ALEXANDER POLYNOMIALS OF TWO-BRIDGE LINKS

TAIZO KANENOBU

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Abstract

We provide an algorithm for calculating the Alexander polynomial of a two-bridge link by putting every two-bridge link in a special type of Conway diagram. Using this algorithm, some necessary conditions for a polynomial to be the Alexander polynomial of a two-bridge link are given, in particular, certain alternating and monotonicity conditions on the coefficients, analogous to corresponding known properties of the reduced Alexander polynomial.


Hartley [4] gave a necessary condition for a polynomial to be the Alexander polynomial of a two-bridge knot or the reduced Alexander polynomial of a two-bridge link. He showed how the coefficients of the polynomial may be read straight from the extended diagram, which is derived from Schubert's normal form of a two-bridge knot or link, and showed the following theorem: If \( \Delta(t) = \sum_{i=0}^{n} (-1)^i a_i t^i \) where \( a_i > 0 \), is the Alexander polynomial of a two-bridge knot or the reduced Alexander polynomial of a two-bridge link, then for some integer \( s \), \( a_0 < a_1 < \cdots < a_s = a_{s+1} = \cdots = a_{n-s} > \cdots > a_n \). On the other hand, using surgery techniques, Bailey [1] presented an algorithm for calculating the Alexander polynomial of a two-bridge link from Conway diagram. As a corollary to this he proved a conjecture of Kidwell about the linking complexity or geometric intersection numbers of a link in the special case of two-bridge links.

The main results of this paper are Theorems 1 and 3. The former provides another algorithm for calculating the Alexander polynomial of a two-bridge link from a special type of Conway diagram. The latter gives some necessary conditions for a polynomial to be the Alexander polynomial of a two-bridge link. These
conditions are analogous to Hartley's theorem above. Theorem 2 and Corollary 1 also give some properties of the Alexander polynomial of a two-bridge link, including the Torres condition [8]. Corollary 2 is the above-mentioned conjecture of Kidwell in the case of two-bridge links.

In Section 2, we give some lemmas for Theorems 1 and 2. In Section 3, we summarize some properties of two-bridge links. In Section 4, we state the above-mentioned results. In Section 5, we prove Theorem 3.

1. Preliminaries

In this paper, a link \(L\) will mean a piecewise linear embedding of two oriented circles \(K_1\) and \(K_2\) in the 3-sphere \(S^3\). Two links \(L\) and \(L'\) are called equivalent, if there is an orientation preserving autohomeomorphism of \(S^3\), which maps \(L\) onto \(L'\). The Alexander polynomial \(\Delta(x, y)\) of \(L\) is an element of the polynomial ring \(\mathbb{Z}[x, y]\), and is determined only up to multiplication by a unit \(\pm xy\). Let \(G = \pi_1(S^3 - L)\), and let \(G'\) be its commutator subgroup. Then \(\Delta = \mathbb{Z}[G/G']\); the basis \(\{x, y\}\) of \(G/G'\) is always taken to be represented by the meridians of \(K_1\) and \(K_2\) respectively.

Throughout this paper, we will often abbreviate a polynomial \(f(x, y)\) in \(\Delta\) to \(f\) and will use the following notation:

\[
F_n(x, y) = \begin{cases} 
\sum_{i=0}^{n-1} (xy)^i & \text{if } n > 0, \\
0 & \text{if } n = 0, \\
-\sum_{i=-n}^{-1} (xy)^i & \text{if } n < 0.
\end{cases}
\]

In the figures of this paper we will use the concept of a tangle [2], which is a portion of the link diagram containing two arcs. An integral tangle, which is represented by a circle labeled "\(i\)" or "\(-i\)" where \(i\) is a non-negative integer, is a 2-braid with \(i\) or \(-i\) crossings, in the manner indicated in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tangle.png}
\caption{Figure 1}
\end{figure}
2. Lemmas

**Lemma 1.** Let \( L(q, r, s, t) \) be a link as shown in Figure 2, where \( T \) is any tangle. Let \( \Delta(q,r,s,t) \) be the Alexander polynomial of \( L(q, r, s, t) \). If we set \( \Delta_0 = \Delta(q,r,0,0) \) and \( \Delta_{00} = \Delta(0,0,0,0) \), then

\[
\Delta = \{s(x - 1)(y - 1)F_t + 1\}\Delta_0 + \frac{F_t}{F_r} (xy)'(\Delta_0 - \Delta_{00}),
\]

where \( r \neq 0 \).

