BROWN-HALMOS TYPE THEOREMS OF WEIGHTED TOEPLITZ OPERATORS

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ABSTRACT. The spectra of the Toeplitz operators on the weighted Hardy space \( H^2(Wd\beta/2\pi) \) and the Hardy space \( H^p(d\beta/2\pi) \), and the singular integral operators on the Lebesgue space \( L^p(d\beta/2\pi) \) are studied. For example, the theorems of Brown-Halmos type and Hartman-Wintner type are studied.

1. Introduction. Let \( m \) be the normalized Lebesgue measure on the unit circle \( T \) and let \( W \) be a non-negative integrable function on \( T \) which does not vanish identically. Suppose \( 1 \leq p \leq \infty \). Let \( L^p(W) = L^p(Wd\beta) \) and \( \mathcal{P}(W) = L^p \) when \( W \equiv 1 \). Let \( \mathcal{H}^p(W) \) denote the closure in \( L^p(W) \) of the set \( \mathcal{P} \) of all analytic polynomials when \( p \neq \infty \). We will write \( \mathcal{H}^p(W) = \mathcal{H}^p \) when \( W \equiv 1 \), and then this is a usual Hardy space. \( \mathcal{H}^\infty \) denotes the weak \( \mathcal{P} \) closure of \( \mathcal{P} \) in \( L^\infty \). \( \mathcal{P} \) denotes the projection from the set \( C \) of all trigonometric polynomials to \( \mathcal{P} \). For \( 1 \leq p < \infty \), \( \mathcal{P} \) can be extended to a bounded map of \( L^p(W) \) onto \( \mathcal{H}^p(W) \) if and only if \( W \) satisfies the condition

\[
(A_p) \quad \sup_I \left( \frac{1}{|I|} \int_I W \, dm \right) \left( \frac{1}{|I|} \int_I W^{-1/p} \, dm \right)^{p-1} < \infty
\]

where the supremum is over all intervals \( I \) of \( T \). This is the well known theorem of Hunt, Muckenhoupt and Wheeden [7], which is a generalization of the theorem of Helson and Szegő [6].

In this paper, we assume that the weight \( W \) satisfies the condition \((A_p)\). For \( \phi \) in \( L^\infty \), the Toeplitz operator \( T^W_\phi \) is defined as a bounded map on \( \mathcal{H}^p(W) \) by

\[
T^W_\phi f = \mathcal{P}(\phi f).
\]

For \( \alpha \) and \( \beta \) in \( L^\infty \), the singular integral operator \( S^W_{\alpha\beta} \) is defined as a bounded map on \( L^p(W) \) by

\[
S^W_{\alpha\beta} f = \alpha \mathcal{P}f + \beta (I - \mathcal{P})f
\]

where \( I \) is an identity operator. If \( W \equiv 1 \), we will write \( T^W_\phi = T_\phi \) and \( S^W_{\alpha\beta} = S_{\alpha\beta} \). Almost all results in this paper will be essentially shown using the following theorems. They are called the theorems of Widom, Devinatz and Rochberg (cf. [1], [10] and [9]).

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THEOREM A. Suppose $1 < p < \infty$ and $W = |h|^p$ satisfies the condition $(A_p)$, where $h$ is an outer function in $H_p$. Then the following conditions on $\phi$ and $W$ are equivalent.

1. $T^W_\phi$ is an invertible operator on $H_p(W)$.
2. $\phi = k(h_0^* / h_0)(h/h_0)$, where $k$ is an invertible function in $H^\infty$ and $h_0$ is an outer function in $H_p$ with $|h_0|^p$ satisfying the condition $(A_p)$.
3. $\phi = \gamma \exp(U - i\bar{V})$, where $\gamma$ is constant with $|\gamma| = 1$, $U$ is a bounded real function, $V$ is a real function in $L^1$ and $V \exp(\xi V)$ satisfies $(A_p)$.

THEOREM B. Suppose $1 < p < \infty$ and $W = |h|^p$ satisfies the condition $(A_p)$, where $h$ is an outer function in $H_p$. $S^W_{\alpha\beta}$ is invertible in $L^p(W)$ if and only if both $\alpha$ and $\beta$ are invertible in $L^\infty$ and $\alpha / \beta = \gamma \exp(U - i\bar{V})$, where $\gamma$ is constant with $|\gamma| = 1$, $U$ is a bounded real function, $V$ is a real function in $L^1$ and $V \exp(\xi V)$ satisfies $(A_p)$.

THEOREM C. Suppose $T_\phi$ and $S_{\alpha\beta}$ are on $L^2$, where $\phi$, $\alpha$ and $\beta$ are invertible functions in $L^\infty$.

1. $T_\phi$ is invertible if and only if $\phi$ has the form: $\phi = |\phi| e^t$ where $t$ is a real function in $L^1$ such that

$$||t||' = \inf\{||t - t' - t||_{L^\infty} : s \in L^\infty \text{ and } a \in R\} < \pi/2$$

2. $S_{\alpha\beta}$ is invertible if and only if $\alpha / \beta$ has the form: $\alpha / \beta = |\alpha / \beta| e^t$ where $t$ is the same to that of (1). Hence $S_{\alpha\beta}$ is invertible if and only if $T_{\alpha / \beta}$ is invertible.

In this paper, we are interested in $\sigma(T^W_\phi)$ and $\sigma(S^W_{\alpha\beta})$, that is, the spectra of $T^W_\phi$ and $S^W_{\alpha\beta}$.

