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FORBIDDEN SUBCATEGORIES OF NON-POLYNOMIAL GROWTH TAME SIMPLY CONNECTED ALGEBRAS

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ABSTRACT. Let k be an algebraically closed field and $A = kQ/I$ be a basic finite dimensional k-algebra such that Q is a connected quiver without oriented cycles. Assume that A is strongly simply connected, that is, for every convex subcategory B of A the first Hochschild cohomology $H^1(B,B)$ vanishes. The algebra A is sincere if it admits an indecomposable module having all simples as composition factors. We study the structure of strongly simply connected sincere algebras of tame representation type. We show that a sincere, tame, strongly connected algebra A which contains a convex subcategory which is either representation-infinite tilted of type $\tilde{E}_p$, $p = 6,7,8$, or a tubular algebra, is of polynomial growth.

The class of finite dimensional algebras over an algebraically closed field k may be divided into two disjoint classes: First, there are tame algebras for which the indecomposable modules occur, in each dimension, in a finite number of discrete and one-parameter families. Second, there are wild algebras whose representation theory is at least as complicated as the study of finite dimensional vector spaces together with two non-commuting endomorphisms, for which the classification of indecomposable up to isomorphism is a well-known unsolved problem. We are interested in the classification of tame simply connected algebras and their representations. Our interest in this problem is motivated by the fact that often a convenient way to determine whether a given algebra A is tame (and to compute its representations) consists in finding a simply connected cover of a suitable degeneration of A. It is known to be the case for all representation-finite algebras [5, 7], and it is expected to be true for tame algebras (for some special cases see [9, 11, 20]). It is also expected that a simply connected algebra A is tame it and only if the Tits form q_A of A is weakly non-negative, and, if this is the case, the one-parameter families correspond to the generic positive null vectors, see [14]. Following [1], by a simply connected algebra A, we mean a basic triangular algebra such that, for any presentation $A \xrightarrow{\sim} kQ/I$ of A as a bound quiver algebra, the fundamental group $\Pi_1(Q,I)$ of (Q,I) is trivial, or equivalently A does not admit proper Galois coverings. The class of simply connected algebras consists of algebras of finite global dimension, and includes the tilted algebras of Euclidean types $\tilde{\mathbf{D}}_n$, $n \geq 4$, $\tilde{\mathbf{E}}_p$, $p = 6,7,8$, and Ringel's tubular algebras [18]. We consider here the simply connected algebras for which every convex subcategory is also simply connected, and call them strongly simply connected algebras. It is known [21] that a triangular algebra A is strongly simply connected if and only if,

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for every convex subcategory C of A, the first Hochschild cohomology group $H^1(C,C)$ vanishes.

The representation theory of strongly simply connected algebras is best understood in case A is of polynomial growth (see [15, 22, 23]), that is, there is a natural number m such that the indecomposable A-modules occur, in each dimension d, in a finite number of discrete and at most d^m one-parameter families. But the knowledge of non-polynomial growth tame (strongly) simply connected algebras is rather poor. In the study of indecomposable modules over such algebras we may restrict to the sincere algebras, that is, algebras which admit indecomposable finite dimensional modules having all simple modules as composition factors. Our main result gives crucial information on the structure of non-polynomial growth, tame, sincere, strongly simply connected algebras.

THEOREM. *Let A be a sincere, tame, strongly simply connected algebra which contains a convex subcategory which is either representation-infinite tilted of type $\tilde{\mathbf{E}}_p$, $p = 6,7,8$, or a tubular algebra. Then A is either tilted algebra or a coil algebra. In particular A is of polynomial growth.*

We have the following direct consequence:

COROLLARY. *Let A be a sincere, non-polynomial growth tame strongly simply connected algebra. Then every minimal representation-infinite convex subcategory of A is a concealed algebra of type $\tilde{\mathbf{D}}_m$, $m \geq 4$.*

The paper is organized as follows. In a preliminary Section 1, we shall recall some concepts and results necessary for the proof of our main result. Section 2 consists of some preparatory lemmas on strongly simply connected algebras. We devote Section 3 to coil enlargements of algebras and their tame one-point extensions. In Sections 4 and 5 we present two essential parts of the proof of the above theorem, related respectively with tilted and coil algebras. In Section 6 we prove the theorem and present some of its consequences.

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1. **Preliminaries.**

1.1. *Notation.* Throughout this article, k will denote a fixed algebraically closed field. By an algebra A is meant a finite dimensional associative k-algebra with an identity, assumed moreover to be basic and connected: In this case, it is known that there exists a connected bound quiver (Q_A, I) and an isomorphism $A \xrightarrow{\sim} kQ_A/I$. Equivalently, $A = kQ_A/I$ can be considered as a k-category, of which the object class is the set of the vertices of Q_A, and the set $A(x,y)$ of morphisms from x to y is the quotient of the k-vector space $kQ_A(x,y)$ of all linear combinations of paths in Q_A from x to y by the subspace $I(x,y) = I \cap kQ_A(x,y)$, see [10]. A full subcategory C of A is said to be *convex* if any path in Q_A with source and target in C lies entirely in Q_C. If Q_A has no oriented cycle, A is said to be *triangular*.

By an A-module we mean a finitely generated right A-module, and we denote by $\operatorname{mod} A$ their category. Then $\operatorname{ind} A$ is the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules. A path in $\operatorname{mod} A$ is a sequence $M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_t$ of non-zero non-isomorphisms between indecomposable A-modules. If $M_t \cong M_0$, such a path is said to be a *cycle*. An indecomposable A-module M is said to be *directing* if it does not lie on any cycle in $\operatorname{mod} A$. For each vertex i of Q_A, we shall denote by e_i the corresponding primitive idempotent of A, by $S_A(i)$ the simple module $e_iA / e_i \operatorname{rad} A$, by $P_A(i)$ the projective cover of $S_A(i)$, and by $I_A(i)$ the injective envelope of $S_A(i)$. For an A-module M, its support $\operatorname{Supp} M$ is the full subcategory of A consisting of all objects $i \in Q_A$ such that $\operatorname{Hom}_A\big(P_A(i), M\big) \neq 0$. A module M is called *sincere* if $\operatorname{Supp} M = A$. Finally, an algebra A having an indecomposable sincere module is said to be sincere.

We shall denote by Γ_A the Auslander-Reiten quiver of A whose vertices are the isomorphism classes of indecomposable A-modules, arrows are the irreducible maps and whose translations τ_A, τ_A^- are the Auslander-Reiten translations $D\operatorname{Tr}$, $\operatorname{Tr} D$, respectively. We shall agree to identify the vertices of Γ_A with the corresponding indecomposable A-modules.

Finally, by a component of Γ_A we mean a connected component. For a background on Auslander-Reiten theory we refer to [10, 17].

1.2. *One-point extensions of algebras.* Let B be an algebra and M be a B-module. The one-point extension of B by M is the algebra

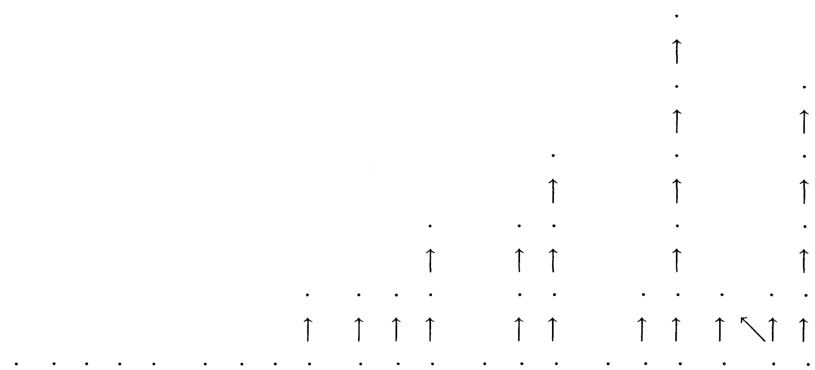
$$B[M] = \begin{bmatrix} k & M \\ 0 & B \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver $Q_{B[M]}$ of $B[M]$ contains the quiver Q_B as a full convex subquiver and there is an additional (extension) vertex which is a source. We may identify the $B[M]$-modules with the triples (V, X, φ) where V is a finite dimensional k-vector space, X a B-module, and $\varphi\colon V \longrightarrow \operatorname{Hom}_B(M, X)$ a k-linear map. A $B[M]$-linear map $(V, X, \varphi) \longrightarrow (V', X', \varphi')$ is thus a pair (f, g) where $f\colon V \longrightarrow V'$ is a k-linear map and $g\colon X \longrightarrow X'$ is a B-linear map such that $\varphi' f = \operatorname{Hom}_B(M, g)\varphi$. With $\operatorname{mod} B[M]$ one associates a vector space category $\operatorname{Hom}_B(M, \operatorname{mod} B)$ in the following sense.