![Figure 2]

**Lemma 2.** Besides the notation in Lemma 1, let \( \Delta'_0 = \Delta(q,r,0,t) \) and \( \Delta'(0) = \Delta(q,r,s,t) \). Then

\[
\Delta = s(x - 1)(y - 1)F_t\Delta_0 + \Delta'_0;
\]

\[
\Delta^{(t)} = F_t\Delta^{(1)} - xyF_t^{-1}\Delta_0;
\]

\[
\Delta^{(t)} + xy\Delta^{(t-2)} = (1 + xy)\Delta^{(t-1)}.
\]

**Remark.** (1) In the above notation \( \Delta^{(t)} = \Delta \) and \( \Delta(0) = \Delta_0 \).

(2) (2.4) is a special case of Conway’s result [2, page 338], see also [5, page 462].

Lemma 1 can be shown by using Fox’s free differential calculus, see [3], [8]. The proofs of these lemmas are standard, so we omit them.

3. Two-bridge links

According to Conway [2], every two-bridge link can be put in the form as shown in Figure 3. It will be denoted by \( C(a_1, a_2, \ldots, a_n) \), including the indicated orientation of each component. The diagram is slightly different in the cases \( n = 2k \) and \( n = 2k + 1 \), as indicated in Figure 3. From this projection we can see that a two-bridge link is a link with two components each of which is a trivial,
knot. Moreover a two-bridge link is interchangeable, that is, there is an isotopy of $S^3$ which interchanges the two components. This follows immediately from Schubert's normal form [6], or Bailey [1, page 48] also proves this using Conway's diagram.

Let $\alpha(>0)$ and $\beta$ be the coprime integers computed from the continued fraction:

$$\frac{\alpha}{\beta} = a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$  

Then $\alpha$ is even and $0 < |\beta| < \alpha$. This link is equivalent to the link with Schubert's normal form $(\alpha, \beta)$, denoted by $S(\alpha, \beta)$ endowed with suitable orientations. According to Schubert [6, page 144], $S(\alpha, \beta)$ and $S(\alpha', \beta')$ are equivalent if and only if $\alpha = \alpha'$ and $\beta \equiv \beta' \pmod{2\alpha}$. Furthermore, if $\beta' \equiv \beta + \alpha \pmod{2\alpha}$ or $\beta\beta' \equiv \alpha + 1 \pmod{2\alpha}$, then $S(\alpha, \beta)$ differs from $S(\alpha, \beta')$ only by the orientation of one of the components (see [7, page 7]).

The two-fold cover of $S^3$ branched over $S(\alpha, \beta)$ is the lens space $L(\alpha, \beta)$, see [2], [6], [7]. If we neglect the difference between $S(\alpha, \beta)$ and $S(\alpha, -\beta)$ and the orientations of $S(\alpha, \beta)$, this sets up a one-to-one correspondence between two-bridge links and the lens spaces up to homeomorphism.
We can obtain easily another continued fraction:

\[
\frac{a}{b} = 2b_1 + \frac{1}{2b_2 + \cdots + \frac{1}{2b_m}},
\]

where \(m\) is odd. \(C(2b_1, 2b_2, \ldots, 2b_m)\) is then equivalent to \(C(a_1, a_2, \ldots, a_n)\) and will be denoted by \(D(b_1, b_2, \ldots, b_m)\). In the following we will consider every two-bridge link to be put in this form (see [7, page 13]).

4. Main theorems

From Lemma 1, we have

**Theorem 1.** Let \(L_0 = D(0)\) and for \(n \geq 1\) let

\[
L_n = D(\sum_{i=1}^{n} p_i, \sum_{j=1}^{n-1} q_j) = \prod_{i=1}^{n} p_i \prod_{j=1}^{n-1} q_j \neq 0.
\]

Let \(\Delta_n(x, y)\) be the polynomial inductively defined as follows:

\[
\begin{align*}
\Delta_0 &= 0; \\
\Delta_1 &= F_{p_1}; \\
\Delta_n &= \{q_{n-1}(x-1)(y-1)F_{p_n} + 1\} \Delta_{n-1} \\
&\quad + (xy)^{p_n-1} \frac{F_{p_n}}{F_{p_n-1}} (\Delta_{n-1} - \Delta_{n-2}), \quad \text{for } n \geq 2.
\end{align*}
\]

Then \(\Delta_n(x, y)\) is the Alexander polynomial of \(L_n\).