For $\alpha = \alpha_1 + i\alpha_2 \in C$ and $\beta = \beta_1 + i\beta_2 \in C$, put $\langle \alpha, \beta \rangle = \alpha_1 \beta_1 + \alpha_2 \beta_2$ and $\theta(\alpha, \beta) = \arccos((\langle \alpha, \beta \rangle / |\alpha||\beta|))$ for $\alpha \neq 0$ and $\beta \neq 0$. Set

$$\ell^*_\alpha = \{z \in C : \langle z, \alpha \rangle \geq 1\} \quad \text{and} \quad \ell_\alpha = \{z \in C : \langle z, \alpha \rangle \leq 1\}$$

and $E^i_{\alpha\beta}$ denotes $\ell^*_\alpha \cap \ell^j_\beta$ where $i = +$ or $-$ and $j = +$ or $-$. For each pair $(\alpha, \beta)$,

$$C = E^+_{\alpha\beta} \cup E^-_{\alpha\beta} \cup E^+_{\alpha\beta} \cup E^-_{\alpha\beta}$$

and if $\ell = -i$ and $m = -j$, then

$$\mathcal{E}^{(\ell, m)} = C \setminus \mathcal{E}^{(\ell, m)} \supset E^i_{\alpha\beta}$$

For any bounded subset $E$ in $C$, there exists a pair $(\alpha, \beta)$ such that $E^i_{\alpha\beta} \supseteq E$ for some $(i, j)$. In fact, there are a lot of such pairs $(\alpha, \beta)$. Now we can define a set which contains $E$ and is important in this paper. When $|\theta(\alpha, \beta)| = \pi - 2t$ and $0 \leq t < \pi/2$, put

$$h^t(E) = \{t \in C : \mathcal{E}^{(\ell, m)} \supseteq E \text{ and } \ell = -i, m = -j\}$$

for a subset $E$ in $C$. If $t < s$, then $h^t(E) \subseteq h^s(E)$. If $t = 0$, then $h^0(E)$ is the closed convex hull of $E$. For example, if $E = [a, b]$ then

$$h^t(E) = \Delta(c, r) \cap \Delta(\bar{c}, r)$$
Toeplitz operators on 
shows the known result [11] such that if 
$\hat{u}$
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piecewise continuous symbol $u$ is a real function and $\hat{u}$
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$D$
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$A$
the condition (2) and (3) of Theorem 1 is a theorem of Hartman and Wintner (cf. [2, Theorem 7.20]) showed that $\sigma(T_\phi) = h^0(\mathcal{R}(\phi))$. In this paper, for real symbols we try to describe the spectra of Toeplitz operators on $H^2(W)$ and $H^p$ and for singular integral operators on $L^2$. When $\phi$ is a real function and $T_\phi$ is a Toeplitz operator on $H^2$, Hartman and Wintner (cf. [2, Theorem 7.20]) showed that $\sigma(T_\phi) = h^0(\mathcal{R}(\phi))$. In this paper, we study symbols $\phi$ such that $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary weight $W$.

Now we collect the notations which will be used in this paper. $R$ is the set of all real numbers and $X_R$ denotes the set of the real parts of all elements in $X$. $|X|^{t}$ denotes the closure of $X$. $D$ is the open unit disc. $C$ is the set of all continuous functions on $T$. If $v$ is a real function in $L^1$, then $\bar{v}$ denotes the harmonic conjugate function with $v(0) = 0$.

2. Toeplitz operators on $H^2(W)$. In this section, we fix arbitrary weight $W$ satisfying the condition (A2) or equivalently, a Helson-Szegö weight $W$. We call $W$ a Helson-Szegö weight when $W = e^{a+\bar{v}}$, $u$ and $v$ are functions in $L^\infty_R$ and $\|v\|_\infty < \pi / 2$. For a Helson-Szegö weight $W = e^{a+\bar{v}}$, put

$$t_W = \|v\|' = \inf\{|v - s - a|_\infty ; s \in L^\infty_R, a \in R\}.$$

When $W = 1$, (1) of Theorem 1 is a theorem of Brown and Halmos (cf. [2, Corollary 7.19]) and (2) and (3) of Theorem 1 is a theorem of Hartman and Wintner (cf. [2, Theorem 7.20]).

When $\phi$ is a piecewise continuous function, $\sigma(T_\phi^W)$ is described when $W$ is arbitrary weight [11]. The symbol $\phi$ in Corollary 2 and (3) of Corollary 3 is not necessarily piecewise continuous. It is known that $\sigma(T_\phi^W) \neq \sigma(T_\phi)$ for some weight $W$ and some piecewise continuous symbol $\phi$ (cf. [4]). In Theorem 2, we determine weight $W$ such that $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary symbol $\phi$ in $L^\infty$ and study symbols $\phi$ such that $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary weight $W$. Spitkovsky [13] showed that the set of all weights $W$ for which $\sigma(T_\phi^W) = \sigma(T_\phi)$ for all $\phi$ in $L^\infty$ does not depend on $p$. (1) of Corollary 3 is related with a particular (corresponding to $p = 2$) case of [3, Theorem 6.1 and Corollary 6.2]. For if log $W \in VMO$ then log $W = u + \bar{v}$ for some real functions $u$ and $v$ in $C$. (2) of Corollary 3 shows the known result [11] such that if $\phi$ is continuous, then $\sigma(T_\phi^W) = \sigma(T_\phi)$ for arbitrary weight $W$.

**Theorem 1.** Let $\phi$ be a function in $L^\infty$, let $W$ be a Helson-Szegö weight and $t = t_W$.

1. $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^W) \subseteq h^0(\mathcal{R}(\phi))$.