A vector space category $\mathbf{K}$ is a Krull-Schmidt k-category together with a faithful functor $|-|\colon \mathbf{K} \longrightarrow \operatorname{mod} k$. The subspace category $\mathcal{U}(\mathbf{K})$ of $\mathbf{K}$ is defined as follows: its objects are triples (V, Y, φ) where V is a finite dimensional k-vector space, Y an object of $\mathbf{K}$, and $\varphi\colon V \longrightarrow |X|$ a k-linear map. A morphism $(V, Y, \varphi) \longrightarrow (V', Y', \varphi')$ in $\mathcal{U}(\mathbf{K})$ is a pair (f, h) where $f\colon V \longrightarrow V'$ is a k-linear map and $h\colon Y \longrightarrow Y'$ is a morphism in $\mathbf{K}$ such that $\varphi' f = |h|\varphi$. Then $\operatorname{Hom}_B(M, \operatorname{mod} B)$ is the vector space category whose objects are of the form $\operatorname{Hom}_B(M, X)$ with X in $\operatorname{mod} B$, and morphisms are of the form $\operatorname{Hom}_B(M, g)\colon \operatorname{Hom}_B(M, X) \longrightarrow \operatorname{Hom}_B(M, X')$ with $g\colon X \longrightarrow X'$ a morphism in $\operatorname{mod} B$. Then $|\operatorname{Hom}_B(M, X)|$ is the underlying k-vector space of $\operatorname{Hom}_B(M, X)$. We have then the reduction functor $\phi_M^B\colon \operatorname{mod} B[M] \longrightarrow \mathcal{U}\big(\operatorname{Hom}_B(M, \operatorname{mod} B)\big)$ which assigns to each $B[M]$-module (V, X, φ) the object $\big(V, \operatorname{Hom}_B(M, X), \varphi\big)$ and to each $B[M]$-homomorphism

$(f,g)\colon (V,X,\varphi) \longrightarrow (V',X',\varphi')$ the morphism $\big(f, \operatorname{Hom}_B(M,g)\big)$. It is well-known (see [19, (17.3)]) that ϕ_M^B is full, dense and induces a representation equivalence between the full subcategory of $\operatorname{mod} B[M]$ consisting of modules having no direct summands of the form $(0,X,0)$ and the category $\mathcal{U}\big(\operatorname{Hom}_B(M, \operatorname{mod} B)\big)$.

A vector space category $\mathbf{K}$ is said to be *linear* if $\dim_k |X| \leq 1$ for every object X. A linear vector space category is of tame representation type if and only if it does not contain as a full subposet one of the posets of the following Nazarova's list:



(1, 1, 1, 1, 1) (1, 1, 1, 2) (2, 2, 3) (1, 3, 4) (1, 2, 6) $(N, 5)$

See [17].

1.3. *Splitting lemma.* The following lemma gives some necessary conditions for an algebra A to be sincere.

LEMMA. *Let A be a triangular algebra and $B = B_0, B_1, \ldots, B_s = A$ a family of convex subcategories of A such that, for each $0 \leq i < s$, either $B_{i+1} = B_i[M_i]$ or $B_{i+1} = [M_i]B_i$ for some indecomposable B_i-module M_i. Assume moreover that* $\operatorname{ind} B$ *admits a splitting* $\operatorname{ind} B = \mathcal{P} \vee \mathcal{J}$ *where $\mathcal{P}$ and $\mathcal{J}$ are full subcategories of* $\operatorname{ind} B$*, and the following conditions are satisfied:*

1) $\operatorname{Hom}_B(\mathcal{J}, \mathcal{P}) = 0$

2) For each i such that $B_{i+1} = B_i[M_i]$, $M_i|_B$ belongs to $\operatorname{add} \mathcal{J}$

3) For each i such that $B_{i+1} = [M_i]B_i$, $M_i|_B$ belongs to $\operatorname{add} \mathcal{P}$.

4) There are i and j such that $M_i \in \mathcal{J}$ and $M_j \in \mathcal{P}$.

Then A is not sincere.

PROOF. We know that Q_A has no oriented cycle and Q_B is a convex subquiver of Q_A. Denote by $x_1, \ldots, x_r$ (resp. $y_1, \ldots, y_t$) the set of all vertices in Q_A being sources (resp. targets) of arrows with target (resp. source) in Q_B. Observe that, by (4), both sets are not empty. For each i, denote by B_i^+ (resp. B_i^-) the maximal convex subcategory of B_i which does not contain $y_1, \ldots, y_t$ (resp. $x_1, \ldots, x_r$). Moreover, let $\mathcal{P}_i$ (resp. $\mathcal{J}_i$) be the full subcategory of $\operatorname{ind} B_i^-$ (resp. of $\operatorname{ind} B_i^+$) consisting of modules X such that $X|_B \in \operatorname{add} \mathcal{P}$

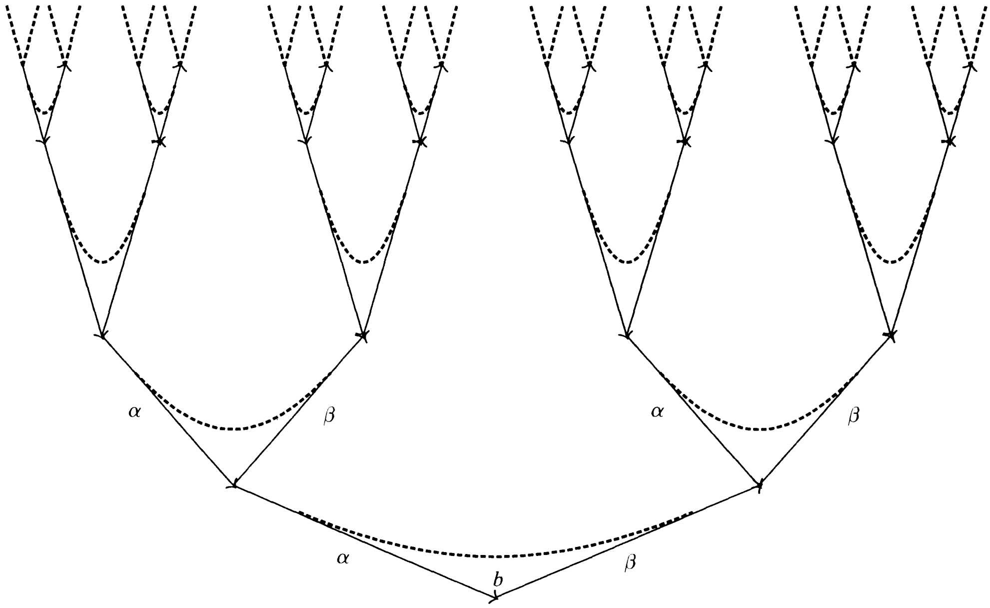
(resp. $X|_B \in \operatorname{add} \mathcal{J}$). We claim that, for each $0 \leq i \leq s$, we have $\operatorname{ind} B_i = \mathcal{P}_i \vee \mathcal{J}_i$ and $\operatorname{Hom}_{B_i}(\mathcal{J}_i, \mathcal{P}_i) = 0$. It is obvious for $i = 0$. Assume that it is true for some $0 \leq i < s$. Consider the case $B_{i+1} = B_i[M_i]$. By our assumption, we know that M_i belongs to $\mathcal{P}_i$ or $\mathcal{J}_i$. If $M_i \in \mathcal{P}_i$ then $M_i \in \operatorname{ind} B_i^-$ and $M_i|_B = 0$, by (2). Similarly, if $M_i \in \mathcal{J}_i$ then $M_i \in \operatorname{ind} B_i^+$ and $M_i|_B \in \operatorname{add} \mathcal{J}$. Take now an arbitrary module $Z = (V, X, \varphi)$ in $\operatorname{ind} B_{i+1}$. We have three cases to consider: Suppose first that $Z = (k, 0, 0)$. Then $Z \in \mathcal{P}_{i+1}$, if $M_i \in \mathcal{P}_i$, or $Z \in \mathcal{J}_{i+1}$, if $M_i \in \mathcal{J}_i$. Let now $V = 0$. Then $Y \in \operatorname{ind} B_i = \mathcal{P}_i \vee \mathcal{J}_i$, and hence Y belongs to $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$ or to $\mathcal{J}_i \subseteq \mathcal{J}_{i+1}$. Assume now that $\varphi \neq 0$. Let $X = X_1 \oplus \cdots \oplus X_s$ be a decomposition of X into a direct sum of indecomposable B_i-modules. Since Z is in $\operatorname{ind} B_{i+1}$ and $\varphi \neq 0$, we get that $\operatorname{Hom}_{B_i}(M_i, X_j) \neq 0$ for any $1 \leq j \leq s$. If $M_i \in \mathcal{P}_i$, then since $M_i|_B = 0$, $M_i \in \operatorname{ind} B_i^-$ and $\operatorname{ind} B_i = \mathcal{P}_i \vee \mathcal{J}_i$, we get that the modules $X_1, \ldots, X_s$ belong to $\mathcal{P}_i$, and hence $Z = (V, X, \varphi)$ belongs to $\mathcal{P}_{i+1}$. Finally, if $M_i \in \mathcal{J}_i$ then, since $M_i|_B \in \operatorname{add} \mathcal{J}$, $M_i \in \operatorname{ind} B_i^+$, $\operatorname{ind} B_i = \mathcal{P}_i \vee \mathcal{J}_i$ and $\operatorname{Hom}_{B_i}(\mathcal{J}_i, \mathcal{P}_i) = 0$, we get that the modules $X_1, \ldots, X_s$ belong to $\mathcal{J}_i$ and hence $Z = (V, X, \varphi)$ belongs to $\mathcal{J}_{i+1}$. This proves that $\operatorname{ind} B_{i+1} = \mathcal{P}_{i+1} \vee \mathcal{J}_{i+1}$. Moreover, if $Y \in \mathcal{P}_{i+1}$ and $Z \in \mathcal{J}_{i+1}$, then $\operatorname{Hom}_{B_{i+1}}(Z, Y) = \operatorname{Hom}_B(Z|_B, Y|_B) = 0$ because $\operatorname{Hom}_B(\mathcal{J}, \mathcal{P}) = 0$. Thus $\operatorname{Hom}_B(\mathcal{J}_{i+1}, \mathcal{P}_{i+1}) = 0$. The proof in the case $B_{i+1} = [M_i]B_i$ is similar.