In the following, by the Alexander polynomial of a two-bridge link we mean the polynomial defined in Theorem 1 and we will use the following notation besides that in Theorem 1. Let \(\Delta_n^{(p)}\) be the Alexander polynomial of \(D(p_1, q_1, p_2, q_2, \ldots, p_{n-1}, q_{n-1}, p_n)\); thus \(\Delta^{(p_n)} = \Delta_n\) and \(\Delta^{(0)} = \Delta_{n-1}\). Let \(l_n = \sum_{i=1}^{n} p_i\), that is, the linking number of \(L_n\). Let \(\bar{l}_n = \sum_{i=1}^{n} |p_i|\).

From Lemma 2, we have

**Theorem 2.**

\[
\begin{align*}
\Delta_n &= q_{n-1}(x-1)(y-1)F_{p_n} \Delta_{n-1} + \Delta^{(p_n-1+p_n)}; \\
\Delta^{(p)} &= F_p \Delta_n^{(1)} - xyF_{p-1} \Delta_{n-1}; \\
\Delta^{(p)} + xy\Delta^{(p-2)} &= (1 + xy)\Delta^{(p-1)}.
\end{align*}
\]
Using (4.4) or Theorem 1 we can easily prove each of the following formulae.

**COROLLARY 1.**

(4.5) \[ \Delta_n(x, y) = \Delta_n(y, x); \]
(4.6) \[ \Delta_n(x, y) \equiv F_n(x, y) \mod (x - 1)(y - 1); \]
(4.7) \[ \Delta_n(x, y) = (xy)^{-1}\Delta_n(x^{-1}, y^{-1}). \]

The fact that a two-bridge link is interchangeable assures us of (4.5). From (4.6), we have immediately

(4.8) \[ \Delta_n(x, 1) = F_n(x, 1). \]

(4.7) and (4.8) constitute the Torres conditions [8] for two-bridge links.

**DEFINITION 1.** Let \( f(x, y) \) be a polynomial in \( \Lambda \). If \( f(x, y) \neq 0 \), then \( \deg_x f = \) (maximum \( x \)-power of any term of \( f \)) minus (minimum \( x \)-power of any term of \( f \)). If \( f(x, y) = 0 \), then \( \deg_x f = -1 \). We define \( \deg_y f \) in the same way.

**DEFINITION 2.** \( \Lambda^+(r, s) \) denotes the set of all polynomials \( f(x, y) = \sum_{i+j<s} a_{ij}x^iy^j \) in \( \Lambda \) satisfying the following conditions.

(i) \( \deg_x f = \deg_y f = s - r. \)

(ii) Both

\[
\begin{bmatrix}
    a_{sr} & \cdots & a_{ss} \\
    \vdots & \ddots & \vdots \\
    a_{rr} & \cdots & a_{rs}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
    a_{rr} & \cdots & a_{rs} \\
    \vdots & \ddots & \vdots \\
    a_{sr} & \cdots & a_{ss}
\end{bmatrix}
\]

are symmetric matrices.

(iii) \( a_{ij} \geq 0 \) if \( i + j \) is even, and \( a_{ij} \leq 0 \) if \( i + j \) is odd.

(iv) Let \( b_{ij} = a_{i+r,j+r} \). Then \( |b_{k,0}| \leq |b_{k-1,1}| \leq \cdots \leq |b_{k-u,v}| \), and \( |b_{k,0}| \leq |b_{k+1,1}| \leq \cdots \leq |b_{k+u,v}| \) for \( 0 \leq k \leq s - r \), where \( u = \lfloor k/2 \rfloor \) and \( v = \lfloor (s - r - k)/2 \rfloor \).

Furthermore \( \Lambda^{-1}(r, s) \) denotes the set of all polynomials \( f(x, y) \) in \( \Lambda \) such that \(-f(x, y) \in \Lambda^+(r, s)\).

