}$, and hence $\langle g, \xi \rangle \geqslant \min\{\langle g, \omega \rangle, \langle \xi, \omega \rangle\} - \hat{\delta} \geqslant R$. This justifies the first inclusion in (12). For the second one, let $\xi \in \partial\Gamma$ with $\langle g, \xi \rangle \geqslant R$; then $\langle \xi, \omega \rangle \geqslant \min\{\langle g, \xi \rangle, \langle g, \omega \rangle\} - \hat{\delta} \geqslant R - \hat{\delta}$, and hence $\hat{d}_\varepsilon(\xi, \omega) \leqslant e^{-\varepsilon(R-\hat{\delta})}$. The claimed measure-theoretic estimate follows from (12) together with the Ahlfors regularity of μ.

From the above claim, it follows that there exists a positive constant $\rho = \rho(\varepsilon)$ such that the sets

$$A_k := \{\xi \in \partial\Gamma : (k+1)\rho > \langle g, \xi \rangle \geqslant k\rho\}$$

satisfy $\mu(A_k) \asymp_\varepsilon K^{-k\rho}$ for $0 \leqslant k \leqslant N(g)$, where $N(g) := \lfloor (M(g) - \hat{\delta})/\rho \rfloor - 1$. Splitting over the annular sets A_k, we have

$$\|c_g\|_{L^p(\nu)}^p \succcurlyeq \sum_{0 \leqslant j < k \leqslant N(g)} \iint_{\substack{\xi \in A_k \\ \omega \in A_j}} |\langle g, \xi \rangle - \langle g, \omega \rangle|^p \, K^{\,2\min\{\langle g, \xi\rangle, \langle g, \omega\rangle\}} \, d\mu(\xi) \, d\mu(\omega).$$

For $\xi \in A_k$ and $\omega \in A_j$ with $k > j$, we have $|\langle g, \xi \rangle - \langle g, \omega \rangle| = \langle g, \xi \rangle - \langle g, \omega \rangle \geqslant (k - j - 1)\rho$ and $\min\{\langle g, \xi \rangle, \langle g, \omega \rangle\} = \langle g, \omega \rangle \geqslant j\rho$. Hence

$$\|c_g\|_{L^p(\nu)}^p \succcurlyeq_\varepsilon \sum_{0 \leqslant j < k \leqslant N(g)} (k - j - 1)^p \, K^{\,2j\rho} \, K^{-k\rho} \, K^{-j\rho}$$

$$= K^{-\rho} \sum_{0 \leqslant j \leqslant k \leqslant N(g)-1} (k - j)^p \, K^{-(k-j)\rho} \geqslant K^{-2\rho}(N(g) - 1),$$

where the last inequality follows from a simple recurrence, as in the proof of Lemma 5. Summarizing, we have that $\|c_g\|_{L^p(\nu)}^p \geqslant a \, M(g) - b$ for some positive constants $a = a(\varepsilon), b = b(\varepsilon)$.

Finally, we claim that $M(g) + M(g^{-1}) \geqslant \hat{l}(g)$. Since $\|c_{g^{-1}}\|_{L^p(\nu)} = \| - g^{-1} \cdot c_g\|_{L^p(\nu)} = \|c_g\|_{L^p(\nu)}$, the desired lower bound

$$\|c_g\|_{L^p(\nu)}^p \geqslant \frac{a}{2} \, \hat{l}(g) - b$$

immediately follows. As for the claim, it is a consequence of the fact that $\langle g, \xi \rangle + \langle g^{-1}, g^{-1}\xi \rangle = \hat{l}(g)$ for all $\xi \in \partial\Gamma$. This can be seen by plugging in formula (10) in the identity $|(g^{-1})'| = 1/g.|g'|$. More directly, one can check that $\langle g, x \rangle + \langle g^{-1}, g^{-1}x \rangle = \hat{l}(g)$ for all $x \in \Gamma$. $\qquad\square$

7.3 Hyperbolic dimension

The best integrability exponent p in sight is obtained by taking the visual parameter ε to be 1. Then we obtain a proper action of Γ on $L^p(\partial\Gamma \times \partial\Gamma)$ for every $p > \hat{e}(\Gamma)$; recall, $\hat{e}(\Gamma)$ is the

growth exponent of Γ with respect to the hat metric $\hat{d}$. However, we can do slightly better. Consider the infimum of all Hausdorff dimensions of the boundary with respect to visual metrics $\exp(-\varepsilon\langle\cdot,\cdot\rangle)$ which come from some hat metric, i.e., a metric satisfying properties (i), (ii), and (iii) of Theorem 16. This infimum is Mineyev's *hyperbolic dimension* (cf. [Min07, § 10]). The notion of hyperbolic dimension is inspired by Pansu's conformal dimension, and one can see that the hyperbolic dimension is at least as large as the Ahlfors regular, conformal dimension mentioned in the introduction.