Therefore, $\operatorname{ind} A = \mathcal{P}_s \vee \mathcal{J}_s$, $\operatorname{Hom}_A(\mathcal{J}_s, \mathcal{P}_s) = 0$, and by (4), $B \neq B_s^+$ and $B \neq B_s^-$. Since $\mathcal{P}_s$ consists of B_s^+-modules and $\mathcal{J}_s$ of B_s^--modules, we conclude that A is not sincere. ■

1.4. *Tame algebras.* Let A be an algebra and $k[X]$ be the polynomial algebra in one variable. Then A is said to be *tame* if, for any dimension d, there is a finite number of $k[X] - A$-bimodules M_i, $1 \leq i \leq n_d$, which are finitely generated and free as left $k[X]$-modules and such that all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $k[X]/(X - \lambda) \otimes_{k[X]} M_i$ for some $\lambda \in k$ and some i. Let $\mu_A(d)$ be the least number of $k[X] - A$-bimodules M_i satisfying the above condition. Then A is said to be of *polynomial growth* (resp. domestic) if there is a natural number m such that $\mu_A(d) \leq d^m$ (resp. $\mu_A(d) \leq m$) for any $d \geq 1$. Observe that, from the validity of the second Brauer-Thrall conjecture, A is representation-finite if and only if $\mu_A(d) = 0$ for any $d \geq 1$.

1.5. *Tilted algebras.* Let $H = k\Delta$ be a hereditary algebra and T a (multiplicity-free) tilting H-module, that is, $\operatorname{Ext}_A^1(T, T) = 0$ and T is a direct sum of n pairwise non-isomorphic indecomposable H-modules, where n is the number of vertices of Δ. Then $B = \operatorname{End}_A(T)$ is called a *tilted algebra* of type Δ. If Δ is one of the Euclidean quivers $\tilde{\mathbf{A}}_p$, $\tilde{\mathbf{D}}_q$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_7$, or $\tilde{\mathbf{E}}_8$ and T is preprojective (direct sum of modules lying in the τ_H^- orbits of projective modules), then $C = \operatorname{End}_H(T)$ is called *tame concealed*. In this case, Γ_C consists of a preprojective component $\mathcal{P}$ containing all projective modules, a preinjective component $\mathcal{J}$ containing all injective modules, and a family $\mathcal{T}_\lambda$, $\lambda \in \mathbf{P}_1(k)$, of stable tubes, forming the class of indecomposable regular C-modules, see [18]. We note also that by [12] every tame tilted algebra is domestic.

1.6. *Tubular extensions of tame concealed algebras.* Consider the following infinite tree



bound by all possible relations of the form $\alpha\beta$. A branch is the k-category given by a finite connected full bound subquiver of the above tree containing the root b. Let now C be a tame concealed algebra and $\mathcal{T}_\lambda$, $\lambda \in \mathbf{P}_1(k)$, its family of stable tubes. Take a sequence $E = (E_1, \ldots, E_s)$ of pairwise non-isomorphic C-modules which are simple among the regular modules, and a family $K = (K_1, \ldots, K_s)$ of branches, say with the roots $b_1, \ldots, b_s$, respectively. The tubular extension $\Lambda = C[E, K]$ of C by E and K is the category whose set of objects is the disjoint sum of the sets of objects of $C, K_1, \ldots, K_s$. The morphism sets are such that $\Lambda(x,y) = C(x,y)$ if $x, y \in C$; $\Lambda(x,y) = K_i(x,y)$ if $x, y \in K_i$, $\Lambda(x,y) = 0$ if $y \in K_j$ and $x \in C \cup K_i$, $i \neq j$; $\Lambda(x,y) = E_i(y) \otimes_k K_i(x, b_i)$ if $x \in K_i$, $y \in C$. Let r_λ denote the rank of the tube $\mathcal{T}_\lambda$, $\lambda \in \mathbf{P}_1(k)$. The tubular type $n_\Lambda = (n_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ is defined by $n_\lambda = r_\lambda + \sum_{E_i \in \mathcal{T}_\lambda} |K_i|$. Since almost all n_λ are equal to 1, we shall write, instead of $(n_\lambda)_{\lambda \in \mathbf{P}_1(k)}$, the (finite) sequence containing at least two n_λ, including all those which are larger than 1, arranged in non-decreasing order. A tubular extension Λ of C is called *domestic* (resp. tubular) if its tubular type is one of the following: (p,q), $1 \leq p \leq q$, $(2,2,r)$, $2 \leq r$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$ (resp. $(3,3,3)$, $(2,4,4)$, $(2,3,6)$ or $(2,2,2,2)$). Dually, one defines a tubular coextension of C. It was shown in [18, (4.9)] that B is a representation-infinite tilted algebra of Euclidean type with a complete slice in its preinjective (resp. preprojective) component if and only if B is a domestic tubular extension (resp. coextension) of a tame concealed algebra. A tubular (resp. cotubular) algebra is a tubular extension (resp. coextension) Λ of C with n_Λ tubular. By [18, (5.2)], any tubular algebra is cotubular and conversely. We know (see [13, (2.1)]) that for, a tubular extension Λ of C the following conditions are equivalent: (a) Λ is tame; (b) Λ

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is of polynomial growth (c) n_Λ is domestic or tubular. Moreover, Λ is domestic if and only if n_Λ is domestic.

We shall now describe the Auslander-Reiten quiver of a tame tubular extension or coextension of C. Let B be tubular extensions of C, then

$$\Gamma_B = \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \mathcal{J}_0^B,$$

where $\mathcal{P}_0^B$ denotes the preprojective component of Γ_C, $\mathcal{T}_0^B$ is a $\mathbf{P}_1(k)$-family of (ray) tubes obtained from the corresponding tubes in Γ_C by successive ray insertions, and $\mathcal{J}_0^B$ denotes the remaining components of Γ_B. The ordering from the left to the right indicates that there are non-zero morphisms only from any of these classes to itself and from the classes to its right. All projective indecomposable B-modules belong to $\mathcal{P}_0^B \vee \mathcal{T}_0^B$. Similarly, if B is a tubular coextension of C, then

$$\Gamma_B = \mathcal{P}_\infty^B \vee \mathcal{T}_\infty^B \vee \mathcal{J}_\infty^B$$

where $\mathcal{J}_\infty^B$ denotes the preinjective component of Γ_C, $\mathcal{T}_\infty^B$ is a $\mathbf{P}_1(k)$-family of (coray) tubes obtained from the corresponding tubes in Γ_C by successive coray insertions, and $\mathcal{P}_\infty^B$ denotes the remaining components of Γ_B. All injective indecomposable B-modules belong to $\mathcal{T}_\infty^B \vee \mathcal{J}_\infty^B$. If B is a domestic tubular extension (resp. coextension) of C, then $\mathcal{J}_0^B$ (resp. $\mathcal{P}_\infty^B$) is the preinjective (resp. preprojective) component of Γ_B and contains a complete slice [18, (4.9)]. If B is tubular then B is also a tubular coextension of a tame concealed algebra C' and

$$\mathcal{J}_0^B = \Big(\bigvee_{q\in\mathbf{Q}^+} \mathcal{T}_q^B\Big) \vee \mathcal{T}_\infty^B \vee \mathcal{J}_\infty^B$$

$$\mathcal{P}_\infty^B = \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \Big(\bigvee_{q\in\mathbf{Q}^+} \mathcal{T}_q^B\Big)$$

where $\mathbf{Q}^+$ is the set of positive rationals and each $\mathcal{T}_q^B$ is a $\mathbf{P}_1(k)$-family of stable tubes [18, (5.2)].