2. If $\phi$ is real valued, $a = \text{essinf} \phi$ and $b = \text{esssup} \phi$, then

$$\mathcal{R}(\phi) \subseteq \sigma(T_\phi^W) \subseteq \Delta(c, r) \cap \Delta(\bar{c}, r).$$
where \( c = \frac{a+b}{2} - i\frac{a-b}{2}\cos 2t \) and \( r = -\frac{a-b}{2}\sin 2t \).

(3) Suppose \( W = e^{it\phi} \) and \( \lambda \in [a, b] \cap \mathcal{R}(\phi) \) in (2). Then \( \frac{\phi - \lambda}{\frac{\pi}{2} - \lambda} = e^{it} \) and \( \ell = \pi(1 - \chi_E) \) for some measurable set \( E \) in \( T \) with \( 0 < m(E) < 1 \). \( \lambda \in \sigma(T^W_\phi) \) if and only if

\[
\|\pi \chi_E - v\| \geq \frac{\pi}{2}.
\]

**Proof.** In (1) and (2), it is well known that \( \mathcal{R}(\phi) \subseteq \sigma(T^W_\phi) \). Suppose \( W = e^{it\phi} \), \( u \) and \( v \) are functions in \( L^\infty_R \) and \( \|v\|_\infty < \frac{\pi}{2} \), and \( g^2 = e^{it\phi + i(\tilde{u} - v)} \). Then \( W = |g|^2 \).

(1) By Theorem A in Introduction, for \( \lambda \in C \), \( T^W_{\phi - \lambda} \) is invertible if and only if

\[
T^W_{\frac{\phi - \lambda}{|\phi - \lambda|}} \text{ is invertible.}
\]

Suppose \( |\theta(\alpha, \beta)| = \pi - 2t \) and \( \mathcal{R}(\phi) \subseteq \mathcal{T}^U_{\alpha, \beta} \). If \( \lambda \in (\mathcal{T}^U_{\alpha, \beta})_0 \) with \( \ell = -i, m = -j \), then \( T^W_{\phi} \) is invertible. In fact, then \( (\phi - \lambda)/|\phi - \lambda| = e^{ib} \) where \( 0 \leq s_\lambda \leq \pi - 2\epsilon \) a.e. or \( -\pi + 2\epsilon + 2\epsilon \leq s_\lambda \leq 0 \) a.e. for some \( \epsilon > 0 \). Hence \( |s_\lambda - \frac{\pi}{2} + \epsilon| \leq \frac{\pi}{2} - \epsilon \) a.e. or \( |s_\lambda + \frac{\pi}{2} - t - \epsilon| \leq \frac{\pi}{2} - \epsilon \) a.e. Hence

\[
\frac{\phi - \lambda}{|\phi - \lambda|} g = e^{i(s_\lambda + \epsilon - \tilde{u})}
\]

and

\[
\|s_\lambda + v - \tilde{u}\| \leq \frac{\pi}{2} - \epsilon.
\]

Thus \( T^W_{\frac{\phi - \lambda}{|\phi - \lambda|}} \) is invertible by Theorem C and hence \( T^W_{\phi - \lambda} \) is invertible. If \( \lambda \notin \mathcal{R}(\phi) \), then by definition \( \lambda \in \cup \{ (\mathcal{T}^U_{\alpha, \beta})_0 : \mathcal{T}^U_{\alpha, \beta} \supseteq \mathcal{R}(\phi) \text{ and } \ell = -i, m = -j \} \) and \( |\theta(\alpha, \beta)| = \pi - 2t \). By what was just proved, \( \lambda \notin \mathcal{R}(\phi) \). (2) By (1), \( \sigma(T^W_\phi) \subseteq \mathcal{H}^t(\mathcal{R}(\phi)) \subseteq \mathcal{H}^t([a, b]) \) for \( t = t_W \). It is elementary to see that \( \mathcal{H}^t([a, b]) \subseteq \Delta(c, r) \cap \Delta(\tilde{c}, r) \). (3) The first part is clear. The second statement is a result of Theorems A and C.

**Corollary 1.** Suppose \( \phi = aX_E + bX_E \) where \( a \) and \( b \) are real numbers, \( a \neq b \) and \( 0 < m(E) < 1 \). Let \( W = e^{it\phi} \), then \( \sigma(T^W_\phi) \subseteq [a, b] \) if and only if \( \|\pi \chi_E - v\| \geq \frac{\pi}{2} \).

**Corollary 2.** Let \( E \) be a measurable set with \( 0 < m(E) < 1 \). Suppose \( W = e^{it\phi} \) and \( \phi \) satisfy the following (i) and (ii):

(i) \( W = e^{it\phi} \) where \( u \in T^\infty_R, \tilde{v} = d(\chi_E - \chi_E) + q \in C_R \) and \( d \) is a constant with \( 0 < d < \frac{\pi}{2} \).

(ii) \( \phi = aX_E + bX_E \) where \( a \) and \( b \) are real numbers.

Then \( t_W = d \),

\[
\sigma(T^W_\phi) = \left\{ \lambda \in C : \text{arg} \left( \frac{a - \lambda}{b - \lambda} \right) = \pi - 2d \right\}.
\]

and

\[
\mathcal{H}^t(\mathcal{R}(\phi)) = \left\{ \lambda \in C : \text{arg} \left( \frac{a - \lambda}{b - \lambda} \right) = \pi - 2d \text{ or } \pi + 2d \right\}.
\]
PROOF. Put $v_0 = \frac{\pi}{2}(\chi_E - \chi_{E'})$, then $h^2 = e^{i\pi - i\theta}$ and $|h|^2 = e^\theta = i(\chi_E - \chi_{E'})$. If $\|\chi_E - \chi_{E'}\| < 1$, then $|h|^2 = e^{i\theta}$ is a Helson-Szegö weight and so $\|h|^2/z + H^\infty = 1$ (see [3, Chapter IV, Theorem 3.1]). On the other hand, $\|h|^2/z + H^\infty = \|i(\chi_E - \chi_{E'}) + zH^\infty\| = 1$. This contradiction shows that $\|\chi_E - \chi_{E'}\| = 1$. Thus

$$t_W = \inf \{\|\chi_E - \chi_{E'} - \bar{s} - a\|_\infty : s \in L^\infty_R, a \in R\}$$

$$= d \inf \{\|\chi_E - \chi_{E'} - \bar{s} - a\|_\infty : s \in L^\infty_R, a \in R\} = d.$$ 