Let $\hbar(\Gamma)$ denote the hyperbolic dimension of a non-elementary hyperbolic group Γ. Then $\hbar(\Gamma)$ is a non-negative real number equal to 0 if Γ is virtually free, and greater or equal to 1 otherwise [Min07, Theorem 22 and Corollary 23].

Example 21. Consider the rank-1 symmetric space $\mathbb{H}^n_K$, where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $n \geqslant 2$, or $K = \mathbb{O}$ and $n = 2$, normalized so that the maximal sectional curvature is -1. If Γ is a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n_K)$ then, by a result of Pansu [Pan89], we get that $\hbar(\Gamma) = nk + k - 2$ where $k = \dim_{\mathbb{R}} K$.

With the notion of hyperbolic dimension at hand, our main result can be neatly stated as follows.

THEOREM 22. *Let Γ be a non-elementary hyperbolic group. Then Γ admits a proper affine isometric action on $L^p(\partial\Gamma \times \partial\Gamma)$ for all $p > \hbar(\Gamma)$.*

In particular, Theorem 22 together with the Fisher–Margulis result that a group with property (T) is $L^{2+\varepsilon}$-rigid (see [BFGM07, Theorem 1.3]) imply the following consequence: if Γ is an infinite hyperbolic group with property (T), then $\hbar(\Gamma) > 2$.

8. An ℓ^p-cohomological interpretation

In this section we prove Theorem 3 from the Introduction.

8.1 Besov spaces

Let us return to the setup of § 5: (X, d) is a compact metric space of Hausdorff dimension $D \in (0, \infty)$, whose D-dimensional Hausdorff measure μ_D is Ahlfors regular.

Following Bourdon and Pajot [BP03], we consider the *Besov space* $B_p(X)$. The Besov seminorm of a measurable function $f : X \to \mathbb{R}$ is given by

$$\|f\|_{B_p} = \left(\iint |f(x) - f(y)|^p \, d^{-2D}(x, y) \, d\mu_D(x) \, d\mu_D(y) \right)^{1/p}.$$

Note that $\|f\|_{B_p} = 0$ if and only if f is a.e. constant. The Besov space $B_p(X)$ is the space of functions having finite Besov seminorm, modulo a.e. constant functions. Equipped with the induced Besov norm, $B_p(X)$ is a Banach space. There is an obvious isometric embedding

$$B_p(X) \longrightarrow L^p(X \times X, \nu), \quad [f] \longmapsto ((x, y) \mapsto f(x) - f(y))$$

where, we recall, $d\nu(x, y) = d^{-2D}(x, y) \, d\mu_D(x) \, d\mu_D(y)$. For $p > D$, we also have a continuous embedding

$$\mathrm{Lip}(X) \hookrightarrow B_p(X)$$

where $\mathrm{Lip}(X)$ is the space of Lipschitz functions modulo constants, equipped with the induced Lipschitz norm. Indeed, if f is Lipschitz on X then $\|[f]\|_{B_p} \leqslant \|[f]\|_{\mathrm{Lip}} \|d\|_{L^p(\nu)}$, and Lemma 11 says that $d \in L^p(X \times X, \nu)$ whenever $p > D$.

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Now we see that the affine isometric action of the Möbius group Möb(X) on the L^p-space $L^p(X \times X)$ is, in essence, an affine isometric action on the Besov space $B_p(X)$. More precisely, for each $p > D$, the map $g \mapsto \log |(g^{-1})'|$ is a (Lipschitz) cocycle for the linear isometric action of Möb(X) on $B_p(X)$.