**THEOREM 3.** For \( n \geq 1 \), \( \Delta_n \in \Lambda^\epsilon(r_n, s_n) \), where

\[ \epsilon_n = \prod_{i=1}^{n} \frac{p_i}{|p_i|} \prod_{j=1}^{n-1} \frac{q_j}{|q_j|}, \quad r_n = \frac{l_n - \tilde{l}_n}{2} \quad \text{and} \quad s_n = \frac{l_n + \tilde{l}_n}{2} - 1. \]

\( [\cdot] \) denotes the Gaussian symbol.
Note that \( r_n \leq 0 \leq s_n \), \( r_n - r_{n-1} = \frac{1}{2}(p_n - |p_n|) \) and \( s_n - s_{n-1} = \frac{1}{2}(p_n + |p_n|) \).

The proof of Theorem 3 will be given in Section 5.

Let \( \Delta(t) = \sum_{i=0}^{m} (-1)^i a_i t^i \), where \( m \) is odd, be the reduced Alexander polynomial of \( L_n \). Since \( \Delta(t) = e_n t^{2r_n}(1 - t)\Delta_n(t, t) \), we have \( 0 < a_0 \leq a_1 \leq \cdots \leq a_{(m-1)/2} \) and \( a_k = -a_{m-k} \) from Theorem 3. This is a weaker result than that of Hartley [4] stated in the beginning of this paper.

For the sake of Corollary 2 below, we need some preliminaries.

**Definition 3.** Let \( L = K_1 \cup K_2 \) be a link and \( S \) be a Seifert surface for \( K_1 \) with \( S \) and \( K_2 \) in general position. If \( \gamma_S = 2(\text{genus of } S) + (\text{the number of times } K_2 \text{ intersects } S) \), then \( \gamma_1 = \min_S \gamma_S \) is the linking complexity of \( K_2 \) with \( K_1 \). We define \( \gamma_2 \) in the same way. We call the ordered pair \( (\gamma_1, \gamma_2) \) the linking complexity of the link \( L \).

This definition follows Bailey [1, page 45], see also [5].

**Proposition.** (Kidwell) If \( \Delta(x, y) \) is the Alexander polynomial of a link \( L \) with linking complexity \( (\gamma_1, \gamma_2) \), then \( \gamma_1 - 1 \geq \deg_x \Delta(x, y) \).

**Proof.** See [1, page 46].

**Corollary 2.** Let \( (\gamma_1, \gamma_2) \) be the linking complexity of \( L_n \). Then

(4.9) \[ \gamma_1 = \gamma_2; \]

(4.10) \[ \deg_x \Delta_n(x, y) + 1 = \gamma_1 = \tilde{t}_n. \]

**Remark.** The first equality of (4.10) is Proposition 6 of [1, page 57].

**Proof.** (4.9) follows from interchangeability of a two-bridge link or (4.10). For (4.10), from the diagram of \( L_n \), we see that \( \gamma_1 \leq \tilde{t}_n. \) By Theorem 3, \( \deg_x \Delta_n + 1 = \tilde{t}_n \) and by Proposition, \( \gamma_1 \geq \deg_x \Delta_n + 1 \).

**5. Proof of Theorem 3**

In this section we use the following trivial lemma without mention.

**Lemma 3.** Let \( f \in \Lambda'(r, s) \) and \( g \in \Lambda'(r - k, s + k) \) \((k \geq 0)\). Then \( f + g \in \Lambda'(r - k, s + k) \).
**Lemma 4.** Let \( f \in \Lambda^e(r, s) \). Then

\[
F_n f \in \begin{cases} 
\Lambda^e(r, s + n - 1) & \text{if } n > 0, \\
\Lambda^{-e}(r + n, s - 1) & \text{if } n < 0,
\end{cases}
\]

\[
G_n f \in \Lambda^{(-1)^n-1}(r, s + n - 1) & \text{if } n > 0,
\]

where \( G_n(x, y) = x^{n-1}F_n(x^{-1}, y) \).

**Proof.** We show that \( f \in \Lambda^e^{+1}(r, s) \) implies \( F_n f \in \Lambda^e^{+1}(r, s + n - 1) \) if \( n > 0 \). The other cases can be proved similarly. It is clear that \( F_n f \) satisfies the conditions (i), (ii), (iii) and the first inequality of (iv) in Definition 2 for \( \Lambda^e^{+1}(r, s + n - 1) \). The second inequality of (iv) can be reduced to the sublemma below.