Put $g^2 = e^{\alpha + i(v - \theta)}$, then $\bar{g}/g = e^{i(v - \theta)} = \exp i(\bar{s} - d(\chi_E - \chi_{E'}) - \theta)$. If $\lambda \not\equiv a$ and $\lambda \not\equiv b$, then

$$\frac{\phi - \lambda}{|\phi - \lambda|} = \frac{a - \lambda}{|a - \lambda|} \chi_E + \frac{b - \lambda}{|b - \lambda|} \chi_{E'}$$

$$= \exp i(\alpha \chi_E + b(\lambda) \chi_{E'})$$

where $a(\lambda) = \arg(a - \lambda)$ and $b(\lambda) = \arg(b - \lambda)$. Thus $\phi(\lambda) \bar{g}/|\phi - \lambda| g = \exp i(\alpha \chi_E + b(\lambda) \chi_{E'} + \bar{s} - d(\chi_E - \chi_{E'}) - \theta)$. Since $q \in C_R$, by the first part of the proof,

$$\inf \{\|a(\lambda) \chi_E + b(\lambda) \chi_{E'} - d(\chi_E - \chi_{E'}) + \bar{s} - a\|_\infty : s \in L^\infty_R, a \in R\}$$

$$= \frac{a(\lambda) - b(\lambda)}{2} - d \inf \{\|\chi_E - \chi_{E'} - \bar{s} - a\|_\infty : s \in L^\infty_R, a \in R\}$$

$$= \frac{a(\lambda) - b(\lambda)}{2} - d = \frac{1}{2} \arg \frac{a - \lambda}{b - \lambda} + 2d.$$ 

Thus, by (1) of Theorem 2, $W \not\equiv \sigma(T^W_\phi)$ if and only if $\arg \frac{a - \lambda}{b - \lambda} - 2d \not\equiv \pi$. If $\arg \frac{a - \lambda}{b - \lambda} > 0$, then $\arg \frac{a - \lambda}{b - \lambda} - 2d \not\equiv \pi$ because $d > 0$, and hence $\sigma(T^W_\phi) = \{\lambda \not\equiv C \cup \arg \frac{a - \lambda}{b - \lambda} = \pi - 2d\}$. The description of $h^d(R(\phi))$ is a result of (2) of Theorem 1.

THEOREM 2. Let $\phi$ be a function in $L^\infty$ and let $W$ be a Helson-Szegö weight.

1. $t_W = 0$ if and only if $\sigma(T^W_\phi) = \sigma(T_\phi)$ for arbitrary symbol $\phi$ in $L^\infty$.

2. $\sigma(T_\phi) \supseteq \sigma(T^W_\phi)$ for arbitrary Helson-Szegö weight $W$ if and only if for any $\lambda \not\equiv \sigma(T_\phi)$, $\frac{a - \lambda}{b - \lambda} = e^{i\theta}$ and $\|e^\theta\| = 0$.

PROOF. (1) Suppose $W = e^{i\theta}$, $t_W = 0$ and $g^2 = e^{i\alpha + i(v - \theta)}$. If $\lambda \not\equiv \sigma(T_\phi)$, then by Theorem 2, $\frac{a - \lambda}{b - \lambda} = e^{i\theta}$ and $\|e^\theta\| < \pi/2$. Hence

$$\frac{\phi - \lambda}{|\phi - \lambda|} g = \exp i(\ell + \bar{s} - \theta)$$

and since $t_W = 0$,

$$\inf \{\|\ell + \bar{s} - \theta - a\|_\infty : s \in L^\infty_R, a \in R\}$$

$$= \inf \{\|\ell - a\|_\infty : s \in L^\infty_R, a \in R\}$$

$$< \frac{\pi}{2}.$$
Thus $\lambda \not\in \sigma(T^W_\phi)$ by Theorems A and C. Similarly we can show that if $\lambda \not\in \sigma(T^W_\phi)$ then $\lambda \in \sigma(T_\phi)$. Suppose $\sigma(T^W_\phi) = \sigma(T_\phi)$ for arbitrary symbol $\phi$ in $L^\infty$. If $t = t_0$ is nonzero and $W = e^{i\tau^t}$ is a Helson-Szegö weight, then $T_\phi$ is invertible where $\phi = e^{-ikv}$ and $k = \pi/2t - 1$. For inf $\{\|k\ell - \tilde{a}\|_{\infty} : s \in L^\infty_R$ and $a \in R\} = kt = \pi/2 - 1$. On the other hand, $T^W_\phi$ is not invertible. For

$$\frac{\bar{\phi}}{|\phi|} = \exp i(\tilde{a} - (k + 1)v)$$

and

$$\inf \{\|\tilde{a} - (k + 1)v - \tilde{a}\|_{\infty} : s \in L^\infty_R$ and $a \in R\} = (k + 1)t = \frac{\pi}{2}$$

where $\tilde{g}^2 = e^{i\tau^t(\tilde{a} - v)}$.