Moving to the context of hyperbolic groups, recall that the boundary $\partial\Gamma$ of a non-elementary hyperbolic group Γ is metrized by some visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon\langle\cdot,\cdot\rangle)$. However, (9) shows that the Γ-invariant Besov norm on the boundary $\partial\Gamma$ is given by

$$\|f\|_{B_p} = \left(\iint |f(\xi) - f(\omega)|^p \exp(2\hat{e}(\Gamma)\langle\xi, \omega\rangle)\, d\mu(\xi)\, d\mu(\omega)\right)^{1/p},$$

independent of the choice of visual parameter ε. By (10), the cocycle $g \mapsto \log |(g^{-1})'|$ becomes $g \mapsto \hat{\beta}(g,\cdot) = 2\langle g,\cdot\rangle - \hat{l}(g)$. This too could be called the Busemann cocycle (compare the discussion in §7.2), and it is a Lipschitz cocycle for any choice of visual metric $\hat{d}_\varepsilon = \exp(-\varepsilon\langle\cdot,\cdot\rangle)$. When we work modulo constant functions, as we do in the Lipschitz space Lip and in the Besov spaces B_p, this simple Busemann cocycle simplifies further to $g \mapsto [\langle g,\cdot\rangle]$. We thus have the following proposition.

PROPOSITION 23. *Let Γ be a non-elementary hyperbolic group. Then the linear isometric action of Γ on $B_p(\partial\Gamma)$ admits $g \mapsto [\langle g,\cdot\rangle]$ as a proper cocycle for all $p > \hat{e}(\Gamma)$.*

8.2 ℓ^p-cohomology in degree one

Let Γ be a finitely generated group, and consider the Cayley graph of Γ with respect to a finite generating set. We let

$$E_p(\Gamma) = \left\{\phi : \Gamma \to \mathbb{R} : \|\phi\|_{E_p} = \left(\sum_{x\bullet\!\!-\!\!\bullet y} |\phi(x) - \phi(y)|^p\right)^{1/p} < \infty\right\}$$

be the linear space of functions on Γ with p-summable edge differential. The *first ℓ^p-cohomology group* $H^1_{(p)}(\Gamma)$ is defined as the quotient of $E_p(\Gamma)$ by $\ell^p\Gamma + \mathbb{R}$.

The E_p-seminorms with respect to two Cayley graphs of Γ are comparable, meaning that the linear space $E_p(\Gamma)$ and the isomorphism type of the seminormed space $(E_p(\Gamma), \|\cdot\|_{E_p})$ are both independent of the choice of Cayley graph for Γ. Furthermore, the E_p-seminorm is Γ-invariant. All these features are inherited by $H^1_{(p)}(\Gamma)$, equipped with the induced norm. If Γ is non-amenable, then $H^1_{(p)}(\Gamma)$ is a Banach space.

Now let us turn again to non-elementary hyperbolic groups. Applying a beautiful result of Bourdon and Pajot [BP03, Theorems 0.1 and 3.4] to our situation, we obtain the following fact.

PROPOSITION 24. *Let Γ be a non-elementary hyperbolic group. Then the following hold.*

(i) *For each $\phi \in E_p(\Gamma)$, the boundary extension $\phi_\infty(\xi) = \lim_{x\to\xi} \phi(x)$ is a.e. defined on $\partial\Gamma$.*

(ii) *The boundary extension induces a Γ-equivariant Banach space isomorphism $H^1_{(p)}(\Gamma) \to B_p(\partial\Gamma)$, given by $[\phi] \mapsto [\phi_\infty]$.*

In light of the equivariant isomorphism between the first ℓ^p-cohomology group and the boundary Besov space, Proposition 23 can be stated as follows.

THEOREM 25. *Let Γ be a non-elementary hyperbolic group. Then the linear isometric action of Γ on $H^1_{(p)}(\Gamma)$ admits $g \mapsto [\langle g,\cdot\rangle]$ as a proper cocycle for all $p > \hat{e}(\Gamma)$.*

Note that the cocycle in Theorem 25 is given by equivalence classes of functions on Γ, whereas the similar-looking cocycle in Proposition 23 is given by equivalence classes of functions on $\partial\Gamma$. There should be a direct proof for Theorem 25, in fact one involving the Gromov product $(\cdot,\cdot)$ with respect to the word metric on Γ rather than the hat Gromov product $\langle\cdot,\cdot\rangle$.

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Bogdan Nica bogdan.nica@gmail.com

Mathematisches Institut, Georg-August Universität Göttingen,

Bunsenstrasse 3–5, D-37073 Göttingen, Germany