We shall need the following facts.

PROPOSITION. *Let B be a tame tubular extension of a tame concealed algebra C, and M be an indecomposable B-module. Then, with the above notation, the following holds:*

i) If $B[M]$ is tame then M does not belong to $\mathcal{P}_0^B = \mathcal{P}_0^C$.
ii) If B is tubular and $B[M]$ is tame, then M belongs to $\mathcal{T}_\infty^B \vee \mathcal{J}_\infty^B$
iii) If B is tubular and $[M]B$ is tame, then M belongs to $\mathcal{P}_0^B \vee \mathcal{T}_0^B$.
iv) If B is tilted of type $\tilde{\mathbf{E}}_p$, $6 \le p \le 8$, $M \in \mathcal{J}_0^B$, and $[M]B$ tame, then $[M]B$ is a tubular extension of C.
v) If B is tilted of type $\tilde{\mathbf{E}}_p$, $6 \le p \le 8$, $M \notin \mathcal{J}_0^B$, and $B[M]$ tame, then M lies on the mouth of a tube in $\mathcal{T}_0^B$ and $B[M]$ is a tubular extensions of C.

PROOF. (i): see [17, (3.5)] and [2, (3.1)].

(ii) and (iii): see [2, (3.2)].

(iv) and (v): take verbatim the proofs of Lemmas 3.3, 3.4 and 3.5 in [2], using the tameness of $[M]B$ (resp. of $B[M]$) and the fact that, by [17, (3.5)], every one-point extension of a hereditary algebra of type $\tilde{\mathbf{E}}_p$, $6 \le p \le 8$, by an indecomposable regular module of regular length at least 2, is wild.

2. Strongly simply connected algebras.

2.1. Let A be a triangular algebra and $Q = Q_A$. For each vertex x of Q, denote by $Q(x)$ the subquiver of Q obtained by deleting all those vertices of Q being a source of a path in Q with target x (including the trivial path from x to x). Following [6], A is said to have the separation property if, for each vertex x of Q, the radical $\operatorname{rad} P_A(x)$ of the projective A-module $P_A(x)$ at x is a direct sum of pairwise non-isomorphic indecomposable modules whose supports are contained in pairwise different connected components of $Q(x)$. It is known [21] that every algebra A with the separation property is simply connected in the sense of [1], that is, for any presentation $A \xrightarrow{\sim} KQ/I$ of A as a bound quiver algebra, the fundamental group $\Pi_1(Q,I)$ of (Q,I) is trivial. It was shown in [21] that every convex subcategory of A has the separation property if and only it every convex subcategory of A is simply connected. If this is the case, A is said to be *strongly simply connected*. Observe that the opposite algebra to a strongly simply connected algebra is also strongly simply connected. The class of strongly simply connected algebras contains all representation-finite simply connected algebras, algebras whose ordinary quiver is a tree, and all tilted algebras of Euclidean type (resp. tubular algebras) which do not contain convex hereditary subcategories of type $\tilde{\mathbf{A}}_n$.

2.2. PROPOSITION. *Let B be a convex subcategory of a strongly simply connected algebra A. Then there is a sequence $B = \Lambda_0, \Lambda_1, \dots, \Lambda_t = A$ of convex subcategories of A such that, for each $0 \le i < t$, Λ_{i+1} is a one-point extension or coextension of Λ_i by an indecomposable Λ_i-module.*

PROOF. Let $Q = Q_A$ (resp. $Q' = Q_B$) be the quiver of A (resp. $B = \Lambda_0$), then Q' is a convex full subquiver of Q. Without loss of generality, there is a source x_0 of Q not in Q'. Consider the indecomposable decomposition $\operatorname{rad} P_A(x_0) = \bigoplus_{i=1}^{s} R_i$ such that $\operatorname{supp} R_i$ is contained in the connected component Δ_i of $Q(x_0)$. Then $\Delta_i \neq \Delta_j$ for $i \neq j$. Let D_i be the convex subcategory of A with objects Δ_i. Since Q' is connected we may assume it is contained in Δ_1. Therefore B is a convex subcategory of D_1.

If $s = 1$, then $A = D_1[R_1]$ and by induction we obtain the result. Assume that $s > 1$. Choose a sink y_0 in Δ_2 and consider the indecomposable decomposition $I_A(y_0)/S_A(y_0) = \bigoplus_{j=1}^{t} C_j$ such that $\operatorname{supp} C_j$ is contained in the connected component Δ'_j of the quiver obtained from Q by deleting y_0. Then $\Delta'_i \neq \Delta'_j$ for $i \neq j$. Let D'_j be the convex subcategory of A with objects Δ'_j. We may assume that Q' (and x_0) is contained in Δ'_1. If $t = 1$, then $A = [C_1]D'_1$ and we obtain the result by induction. Otherwise, $t > 1$ and we may choose a source x_1 in Δ'_2. Since x_0 is in Δ'_1, then $x_0 \neq x_1$. We repeat the above construction obtaining either $A = E_1[M_1]$ (or $[N_1]E'_1$) for a convex subcategory E_1 (or E'_1) of A and an indecomposable module M_1 (or N_1), or a source x_2 different to x_0 and x_1. This process can only be repeated finitely many times. Thus the result. ∎

2.3. The following technical result will be useful; we will encounter situations such as the one described below in Section 5.

LEMMA. *Let A be a strongly simply connected algebra. Let B be a convex subcategory of A satisfying the following, (1)* $\operatorname{ind} B$ *admits a splitting* $\operatorname{ind} B = \mathcal{P} \vee I$, *where $\mathcal{P}$ and I are full subcategories of* $\operatorname{ind} B$ *with* $\operatorname{Hom}_B(I, \mathcal{P}) = 0$*; (2)there is a vertex x_0 of Q_A not in Q_B such that* $\operatorname{rad} P_A(x_0)|_B$ *has a direct summand N in $\mathcal{P}$; and for any successor y of x_0 not in Q_B,* $\operatorname{rad} P_A(y)|_B \in \operatorname{add}(I)$. *Then there exists an indecomposable B-module R such that $B[R]$ is a convex subcategory of A and* $\operatorname{Hom}_A(R, N) \neq 0$. *In particular, R is in $\mathcal{P}$.*

PROOF. We denote $M = \operatorname{rad} P_A(x_0)$.

Let D be the convex subcategory of A formed by the vertices of B and those vertices of Q_A which are successors of x_0 and predecessors of some vertex in Q_B. Then D is strongly simply connected and by (2.2) there is a sequence of convex subcategories $B = B_0 \subset B_1 \subset \cdots \subset B_s \subset B_{s+1} = D$ of A such that $B_{i+1} = B_i[M_i]$ for some indecomposable B_i-module M_i. Moreover, observe that $M_s = M|_D$ and $\operatorname{rad} P_A(x_0)|_B = \operatorname{rad} P_D(x_0)|B$. Assume that $s \geq 1$.

By (2), $M_i|_B \in \operatorname{add}(I)$ for $i = 0, \ldots, s-1$. Therefore we may proceed as in the splitting lemma (1.3), to construct full subcategories $\mathcal{P}_i$ and I_i of $\operatorname{ind} B_i$ such that $\operatorname{ind} B_i = \mathcal{P}_i \vee I_i$ and $\operatorname{Hom}_{B_i}(I_i, \mathcal{P}_i) = 0$, $i = 0, \ldots, s$. Let y_i be the extension vertex of B_i from B_{i-1}, that is, $\operatorname{rad} P_{B_i}(y_i) = M_{i-1}$. By definition, $P_{B_i}(y_i)$ belongs to I_i. Hence every indecomposable B_i-module X with $\operatorname{Hom}_A\big(P_{B_i}(y_i), X\big) \neq 0$ belongs to I_i. In the last step $y_{s+1} = x_0$ and $\operatorname{Hom}_A\big(P_{B_s}(y_s), M_s\big) \neq 0$. Therefore M_s belongs to I_s and the restriction $M_s|_B \in \operatorname{add}(I)$ which contradicts (2). This shows that $s = 0$ and $D = B[R]$ with $R = M_1$ an indecomposable B-module. Finally, there is an isomorphism $P_D(x_0) \longrightarrow P_A(x_0)|_D$, which implies that $\operatorname{Hom}_A(R, N) \neq 0$. ∎