(2) Suppose for any $\lambda \not\in \sigma(T_\phi)$, $\frac{\bar{\phi}}{|\phi|} e^{it} = e^{it} \lambda$ and inf $\{\|\ell - \tilde{a}\|_{\infty} : s \in L^\infty_R$ and $a \in R\} = 0$. We will show that $\sigma(T_\phi) \geq \sigma(T^W_\phi)$ for arbitrary Helson-Szegö weight $W$. If $\lambda \not\in \sigma(T_\phi)$, $W = e^{i\tau^t}$ is a Helson-Szegö weight and $g^2 = e^{i\tau^t(\tilde{a} - v)}$, then

$$\frac{\phi - \lambda}{|\phi - \lambda|} = e^{i(t\tilde{a} - v)}$$

and inf $\{\|\ell + \tilde{a} - v - \tilde{a}\|_{\infty} : s \in L^\infty_R$ and $a \in R\} < \pi/2$ by the hypothesis. This implies that $\sigma(T^W_\phi) \not\in \lambda$. Conversely suppose that $\sigma(T_\phi) \supseteq \sigma(T^W_\phi)$ for arbitrary Helson-Szegö weight $W$. If $\lambda \not\in \sigma(T_\phi)$, then $\frac{\bar{\phi}}{|\phi|} e^{it} = e^{it} \lambda$ and $0 < m(E) < 1$ and $a, b \in C$ with $a \neq b$, then there exists a Helson-Szegö weight $W$ such that $\sigma(T^W_\phi) \subseteq \sigma(T_\phi)$.

**Proof.** Since $tv = 0$ because $v \in \mathbb{C}_R$, (1) of Theorem 2 implies (1). Suppose $\phi$ is a function in $C$ and $\lambda \not\in \sigma(T^W_\phi)$ for a Helson-Szegö weight $W = e^{i\tau^t}$. Since $R(\phi) \subset \sigma(T^W_\phi)$,

$$\frac{\phi - \lambda}{|\phi - \lambda|} e^{i\tau^t} e^{i(\tilde{a} - v)}$$

where $m$ is an integer, $\ell \in \mathbb{C}_R$ and $g^2 = e^{i\tau^t(\tilde{a} - v)}$. By Theorems A and C, we can show $m = 0$. As $W^* = 1$, (2) of Theorem 2 implies that $\sigma(T_\phi) \supseteq \sigma(T^W_\phi)$ for arbitrary Helson-Szegö weight $W$. The converse is trivial. Suppose $\phi$ is a function in $H^\infty$ and $\lambda \not\in \sigma(T^W_\phi)$ for a Helson-Szegö weight $W = e^{i\tau^t}$. Since $R(\phi) \subset \sigma(T^W_\phi)$, $\phi - \lambda$ is invertible in $L^\infty$.
and so \( \phi - \lambda = qh \) where \( q \) is inner and \( h \) is invertible in \( H^p \). Since \( h = e^{i\ell\theta} \) and \( \ell = \log |h| \in L^\infty \),

\[
\frac{\phi - \lambda}{|\phi - \lambda|} = qe^{i\ell} e^{i(\bar{h} - v)}
\]

where \( g^2 = e^{i\phi + i|h|} \). By Theorems A and C, we can show that \( q \) is constant. As in case \( \phi \in C \), we can show \( \sigma(T^\phi_W) = \sigma(T^\phi) \) for arbitrary Helson-Szegő weight \( W \). This completes the proof of (2). Suppose \( \phi = a_X + b_XE \), \( 0 < m(E) < 1 \) and \( a, b \in C \) with \( a \neq b \). To prove (3), without loss of generality, we may assume that \( a \) and \( b \) are real numbers. By a theorem of Hartman and Wintner (cf. [2, Theorem 7.20]), \( \sigma(T^\phi) = [a, b] \).

If \( \lambda \notin [a, b] \),

\[
\frac{\phi - \lambda}{|\phi - \lambda|} = \exp i\{a(\lambda)X + b(\lambda)X_E \}
\]

where \( a(\lambda) = \arg(a - \lambda) \) and \( b(\lambda) = \arg(b - \lambda) \). By the proof of Corollary 1,

\[
\inf\{\|a(\lambda)X + b(\lambda)X_E - \bar{s} - a\|_\infty : s \in L^\infty_R \text{ and } a \in R\} = \frac{1}{2} \arg \frac{a - \lambda}{b - \lambda} \neq 0
\]

and hence by (2) of Theorem 2, there exists a Helson-Szegő weight \( W \) such that \( \sigma(T^\phi_W) \subseteq \sigma(T^\phi) \).

3. **Toeplitz operators on \( H^p \).** For \( 1 < p < \infty \), \( T^\phi \) denotes a Toeplitz operator on \( H^p \). We will write \( T^\phi = T^\phi \). By a theorem of Widom, Devinatz and Rochberg (cf. [8]), we know the invertibility of \( T^\phi \) and by a theorem of Widom (cf. [2, Corollary 7.46]), \( \sigma(T^\phi) \) is connected. If \( 1 < q < 2 < p < \infty \), then \( A_q \subseteq A_2 \subseteq A_p \). It is more difficult to describe \( \sigma(T^\phi) \) than \( \sigma(T^\phi) \). In this paper, we study only \( \sigma(T^\phi) \). When \( p = 2 \), (1) of Theorem 3 is a theorem of Brown and Halmos and (2) is a theorem of Hartman and Wintner. (3) of Theorem 3 is known in [10] for arbitrary \( 1 < p < \infty \). Our proof is different from it.