**Sublemma.** Let \( f(x) = \sum_{i=0}^{n} a_i x^i \), where \( a_i = a_{n-i} \) and \( 0 < a_0 \leq a_1 \leq \ldots \leq a_{[n/2]} \). Let \( (\sum_{j=0}^{m} x^j) f(x) = \sum_{k=0}^{m+n} b_k x^k \). Then \( b_k = b_{m+n-k} \) and \( 0 < b_0 \leq b_1 \leq \ldots \leq b_{(m+n)/2} \).

We omit the proof, as it is straightforward to prove it directly.

**Lemma 5.** If \( \Delta_{n-1} \in \Lambda^{-e}(r, s - 1) \) and \( \Delta_{n}^{(1)} \in \Lambda^{e}(r, s) \), then

\[
\Delta^{(p)}_n \in \begin{cases} 
\Lambda^e(r, s + p - 1) & \text{if } p > 0, \\
\Lambda^{-e}(r + p, s - 1) & \text{if } p < 0.
\end{cases}
\]

**Proof.** (4.2) in Theorem 2 states that \( \Delta^{(p)}_n = F_p \Delta_{n-1}^{(1)} - xy F_{p-1} \Delta_{n-1} \). The case \( p = 1 \) is the hypothesis. If \( p \geq 2 \), then using Lemma 4, \( F_p \Delta_{n}^{(1)} \in \Lambda^e(r, s + p - 1) \) and \( -xy F_{p-1} \Delta_{n-1} \in \Lambda^{-e}(r + 1, s + p - 2) \). Thus \( \Delta^{(p)}_n \in \Lambda^e(r, s + p - 1) \). If \( p \leq -1 \), then \( F_p \Delta_{n}^{(1)} \), \( -xy F_{p-1} \Delta_{n-1} \in \Lambda^{-e}(r + p, s - 1) \), so \( \Delta^{(p)}_n \in \Lambda^{-e}(r + p, s - 1) \).

**Lemma 6.** Let \( \Delta^{(m)}_n \) be the Alexander polynomial of

\[
D(p_1, q_1, \ldots, p_{n-m}, q_{n-m}, 1, q_{n-m+1}, 1, \ldots, q_{n-1}, 1).
\]

Then we have

\[
\Delta^{(m)}_n = G_{m+1} \Delta_{n-m} - xy G_m \Delta^{(p_{n-m-1})}_{n-m} + (x-1)(y-1) \sum_{k=1}^{m} (q_{n-k} + 1) G_k \Delta_{n-k},
\]

where the last term denotes zero if \( m = 0 \).

**Proof.** We prove (5.1) by induction on \( m \). For \( m = 0 \), it is clear that \( \Delta^{(0)}_n = \Delta_n \). Assume that (5.1) is proved for \( m = 1 \). Substituting \( p_{n-m+1} = 1 \) in
\[ \Delta^{(m-1)}_n \text{ we have} \]
\[ \Delta^{(m)}_n = G_m \Delta^{(1)}_{n-m+1} - xyG_{m-1}\Delta^{(0)}_{n-m+1} \]
\[ + (x-1)(y-1) \sum_{k=1}^{m-1} (q_{n-k} + 1)G_k \Delta_{n-k}. \]

By (4.2), \( \Delta^{(1)}_{n-m+1} = q_{n-m}(x-1)(y-1)\Delta_{n-m} + \Delta^{(p_{n-m}+1)}_{n-m} \). Thus we have
\[ \Delta^{(m)}_n = G_m\{(x-1)(y-1)\Delta_{n-m} + \Delta^{(p_{n-m}+1)}_{n-m}\} - xyG_{m-1}\Delta_{n-m} \]
\[ + (x-1)(y-1) \sum_{k=1}^{m} (q_{n-k} + 1)G_k \Delta_{n-k}. \]

By (4.4), \( \Delta^{(p_{n-m}+1)}_{n-m} = (xy+1)\Delta_{n-m} - xy\Delta^{(p_{n-m}-1)}_{n-m} \). Thus we have
\[ \Delta^{(m)}_n = \{(x+y)G_m - xyG_{m-1}\}\Delta_{n-m} - xyG_m\Delta^{(p_{n-m}-1)}_{n-m} \]
\[ + (x-1)(y-1) \sum_{k=1}^{m} (q_{n-k} + 1)G_k \Delta_{n-k}. \]

Since \( (x+y)G_m - xyG_{m-1} = G_{m+1} \), we have (5.1).