2.4. LEMMA. *Let A be a strongly simply connected algebra. Let B be a convex subcategory of A satisfying the following (1)* $\operatorname{ind} B$ *admits a splitting* $\operatorname{ind} B = \mathcal{P} \vee C \vee I$, *where $\mathcal{P}$, C, I are full subcategories of* $\operatorname{ind} B$ *with* $\operatorname{Hom}_B(C \vee I, \mathcal{P}) = 0 = \operatorname{Hom}_B(I, C)$*; (2)there is a vertex x_0 not in Q_B such that* $\operatorname{rad} P_A(x_0)|_B$ *has a direct summand N with* $\operatorname{Hom}_B(N, C) \neq 0$*; for any proper successor y of x_0 not in Q_B, if* $\operatorname{Hom}_B\big(\operatorname{rad} P_A(y)|_B, C \vee I\big) \neq 0$, *then* $\operatorname{rad} P_A(y)|_B \in \operatorname{add}(I)$. *Then there exists an indecomposable B-module R such that $B[R]$ is a convex subcategory of A and $R \in \mathcal{P} \vee C$.*

PROOF. Let $M = \operatorname{rad} P_A(x_0)$. Let D be the convex subcategory of A formed by the vertices Q_B and those vertices which are successors of x_0 and predecessors of some vertex in Q_B. Consider a sequence of convex subcategories of A, $B = B_0 \subset B_1 \subset \cdots \subset B_s \subset B_{s+1} = D$ such that $B_{i+1} = B_i[M_i]$ for some indecomposable B_i-module M_i. Observe that $M_s = M|_D$.

We construct full subcategories $\mathcal{P}_i$, C_i, I_i of $\operatorname{ind} B_i$ such that $\operatorname{ind} B_i = \mathcal{P}_i \vee C_i \vee I_i$ and $\operatorname{Hom}_{B_i}(C_i \vee I_i, \mathcal{P}_i) = 0 = \operatorname{Hom}_{B_i}(I_i, C_i)$, $i = 0, \ldots, s$. Let y_i be the extension vertex of B_i from B_{i-1}, that is, $\operatorname{rad} P_{B_i}(y_i) = M_i$. Assume that for $i \leq j \leq s-1$, the categories $\mathcal{P}_i$, C_i, I_i are defined and $C_i = C$. Moreover, if $M_i \in \mathcal{P}_i$, then $\operatorname{Hom}_{B_i}(M_i, C_i \vee I_i) = 0$. We prove that the same is satisfied by B_{j+1} (and by M_{j+1} in case $j+1 < s$).

If $M_j \in \mathcal{P}_j$, then $\operatorname{Hom}_{B_j}(M_j, C_j \vee I_j) = 0$. We define $\mathcal{P}_{j+1}$ as the full subcategory of $\operatorname{ind} B_{j+1}$ whose objects are of the form $X = \big(V, X_0, \gamma \colon V \longrightarrow \operatorname{Hom}_{B_j}(M_j, X_0)\big)$ with $X_0 \in \operatorname{add}(\mathcal{P}_j)$. Observe that for any indecomposable $X = (V, X_0, \gamma)$ with $V \neq 0$, $X \in \mathcal{P}_{j+1}$. Set $C_{j+1} = C$ and $I_{j+1} = I_j$. Then $\operatorname{ind} B_{j+1} = \mathcal{P}_{j+1} \vee C_{j+1} \vee I_{j+1}$ is the desired splitting. If $M_j \in C_j \vee I_j$, then $M_j \in I_j$ (by (2), because $C_j = C$). We define I_{j+1} as the full subcategory of $\operatorname{ind} B_{j+1}$ with objects of the form $X = (V, X_0, \gamma)$ with $X_0 \in \operatorname{add}(I_j)$. Set $\mathcal{P}_{j+1} = \mathcal{P}_j$, $C_{j+1} = C$. Then $\operatorname{ind} B_{j+1} = \mathcal{P}_{j+1} \vee C_{j+1} \vee I_{j+1}$ is the desired splitting.

If $j+1 < s$ and $M_{j+1} \in \mathcal{P}_{j+1}$, it is of the form $M_{j+1} = (V, L, \gamma)$ with $L \in \operatorname{add}(\mathcal{P}_j)$. Assume $\operatorname{Hom}_{B_{j+1}}(M_{j+1}, C_{j+1} \vee I_{j+1}) \neq 0$. Since $I_{j+1} = I_j$, there is some $Y_j \in C_j \vee I_j$ such that $\operatorname{Hom}_{B_j}(L, Y_j) \neq 0$. Then either $\mathcal{P}_j = \mathcal{P}_{j-1}$ or $I_j = I_{j-1}$, in any case we get $Y_{j-1} \in C_{j-1} \vee I_{j-1}$ such that $\operatorname{Hom}_{B_j}(L, Y_{j-1}) \neq 0$. Continuing this way, we get that $\operatorname{Hom}_B(M_{j+1}|_B, Y) \neq 0$ for some module $Y \in C \vee I$, a contradiction. Hence $\operatorname{Hom}_{B_{j+1}}(M_{j+1}, C_{j+1} \vee I_{j+1}) = 0$.

At the final step, we get $y_{s+1} = x_0$ and $M_s = M|_D$. If $s \geq 1$, there is some $0 \leq i \leq s$, such that $\operatorname{Hom}_A(P_{B_i}(y_i), M_s) \neq 0$. Since M_s is an indecomposable B_s-module such that $\operatorname{Hom}_B(M_s|_B, C) \neq 0$, then $M_s \in \mathcal{P}_s \vee C$. Therefore $M_i \in \mathcal{P}_s$. Let y_m be a maximal vertex in Q_D not in Q_B, such that y_m is a successor of y_i. Then there is a chain of non-zero maps $P_D(y_m) \longrightarrow \cdots \longrightarrow P_D(y_i)$ and the radical $R = \operatorname{rad} P_D(y_m)$ is an indecomposable B-module in $\mathcal{P}$. This shows that there is a convex subcategory $B[R]$ of A with R in $\mathcal{P}$. In case $s = 0$, then $R = \operatorname{rad} P_D(x_0)$ is indecomposable in $\mathcal{P} \vee C$ and $B[R]$ is a convex subcategory of A. ∎

3. **Coil enlargements of algebras.** We recall in this section the notions of admissible operations, coils, and coil enlargements of algebras, playing an essential role in the proof of our main theorem. For more details on these concepts we refer the reader to [2, 3, 4].

3.1. Let A be a algebra and Γ be a component in Γ_A. For an indecomposable module X in Γ, called the *pivot*, three types of admissible operations are defined, yielding in each case a modified algebra A' of A, and a modified component Γ' of Γ:

ad 1) If the support of $\operatorname{Hom}_A(X, -)|_\Gamma$ is of the form:

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

we set $A' = (A \times D)[X \oplus Y]$, where D is the full $t \times t$ lower triangular matrix algebra, and Y is the unique projective-injective indecomposable D-module. In this case, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \binom{1}{1}\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X_i' = (k, X_i, 1)$ for $i \geq 1$, where Y_j, $1 \leq j \leq t$, denote the indecomposable injective D-modules. If $t = 0$, we set $A' = A[X]$, and the rectangle reduces to the ray formed by modules of the form X_i'.

ad 2) If the support of $\mathrm{Hom}_A(X,-)|_\Gamma$ is of the form:

$$Y_t \longleftarrow \cdots \longleftarrow Y_1 \longleftarrow X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

with $t \geq 1$ (so that X is injective), we set $A' = A[X]$. In this case, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right)$ for $i \geq 1$, $1 \leq j \leq t$, and $X_i' = (k, X_i, 1)$, for $i \geq 0$.

ad 3) If the support of $\mathrm{Hom}_A(X,-)|_\Gamma$ is of the form

$$\begin{array}{ccccccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_t & & & & \\ \uparrow & & \uparrow & & & & \uparrow & & & & \\ X = X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{t-1} & \longrightarrow & X_t & \longrightarrow & \cdots \end{array}$$

with $t \geq 2$ (so that X_{t-1} is injective), we set $A' = A[X]$. In this case, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right)$ for $i \geq 1$, $1 \leq j \leq i$, and $X_i' = (k, X_i, 1)$ for $i \geq 0$.

It was shown in [3] that the component of $\Gamma_{A'}$ containing X is Γ'. The dual coextension operations ad 1*), ad 2*) and ad 3*) are also called *admissible*. We say that X is a *pivoting module* (resp. *copivoting module*) if X can be used as pivot for an admissible operation of type ad 1), ad 2) or ad 3) (resp. ad 1*), ad 2*) or ad 3*)).

3.2. The above admissible operations can be regarded as operations on translation quivers rather than on Auslander-Reiten components (see [2, (2.1)]).