**Theorem 3.** Suppose \( p \geq 2 \) and \( t = (p - 2)\pi / 2p \).

1. If \( \phi \) is a function in \( L^\infty \), then \( \sigma(T^\phi) \subseteq h^t(\mathcal{R}(\phi)) \).

2. If \( \phi \) is a real function in \( L^\infty \), \( a = \text{essinf} \phi \) and \( b = \text{esssup} \phi \), then

\[
[a, b] \subseteq \sigma(T^\phi) \subseteq \Delta(c, r) \cap \Delta(\bar{c}, r)
\]

where \( c = \frac{a + b}{2} + i\frac{a - b}{2\sin t} \cot 2t \) and \( r = -\frac{a - b}{2\sin t} \). In particular, if \( p = 2 \), then \( t = 0 \) and hence \( \sigma(T^\phi) = [a, b] \).

3. If \( \phi \) is a function in \( C \), then \( \sigma(T^\phi) = \sigma(T^\phi) \).

**Proof.** (1) If \( \lambda \notin h^t(\mathcal{R}(\phi)) \), then by definition \( \lambda \in \bigcup \{ \mathcal{E}_{m\beta}^{ij} : \mathcal{E}_{m\beta} \supseteq \mathcal{R}(\phi) \text{ and } \ell = -i, m = -j \} \) and \( \theta(\alpha, \beta) = \pi - 2t \). Hence \( (\phi - \lambda)/|\phi - \lambda| = e^{\ell\beta} \), where \( 0 \leq s_\lambda \leq \pi - 2t - 2\varepsilon \) a.e. or \( -\pi + 2t + 2\varepsilon \leq s_\lambda \leq 0 \) for some \( \varepsilon > 0 \). Put \( v_\lambda = s_\lambda - \frac{\pi}{2} + t + \varepsilon \) or \( v_\lambda = s_\lambda + \frac{\pi}{2} - t - \varepsilon \), then \( \|v_\lambda\|_\infty \leq \frac{\pi}{2} - t - \varepsilon \). Put \( g^2 = e^{-\ell\beta} \), then \( g^2 \) is an outer function and \( |g|^2 = e^{-\ell\beta} \). Then \( \|\frac{1}{2}v_\lambda\|_\infty < \frac{\pi}{2} \) because \( \|v_\lambda\|_\infty < \frac{\pi}{2} - \frac{\pi - 2t}{2\varepsilon} \). Hence \( |g|^p \) satisfies \( A_2 \) condition and so \( |g|^p \) satisfies \( A_p \) condition by (cf. [3, Lemma 6.8]) because \( p > 2 \).
Since \((\phi - \lambda)/|\phi - \lambda| = \alpha(g/g)\) for some constant \(\alpha\) with \(|\alpha| = 1\), Theorem A implies (1).

(2) We may assume that \(\phi\) is not constant. By Theorem A, \(\mathcal{R}(\phi) \subseteq \sigma(T^p_\phi)\). Suppose \(\lambda \in [a, b]\) and \(\lambda \notin \mathcal{R}(\phi)\), then \((\phi - \lambda)/|\phi - \lambda| = 2\chi_E - 1\) for some measurable set \(E\) in \(T\). If \(\lambda \notin \sigma(T^p_\phi)\), then by Theorem A, there exists an outer function \(h_0\) in \(H^p\) such that \(2\chi_E - 1 = \bar{h}_0/h_0\). This implies that \(h_0^2\) is a real function in \(H^1\) because \(p \geq 2\). It is well known that only one real function in \(H^1\) is constant. Hence \(h_0\) is constant and this contradicts that \(\phi\) is not constant. Thus \([a, b] \subseteq \sigma(T^p_\phi)\). Now (1) implies (2).

(3) If \(\lambda \notin \mathcal{R}(\phi)\), then \((\phi - \lambda)/|\phi - \lambda|\) is a continuous function and hence
\[
\frac{\phi - \lambda}{|\phi - \lambda|} = z^\ell e^{i\nu}
\]
where \(\ell\) is an integer and \(\nu\) is a real function in \(C\). Put \(g^2 = e^{-\nu + i\ell}\), then \(|g|^2 = e^{-\nu}\). Since \(\nu\) is continuous, for any \(\epsilon > 0, \tilde{v} = s + t\) where both \(s\) and \(t\) are in \(C\) and \(|\ell| = \epsilon < \tilde{v}\). Suppose \(\ell = 0\). If \(\epsilon < \pi/|p|\), then \(|g|^2 = |s|^2 = \exp(-i\tilde{v}) = \exp(-i\tilde{s} - i\tilde{t})\) and \(|\tilde{g}l|_{\infty} < \tilde{s}\). Hence \(|g|^p\) satisfies \((A_2)\) condition and so \((A_p)\). By Theorem A, \(T^p_{\phi - \lambda}\) is invertible and so \(\lambda \notin \sigma(T^p_\phi)\). Suppose \(\ell \neq 0\). If \(T^p_{\phi - \lambda}\) is invertible, then by Theorem A
\[
\frac{\phi - \lambda}{|\phi - \lambda|} = z^\ell e^{i\nu} = \frac{|k|}{k} \frac{|h|^2}{h^2}
\]
where \(k\) and \(k^{-1}\) are in \(H^\infty\), and \(h\) is an outer function in \(H^p\) with \(|h|^p\) satisfying \((A_p)\) condition. Since \(z^\ell|g|^2 / g^2 = |kh|^2/h^2\), \(z^\ell f\) is a nonnegative function in \(H^{1/2}\) and hence it is constant. This contradicts that \(z^\ell\) is zero on the origin. If \(\ell < 0, z^\ell|1 + z^\ell|/(1 + z^\ell)^2 \geq 0\) and so \((1 + z^\ell)^2 f\) is a nonnegative function in \(H^{1/2}\) and so \(f = c(1 + z^\ell)^2\) for some constant \(c > 0\). This contradicts that \(f^- \notin H^{1/2}\).