Now we are in position to prove Theorem 3. We use induction on \( n \). For \( n = 1 \), the theorem is clear. Assume the theorem proved for \( \Delta_k \), where \( 1 \leq k \leq n - 1 \). Without loss of generality we may suppose that \( q_{n-1} < 0 \). By Lemma 5 we only have to prove for the case \( p_n = 1 \). Then there exists an integer \( m \) such that:

(I) \( 1 \leq m \leq n - 1, p_{n-m+1} = p_{n-m+2} = \cdots = p_{n-1} = 1, p_{n-m} \neq 1 \) and \( q_{n-m}, q_{n-m+1}, \ldots, q_{n-1} < 0 \),

(II) \( 1 \leq m \leq n - 2, p_{n-m} = p_{n-m+1} = p_{n-m+2} = \cdots = p_{n-1} = 1, q_{n-m}, q_{n-m+1}, \ldots, q_{n-1} < 0 \) and \( q_{n-m-1} > 0 \), or

(III) \( m = n - 1, p_1 = p_2 = \cdots = p_{n-1} = 1, q_1, q_2, \ldots, q_{n-1} < 0 \).

To prove Theorem 3, it suffices to prove that \( \Delta_{n-m} \in \Lambda^r(r, s) \) implies \( \Delta_n \in \Lambda^{(1)}(r, s + m) \), where by Lemma 4,
\[ \Delta_n = G_{m+1}\Delta_{n-m} - xyG_m\Delta^{(p_{n-m}-1)}_{n-m} \]
\[ + (x-1)(y-1) \sum_{k=1}^{m} (q_{n-k} + 1)G_k \Delta_{n-k}. \]

By Lemma 4, we have
\[ G_{m+1}\Delta_{n-m} \in \Lambda^{(1)}(r, s + m) \]

By inductive hypothesis, \( \Delta_{n-k} \in \Lambda^{(1)}(r, s + m - k) \) for \( 1 \leq k \leq m \). Then by Lemma 4, \( G_k\Delta_{n-k} \in \Lambda^{(1)}(r, s + m - 1) \); hence we obtain
\[ (x-1)(y-1) \sum_{k=1}^{m} (q_{n-k} + 1)G_k \Delta_{n-k} \biggl\{ \begin{array}{ll} 0 & \text{if } q_{n-k} = -1 \text{ for any } k, \\
\in \Lambda^{(1)}(r, s + m) & \text{otherwise.} \end{array} \]
Case (I). If \( p_{n-m} \neq 1 \), then by inductive hypothesis,
\[
\Delta(p_{n-m}^{-1}) \in \begin{cases} 
\Lambda(r, s - 1) & \text{if } p_{n-m} \geq 2, \\
\Lambda(r - 1, s) & \text{if } p_{n-m} \leq -1.
\end{cases}
\]

Thus, using Lemma 4, we have
\[
(5.5) \quad -xyG_m\Delta(p_{n-m}^{-1}) \in \begin{cases} 
\Lambda^{(-1)m}(r + 1, s + m - 1) & \text{if } p_{n-m} \geq 2, \\
\Lambda^{(-1)m}(r, s + m) & \text{if } p_{n-m} \leq -1.
\end{cases}
\]

Case (II). If \( p_{n-m} = 1 \) and \( q_{n-m-1} > 0 \), then by inductive hypothesis,
\[
\Delta(p_{n-m}^{-1}) = \Delta_{n-m-1} \in \Lambda(r, s - 1).
\]

Thus, using Lemma 4, we have
\[
(5.6) \quad -xyG_m\Delta(p_{n-m}^{-1}) \in \Lambda^{(-1)m}(r + 1, s + m - 1).
\]

Case (III). Since \( m = n - 1 \) and \( p_1 = 1 \), we have
\[
(5.7) \quad -xyG_m\Delta(p_{n-m}^{-1}) = 0.
\]

From (5.2) \( \sim \) (5.7), we have \( \Delta_n \in \Lambda^{(-1)m}(r, s + m) \). This completes the proof of Theorem 3.

References


Department of Mathematics
Kobe University
Kobe 657
Japan

Author’s current address:
Department of Mathematics
Kyushu University 33
Fukuoka 812
Japan