Following [2, 3] a translation quiver Γ is called a *coil* if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ such that Γ_0 is a stable tube and for each $0 \leq i < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation.

Given a coil Γ, the full convex subquiver Γ^{γ} formed by all non-directing vertices in Γ is again a coil (see [3]).

3.3. Let C be a tame concealed algebra and $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ the family of all stable tubes in Γ_C. Following [4], an algebra B is said to be a *coil enlargement* of C if there is a finite sequence of algebras $C = B_0, B_1, \ldots, B_m = B$ such that for each $0 \leq j < m$, B_{j+1} is obtained from B_j by an admissible operation with the pivot in a stable tube of $\mathcal{T}$ or in a component of Γ_{B_j}, obtained from a stable tube of $\mathcal{T}$ by means of the sequence of admissible operations done so far. Observe that, for each $\lambda \in \mathbf{P}_1(k)$, all modules of $\mathcal{T}_\lambda$ are contained in one component, say $\mathcal{C}_\lambda$, of Γ_B which is a coil. It follows that coil enlargements of C using only the operations of type ad 1) (resp. of type ad 1*)) are just tubular extensions (resp. tubular coextensions) of C in the sense of (1.6). By a *coil algebra* we mean a tame coil enlargement of a tame concealed algebra. We have the following facts proved in [4, (3.5), (4.1)].

PROPOSITION. *Let B be a coil enlargement of a tame concealed algebra C. Then, in the above notation, the following holds:*

i) There exists a unique maximal tubular extension B^+ of C which is a convex subcategory of B such that B is obtained from B^+ by a sequence of admissible operations of types ad 1*), ad 2*) *or* ad 3*).

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ii) There is a unique maximal tubular coextension B^- of C which is a convex subcategory of B such that B is obtained from B^- by a sequence of admissible operations of types ad *1),* ad *2) or* ad *3).*

iii) $\Gamma_B = \mathcal{P}_\infty^{B^-} \vee \mathcal{C} \vee \mathcal{I}_0^{B^+}$ *(see (1.6)) where* $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \mathbf{P}_1(K)}$, *is the family of coils obtained from the tubular family* $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbf{P}_1(k)}$, *by the corresponding admissible operations. Moreover* $\mathcal{C}$ *weakly separates* $\mathcal{P}_\infty^{B^-}$ *from* $\mathcal{I}_0^{B^+}$, *that is, every map from* $\mathcal{P}_\infty^{B^-}$ *to* $\mathcal{I}_0^{B^+}$ *factors through* $\operatorname{add}(\mathcal{C})$.

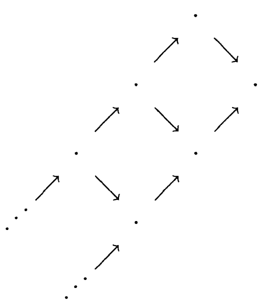
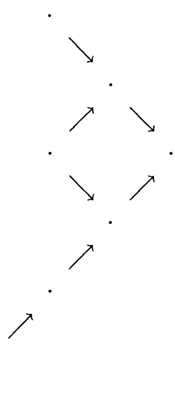
iv) B is tame if and only if B^- and B^+ are tame.

3.4. We shall need also the following proposition:

PROPOSITION. *Let B be a coil enlargement of a tame concealed algebra C of type* $\tilde{\mathbf{D}}_m$, $m \geq 4$, *or* $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$. *Assume that X is an indecomposable B-module lying in a coil Γ of Γ_B such that $\Lambda = [X]B$ is tame and* $\operatorname{Hom}_B(Z,X) \neq 0$ *for some non-directing module Z in Γ. Then*

i) B^- is a tilted algebra of Euclidean type $\tilde{\mathbf{D}}_n$, $n \geq m$, *or* $\tilde{\mathbf{E}}_q$, $6 \leq p \leq q \leq 8$, *with a complete slice in the preprojective component.*

ii) M is either copivoting or B^- is of type $\tilde{\mathbf{D}}_n$ *and the support of* $\operatorname{Hom}_B(-,X)|_\Gamma$ *contains the k-linear category of one of the following posets*

(1)  (2) 

In case (2), there are at least 4 projective indecomposable B^+-modules which are not C-modules. Moreover, there exists an indecomposable B^--module Y lying in the preprojective component of Γ_{B^-} such that $\dim_k \operatorname{Hom}_B(Y,X) = 2$.

PROOF. Since $\operatorname{Hom}_B(Z,X) \neq 0$ for some non-directing module Y of Γ, we conclude that Γ admits an infinite sectional path

$$\Sigma\colon \cdots \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

Put $F = B^-$. We know from Proposition 3.3 that B is obtained from F (resp. Γ is obtained from a coray tube $\mathcal{T}$ in Γ_F) by a sequence of admissible operations of type (ad 1), (ad 2) or (ad 3). Clearly, in the notation of (1.6), $\mathcal{T}$ belongs to the family $\mathcal{T}_\infty^F$. Observe also that infinitely many modules X_i belong to $\mathcal{T}$ and, if s is the minimal index i with this property, then $N = X_s$ is the restriction of X to F. In particular, the one-point coextension $[N]F$ is a convex subcategory of $\Lambda = [X]B$. Since Λ is tame and N belongs to $\mathcal{T}_\infty^F$, we get by Proposition 1.6, that F is not tubular. Therefore, F is a tilted algebra of Euclidean type $\tilde{\mathbf{D}}_n$ or $\tilde{\mathbf{E}}_q$ (see (1.6)) with a complete slice in its preprojective component $\mathcal{P}_\infty^F$. Let Δ be a slice in $\mathcal{P}_\infty^F$ and U the direct sum of modules in Δ. Then U is a tilting module $H = \operatorname{End}_F(U)$ is a hereditary algebra of type Δ, and $M = \operatorname{Hom}_F(U, N)$ is an indecomposable module lying in the stable tube $\mathcal{T}'$ of Γ_H containing all modules $\operatorname{Hom}_F(U, Y)$ with Y from $\mathcal{T}$. Moreover, since $[N]F$ is tame, $[M]H$ is also tame. Suppose now that X is not copivoting. We have two cases to consider. Assume first that $\operatorname{supp}\operatorname{Hom}_B(-, X)|_\Gamma$ contains all modules lying on an infinite sectional path which is parallel to Σ. Then the support of $\operatorname{Hom}_R(-, N)|_{\mathcal{T}}$ contains the k-linear category of a subquiver

$$\begin{tikzcd}
 & & & \cdot \arrow[dr] & \\
 & & \cdot \arrow[ur] \arrow[dr] & & \cdot\ N \\
 & \cdot \arrow[ur] \arrow[dr] & & \cdot \arrow[ur] & \\
\iddots \arrow[ur] & & \cdot \arrow[ur] & & \\
 & \cdot \arrow[ur] & & &
\end{tikzcd}$$

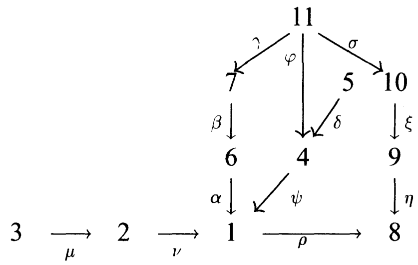
of $\mathcal{T}$. In this case, $M = \operatorname{Hom}_F(U, N)$ does not lie on the mouth of $\mathcal{T}'$, and then, by [17, (3.5)], we get that $\Delta = \tilde{\mathbf{D}}_n$. Hence $F = B^-$ is of type $\tilde{\mathbf{D}}_n$ and $\operatorname{supp}\operatorname{Hom}_B(-, X)$ contains a k-linear subcategory given by a poset of type (1). Suppose now that all non-directing modules from $\operatorname{supp}\operatorname{Hom}_B(-, X)|_\Gamma$ lie on Σ. Then, since X is not copivoting by the structure of Γ we infer that $\operatorname{supp}\operatorname{Hom}_B(-, X)|_\Gamma$ contains a k-linear category of a poset of type (2). We claim that, in this case, F is also of type $\tilde{\mathbf{D}}_n$. Indeed, if F is of type $\tilde{\mathbf{E}}_p$, then $\operatorname{Hom}_B(\operatorname{mod} B, X)$ contains a full subcategory given by a poset

$$\begin{tikzcd}
 & & & \operatorname{Hom}_B(Z_5, X) \arrow[d] \\
\operatorname{Hom}_B(Z_1, X) & \operatorname{Hom}_B(Z_2, X) & \operatorname{Hom}_B(Z_3, X) & \operatorname{Hom}_B(Z_4, X)
\end{tikzcd}$$