4. **Singular integral operators on \(L^2\).** By Theorems A, B and C, we can expect that \(\sigma(S_{\alpha, \beta})\) is strongly related with \(\sigma(T_\alpha)\) and \(\sigma(T_\beta)\). (1) of Theorem 4 is an analogy of a theorem of Brown and Halmos, and (2) of Theorem 4 is an analogy of a theorem of Hartman and Wintner.

**THEOREM 4.** Suppose \(\alpha\) and \(\beta\) are functions in \(L^\infty\).

(1) \(\mathcal{R}(\alpha) \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha, \beta}) \subseteq \sigma(T_\alpha) \cup \sigma(T_\beta)\) where \(t = \pi/4\).

(2) If \(\alpha\) and \(\beta\) are real functions in \(L^\infty\),
\[
\{h(\mathcal{R}(\alpha)) \cap h(\mathcal{R}(\beta))\} \cup \{h^2(\mathcal{R}(\alpha)) \cap h^2(\mathcal{R}(\beta))\} \subseteq \sigma(S_{\alpha, \beta}) \subseteq \Delta(c, r) \cap \Delta(c, r)
\]
where \(a = \max\{\text{essinf} \alpha, \text{essinf} \beta\}, b = \max\{\text{esssup} \alpha, \text{esssup} \beta\}, c = \frac{ab}{b} - i\frac{ab}{b}\) and \(r = \frac{-ab}{2}\).

(3) If \(\beta\) is in \(C\),
\[
\sigma(T_\alpha) \cap \{\lambda \in C : i\langle \beta, \lambda \rangle = 0\} \cup \mathcal{R}(\beta) \subseteq \sigma(S_{\alpha, \beta}) \subseteq \sigma(T_\alpha) \cup \sigma(T_\beta).
\]
(4) If both $\alpha$ and $\beta$ are in $C$, then $\sigma(S_{\alpha,\beta}) = \{\sigma(T_\alpha) \cup \sigma(T_\beta)\} \setminus \{\lambda \in C \mid i_\lambda(\alpha, \lambda) = i_\lambda(\beta, \lambda) \neq 0\}$.

(5) Suppose both $\alpha$ and $\beta$ are in $C$. If $\beta$ is a real function, then $\sigma(S_{\alpha,\beta}) = \sigma(T_\alpha) \cup h(R_{\alpha}(\beta))$ and hence if both $\alpha$ and $\beta$ are real functions, then $\sigma(S_{\alpha,\beta}) = h[R_{\alpha}(\alpha)] \cup h[R_{\beta}(\beta)]$.

(6) If $\alpha$ and $\beta$ are functions in $H^\infty$, then $\sigma(S_{\alpha,\beta}) = [\alpha(D)]^1 \cup [\beta(D)]^1$.

(7) If $\alpha$ and $\beta$ are functions in $H^\infty$, then $\sigma(S_{\alpha,\beta}) = [\alpha(D)]^1 \cup [\beta(D)]^1 \setminus \{\lambda \notin R(\alpha) \cup R(\beta) : T_{\alpha,\beta} \text{ is invertible}\}$ where $q_\lambda$ is the inner part of $\alpha - \lambda$ and $p_\lambda$ is the inner part of $\beta - \lambda$.

(8) If $\alpha$ and $\beta$ are inner functions, and $\text{sing } \alpha \neq \text{sing } \beta$, then $\sigma(S_{\alpha,\beta}) = [D]^1$, while $\text{sing } \alpha \text{ and sing } \beta$ denote the subsets of $\partial D$ on which $\alpha$ and $\beta$ can not be analytically extended, respectively.

**Proof.** (1) By Theorem B, it is clear that $R(\alpha) \cup R(\beta) \subseteq \sigma(S_{\alpha,\beta})$. If $\lambda \not\in h'(R(\alpha) \cup R(\beta))$, then $(\alpha - \lambda) / (\beta - \lambda) = e^{\phi_\lambda}$ and $(\beta - \lambda) / (\beta - \lambda) = e^{\phi_\lambda}$ where $0 \leq s_\lambda, t_\lambda \leq \pi/2$ and $\phi_\lambda = \lambda = \exp(U - i\tilde{V})$.

where $U = \log |\alpha - \lambda| - \log |\beta - \lambda|$ and $\tilde{V} = t_\lambda - s_\lambda$. Then $U$ is bounded and $V = \exp \{t_\lambda - s_\lambda\}$ and $\|t_\lambda - s_\lambda\|_{\infty} \leq \pi/2 - \varepsilon$. By Theorem C, $S_{\alpha - \lambda, \beta - \lambda}$ is invertible.

(2) If $\alpha$ and $\beta$ are real functions and $\lambda \in h(R(\alpha)) \cap h(R(\beta))$, then $\alpha - \lambda$ is a real function which is not nonnegative or nonpositive, and $\beta - \lambda$ is a nonnegative or nonpositive function which is bounded, $(\alpha - \lambda) / (\beta - \lambda)$ is a real function in $L^\infty$ which is not nonnegative or nonpositive. If $S_{\alpha - \lambda, \beta - \lambda}$ is invertible, then by Theorems B and C both $\alpha - \lambda$ and $\beta - \lambda$ are invertible in $L^\infty$, and

$$\frac{\alpha - \lambda}{\beta - \lambda} = e^{i\theta_{\lambda,\beta}}.$$

where $\inf \{\|t - a\|_{\infty} : s \in L^\infty$ and $a \in R\} < \pi/2$. Let $g = e^{\varepsilon t+i\theta_{\lambda,\beta}}$, then $g$ is a real function in $H^1$. Since only one real function in $H^1$ is constant, $g$ is constant and so it contradicts that $(\alpha - \lambda)(\beta - \lambda)/(\beta - \lambda)\alpha - \lambda$ is nonconstant. This implies that $h(R(\alpha)) \cap h(R(\beta)) \subseteq \sigma(S_{\alpha,\beta})$. The same method shows that $h(R(\alpha)) \cap h(R(\beta)) \subseteq \sigma(S_{\alpha,\beta})$. Since $R(\alpha) \cup R(\beta) \subseteq \{a, b\}$, by (1) $\sigma(S_{\alpha,\beta}) \subseteq h(R(\beta))$ where $t = \pi/4$. This implies (2).