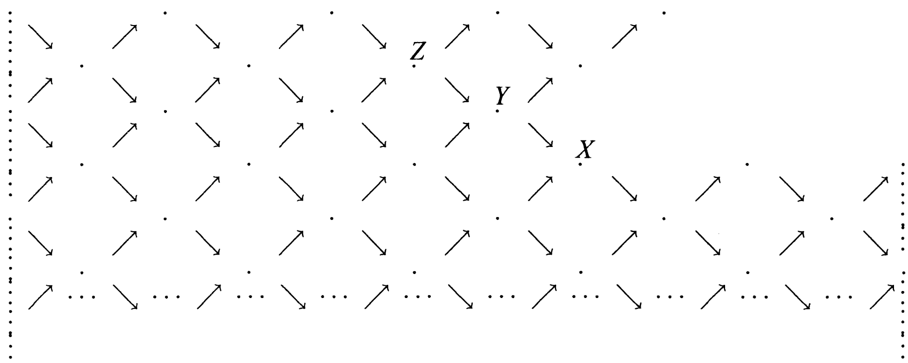
of type $(1, 1, 1, 2)$, where Z_1, Z_2 are from Γ and Z_3, Z_4, Z_5 from $\mathcal{P}_\infty^F$. But then $[X]B$ is not tame, a contradiction. Hence, $F = B^-$ is of type $\tilde{\mathbf{D}}_n$. Moreover in both cases, by [17, (3.5)], there exists Y in $\mathcal{P}_\infty^F$ such that $\dim_k \operatorname{Hom}_F(Y, N) = 2$, and hence $\dim_k \operatorname{Hom}_B(Y, X) \geq 2$, because there is a monomorphism $N \longrightarrow X$. Since $\Lambda = [X]B$ is tame, we get that $\dim_k \operatorname{Hom}_B(Y, X) = 2$. This finishes our proof. ■

We end this section with two typical examples of coil enlargements of concealed algebras which will occur in this paper.

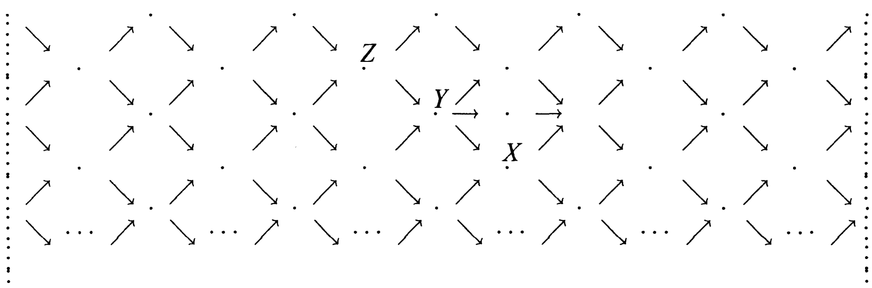
3.5. Let Λ be the bound quiver algebra kQ/I, where Q is the quiver



and I is the ideal in kQ generated by $\nu\rho$, $\delta\psi\rho$, $\gamma\beta\alpha - \varphi\psi$ and $\varphi\psi\rho - \sigma\xi\eta$. Let $C = k\Delta$ by the hereditary (hence concealed) algebra of type $\tilde{\mathbf{E}}_6$ given by the subquiver Δ of Q consisting of the vertices $1, 2, \ldots, 7$. Then, by [8, Tables], Γ_C admits a tube $\mathcal{T}_0$ containing the simple regular C-module X having the space k at the vertices $1, 4, 6$ and 7, and 0 in the remaining vertices of Δ. Consider now the bound quiver algebra $D = kQ'/I'$ where Q' is the subquiver of Q given by the vertices $1, 2, \ldots, 9, 10$, and I' the ideal in kQ generated by $\nu\rho$ and $\delta\psi\rho$. Then D is a cotubular, hence also tubular, algebra of type $(2, 3, 6)$. Moreover, the (coray) tube $\mathcal{T}_0'$ of Γ_D containing all modules of $\mathcal{T}_0$ and the injective D-modules $I_D(8)$, $I_D(9)$ and $I_D(10)$ is of the form



where the vertical dotted lines have to be identified in order to obtain a tube. Then $\Lambda = D[Y]$ is a coil enlargement of C obtained from D by the operation of type ad 2) with the pivot Y. Moreover, the coil $\mathcal{C}_0$ of Γ_Λ containing Y is of the form



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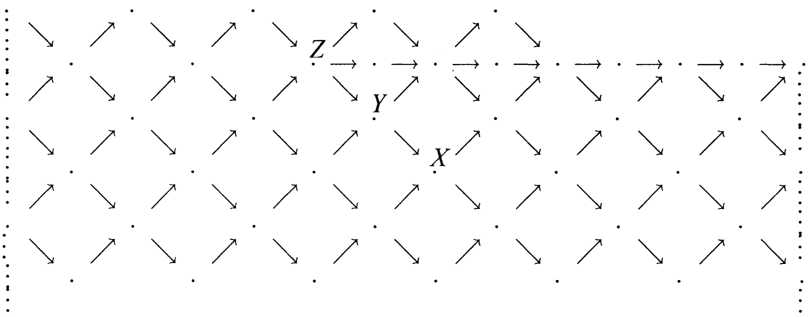
Observe that $\Lambda^- = D$ and Λ^+ is a tubular algebra of type $(2,3,6)$ given by the bound subquiver of (Q,I) formed by all vertices except 8. Moreover, Λ is obtained from Λ^+ by one admissible operation of type ad 2^*) creating the vertex 8. Further, Λ is strongly simply connected and of polynomial growth, by Proposition 3.3 and the polynomial growth of tubular algebras. It is also easy to see that the coil $\mathcal{C}_0$ contains infinitely many sincere indecomposable modules.

Consider now the one-point extensions $R = D[Y]$ of D by the indecomposable module Z having k at the vertices 1, 4, 6, 7, 8 and 9, and 0 in the remaining ones. Then Z is the pivot for an admissible operation of type ad 3), and hence R is a coil enlargement of C. Observe that R is the bound quiver algebra $K\Omega/J$ where Ω is the quiver

$$\begin{tikzcd}
& & & & 11 \arrow[dll, "\gamma"'] \arrow[ddl, "\varphi"] \arrow[ddr, "\sigma"] & \\
& & 7 \arrow[d, "\beta"'] & & 5 \arrow[dl, "\delta"] & 10 \arrow[d, "\xi"] \\
& & 6 \arrow[d, "\alpha"'] & 4 \arrow[dl, "\psi"] & & 9 \arrow[d, "\eta"] \\
3 \arrow[r, "\mu"'] & 2 \arrow[r, "\nu"'] & 1 \arrow[rrr, "\rho"'] & & & 8
\end{tikzcd}$$

and J is the ideal in $K\Omega$ generated by $\nu\rho$, $\delta\psi\rho$, $\gamma\beta\alpha - \varphi\psi$ and $\varphi\psi\rho - \sigma\eta$. Moreover, the coil $\mathcal{C}_0'$ of Γ_Λ containing Z is of form



In this case, $R^- = D$ and R^+ is a tubular algebra of type $(2,3,6)$ given by the bound subquiver of (Q,I) formed by all vertices except 8. Moreover, R is obtained from R^+ by one admissible operation of type ad 3^*) creating the vertex 8. Finally, R is strongly simply connected of polynomial growth, and the coil $\mathcal{C}_0'$ contains infinitely many sincere indecomposable modules.

4. **The tilted case.** We will divide the proof of our main theorem in two parts. In this section we study a particular situation corresponding to the tilted case.

PROPOSITION 4.1. *Let A be a tame, sincere, strongly simply connected algebra. Let B be a convex subcategory of A satisfying the following conditions:*

(i) B is representation-infinite tilted algebra of type $\tilde{\mathbf{E}}_p$, $6 \le p \le 8$ having a complete slice in its preinjective component;

(ii) A admits no convex subcategory of the form $[N]B$ *for some indecomposable B-module N;*

(iii) for any convex subcategory $B[M]$ *of A, M is an indecomposable preinjective B-module.*

Then A is a tilted algebra.

4.1. For the proof of (4.1) we need the following observation:

LEMMA. *Let K be a tilted algebra of type* $\mathbf{A}_t$. *Let* $U_1 \longrightarrow \cdots \longrightarrow U_t$ *be a slice in* Γ_K. *Consider an indecomposable K-module M such that* $\operatorname{Hom}_K(M, U_j) \neq 0$ *for some j. Then one of the following happens,*

(a) $K[M]$ *is a tilted algebra of type* $\mathbf{A}_{t+1}$ *and there is a slice* $U_1 \longrightarrow U_2' \longrightarrow \cdots \longrightarrow U_{t+1}'$ *in* $\Gamma_{K[M]}$ *starting at* U_1;

(b) there are indecomposable modules Y_1, Y_2 *over* $K[M]$ *with trivial endomorphism rings,* $\operatorname{Hom}_{K[M]}(Y_1, Y_2) = 0 = \operatorname{Hom}_{K[M]}(Y_2, Y_1)$ *and* $\dim_k \operatorname{Hom}_{K[M]}(U_1, Y_i) = 1$, $i = 1, 2$.