(3) Suppose $\lambda \in \sigma(T_\alpha) \cap \{\lambda \in C \mid i_\lambda(\beta, \lambda) = 0\}$. Then $\beta - \lambda = |\beta - \lambda|e^{i\varphi}$ and $\varphi \in C$ because $\beta$ is continuous. If $S_{\alpha - \lambda, \beta - \lambda}$ is invertible, then by Theorem B

$$\frac{\alpha - \lambda}{\beta - \lambda} = \gamma e^{(U-i\tilde{V})}$$

where $\gamma$ is constant, $U$ is a bounded real function, $V$ is a real function in $L^1$ and $\exp V$ satisfies $(A_2)$ condition. Hence

$$\alpha - \lambda = \gamma \exp(U + \log |\beta - \lambda| - i(V - \nu)),$$
U + log |β - λ| is in $L^∞$ and $e^{t^α}$ satisfies $(A_2)$ condition because $v \in C$. By Theorem A, this implies that $λ \notin σ(T_α)$. This contradiction shows that $λ \in σ(S_{βλ})$ and hence $σ(T_α) \cap \{ λ \in C : i(β, λ) = 0 \} \cup \{ β \} \subseteq σ(S_{βλ})$. If $λ \notin σ(T_α) \cup σ(T_β)$, then by Theorem C and [2, Corollary 7.28] $α - λ = |α - λ| e^{it}$ and $β - λ = |β - λ| e^{it}$ where

$$\inf \{ \| t - s - a \|_∞ : s \in L^∞_R, and a \in R \} < \pi/2 and \ell \in C. Therefore$$

$$\frac{α - λ}{β - λ} = \frac{|α - λ|}{|β - λ|} e^{i(α - t)}$$

and hence by Theorem C λ $\not\in σ(S_{βλ})$.

(4) If $λ \notin R(α) \cup R(β)$ and $i(α, λ) ≠ i(β, λ)$, then $α - λ = |α - λ| e^{iu}$ and $β - λ = |β - λ| e^{iu}$ where $u$ and $v$ are in $C$, and $λ$ and $t$ are integers with $t ≠ t$. Hence

$$\frac{α - λ}{β - λ} = \frac{|α - λ|}{|β - λ|} e^{i(u - v)}$$

and $λ - t ≠ 0$. By Theorem C, we can show that $λ \notin σ(S_{βλ})$. This implies that $σ(T_α) \cup σ(T_β) \subseteq σ(S_{βλ})$. If $λ \notin σ(T_α) \cup σ(T_β)$, then $α - λ = h$ and $β - λ = k$ where both $h$ and $k$ are invertible in $HF^α$. Hence $α - λ = h$ and $β - λ = k$ and so by Theorem C $λ \in σ(S_{βλ})$. This shows that $α(D) \setminus R(α) \cup R(β) \subseteq σ(S_{βλ})$. By the same method we can show that $β(D) \setminus R(α) \cup R(β) \subseteq σ(S_{βλ})$. By (1), $[α(D)]^α \cup [β(D)]^β \subseteq σ(S_{βλ})$. If $λ \notin [α(D)]^α \cup [β(D)]^β$, then $α - λ = h$ and $β - λ = k$ where both $h$ and $k$ are invertible in $HF^α$. By Theorem C, $λ \notin σ(S_{βλ})$.

(7) If $λ \in [α(D)]^α \setminus R(α) \cup R(β)$, then $α - λ = q_λ h_λ and β - λ = p_λ k_λ where both q_λ and p_λ are inner and both h_λ and k_λ are invertible in $HF^α$. Hence $(α - λ)/(β - λ) = q_λ h_λ/k_λ$. If $T_{q_λ h_λ}$ is not invertible, by Theorem C $λ \notin σ(S_{βλ})$. This implies that $[α(D)]^α \cup [β(D)]^β \setminus \{ λ \notin R(α) \cup R(β) : T_{q_λ h_λ} is invertible \} \subseteq σ(S_{βλ})$. If $λ \notin [α(D)]^α \cup [β(D)]^β$, then $λ \notin σ(S_{βλ})$ as in (6). If $λ \notin R(α) \cup R(β)$ and $T_{q_λ h_λ}$ is invertible, then by Theorem C $λ \notin σ(S_{βλ})$.

(8) $σ(S_{βλ}) \subseteq [D]^α$ by (7) and so if $λ \notin (R(α) \cup R(β)) \cap [D]^α$, then the inner part of $α - λ = q_λ/(1 - λα)$ and the inner part of $β - λ$ is $p_λ/(1 - λβ)$. Then $sing q_λ = sing q = sing p = sing p_λ$. By [6, Theorem 1], $T_{q_λ h_λ}$ is not invertible. By (7), this implies that $σ(S_{βλ}) = [α(D)]^α \cup [β(D)]^β = [D]^β$.

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