PROOF. Assume first $\operatorname{Hom}_K(M, U_i) \neq 0 \neq \operatorname{Hom}_K(M, U_\ell)$ for some $i < \ell$. Then $\operatorname{Hom}_K(M, U_{i+1}) \neq 0$. We consider the indecomposable $K[M]$-modules $Y_1 = U_{i+1}$ and $Y_2 = (k, U_i, \mathbf{1}_{\operatorname{Hom}_K(M,U_i)})$ satisfying the conditions in (b). Otherwise, j is the unique index such that U_j receives morphisms from M. Then in $\Gamma_{K[M]}$ there is a sectional path

$$U_1 \longrightarrow \cdots \longrightarrow U_j \longrightarrow (k, U_j, \mathbf{1}_{\operatorname{Hom}_K(M,U_j)}) \longrightarrow U_{j+1} \longrightarrow \cdots \longrightarrow U_t.$$

In particular, $K[M]$ is tilted of type $\mathbf{A}_{t+1}$. ∎

4.2. *Proof of (4.1).* We know by (1.6) that Γ_B consists of a preprojective component $\mathcal{P}$, a family T_λ, $\lambda \in \mathbf{P}_1(k)$, of (ray) tubes, and a preinjective component I having a section of type $\tilde{\mathbf{E}}_p$. By (iii) we may choose a section Σ of type $\tilde{\mathbf{E}}_p$ in I such that, for any convex subcategory $B[M]$ of A, M is a successor of Σ in I. Denote by $\mathcal{D}$ the full translation subquiver of I given by all predecessors of Σ in I. We infer by (ii) and (iii) that $\mathcal{P}$, T_λ, $\lambda \in \mathbf{P}_1(k)$, and $\mathcal{D}$ are full translation subquivers of Γ_A, and for any path $Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_t$ in $\operatorname{mod} A$ with Z_t in $\mathcal{E} = \mathcal{P} \vee (\bigvee_\lambda T_\lambda) \vee \mathcal{D}$, Z_0 also belongs to $\mathcal{E}$. Denote by $\mathcal{C}$ the component of Γ_A containing $\mathcal{D}$. We may assume that $\mathcal{C}$ contains at least one projective module, because otherwise $\mathcal{C} = I$, $A = B$, and we are done. We shall construct a sequence $\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_m$, $m \geq 1$, of convex subquivers of $\mathcal{C}$ satisfying the following conditions, for any $1 \leq i \leq m$:

(a) Σ_i is a sectional tree subquiver of $\mathcal{C}$ and there exists $r_i \geq 0$ such that $\tau_A^{-r_i}\Sigma_{i-1}$ is a full proper subquiver of Σ_i;

(b) the full translation subquiver $\mathcal{D}_i$ of $\mathcal{C}$ formed by all modules of the form $\tau_A^q X$, $q \geq 0$, $X \in \Sigma_i$, is closed under predecessors in $\mathcal{C}$;

(c) every module X in $\mathcal{D}_i$ such that $\tau_A^q X \neq 0$ for all $q \geq 0$, belongs to the τ_A-orbit of a module in Σ_0;

(d) $\hat{\mathcal{D}}_i = \mathcal{P} \vee (\bigvee_\lambda T_\lambda) \vee \mathcal{D}_i$ is closed under predecessors in $\operatorname{mod} A$;

(e) there is no injective module in $\mathcal{C}$ which is a proper predecessor of some module in Σ_i;

(f) all projective modules in $\mathcal{C}$ belong to $\mathcal{D}_m$.

First we show that this implies that A is a tilted algebra.

Observe that $\Delta = \Sigma_m$ is a finite section of $\mathcal{C}$, and so $\mathcal{C}$ has no oriented cycle. Let Λ be the full subcategory of A given by all objects of B and all $x \in Q_A$ such that $P_A(x)$ is in $\mathcal{C}$. Clearly, Λ is a convex subcategory of A. Moreover, $\mathcal{C}$ is a full component of Γ_Λ. Indeed, if $\operatorname{Hom}_A\big(P_A(y), Y\big) \neq 0$ for $Y \in \mathcal{C}$, $P_A(y) \notin \mathcal{C}$, then there is an infinite path

$$\cdots \longrightarrow Y_{i+1} \longrightarrow Y_i \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = Y$$

in $\mathcal{C}$ such that $\operatorname{Hom}_A\big(P_A(y), Y_i\big) \neq 0$ for any $i \geq 0$. Since $\mathcal{C}$ has only finitely many τ_A-orbits and no oriented cycle, all but finitely many Y_i belong to $\mathcal{D}$, and hence $y \in Q_B$. Further, every non-zero map $P_A(z) \longrightarrow I_A(z)$ factors through a direct sum of modules from Δ, and hence Δ is a sincere family of indecomposable Λ-modules. Finally, Δ is convex in $\operatorname{mod}\Lambda$, because Δ is convex in $\mathcal{C}$ and $\mathcal{E}$ is closed under predecessors in $\operatorname{mod} A$. Therefore, Δ is a slice in $\operatorname{mod}\Lambda$, and Λ is a tilted algebra $\operatorname{End}_H(T)$, where H is a hereditary algebra of type Δ and T is a tilting H-module. Also, $\mathcal{C}$ is the connecting component of Γ_Λ determined by T. Observe that $\Delta = \Sigma_m$ is wild because it contains a proper subquiver $\tau_A^{-t}\Sigma$ of type $\tilde{\mathbf{E}}_p$. Since Λ is tame, as a convex subcategory of A, we get, by [24, (7.6)] (see also [12]), that T has both a non-zero preprojective direct summand and a non-zero preinjective direct summand. But then $\mathcal{C}$ admits at least one injective module, say $I_\Lambda(z) = I_A(z)$. The facts that $\mathcal{C}$ contains an injective module and $\mathcal{P} \vee (\bigvee_\lambda \mathcal{T}_\lambda) \vee \mathcal{C}$ is closed under predecessors in $\operatorname{mod} A$, imply that every indecomposable sincere A-module lies in $\mathcal{C}$. Consequently $A = \Lambda$ and A is a tilted algebra.

We shall construct the required sequence $\Sigma_0, \Sigma_1, \ldots, \Sigma_m$ by induction on i, $0 \leq i \leq m$. Suppose that we have constructed a sequence $\Sigma = \Sigma_0, \ldots, \Sigma_s$ of convex subquivers in $\mathcal{C}$ satisfying the above conditions (a)–(f). If all projective modules belong to $\mathcal{D}_s$, we are done. Assume that $\mathcal{C}$ contains a projective module $P_A(x)$ which is not in $\mathcal{D}_s$. Then there is a smallest $r_s \geq 0$ such that $\tau_A^{-r_s}\Sigma_s$ contains an indecomposable direct summand, say N, of $\operatorname{rad} P_A(x)$. If $\operatorname{rad}_A P(x) = N$, we define Σ_{s+1} as a full subquiver of $\mathcal{C}$ formed by all vertices of $\tau_A^{-r_s}\Sigma_s$ and $P_A(x)$. Obviously, Σ_s satisfies (e). Indeed, suppose that some injective module $I_A(a)$ is a proper predecessor of Σ_{s+1} in $\mathcal{C}$. Then, since $\mathcal{P} \vee (\bigvee_\lambda \mathcal{T}_\lambda) \vee (\mathcal{D}_{s+1} \setminus \Sigma_{s+1})$ is closed under predecessors in $\operatorname{mod} A$, we conclude that $\operatorname{mod} A$ has no path of the form $P_A(x) \longrightarrow W \longrightarrow I_A(a)$. This gives a contradiction because A is sincere, and we are done.

Assume now that $M = \operatorname{rad}_A P(x)$ decomposes as $M = N_1 \oplus \cdots \oplus N_t$ with $t \geq 2$ and $N = N_1, \ldots, N_t$ indecomposable A-modules. Since A is strongly simply connected, the supports of the N_i are pairwise disjoint. Let D be the full subcategory of A given by x and the vertices of $Q_A(x)$. Then $D = E[M]$ where E is the convex subcategory of A given by the vertices of $Q_A(x)$. Moreover, $E = E_1 \times \cdots \times E_t$ with E_i connected and containing the support of N_i, $1 \leq j \leq t$. Since A is tame, the vector space category

$\mathcal{K} = \mathrm{Hom}_E(M, \mathrm{mod}\, E)$ is tame.

Consider the full subquiver Ω of $\mathcal{C}$ formed by all the modules in $\mathcal{C}$ lying on sectional paths with source $N = N_1$ and not passing through $P_A(x)$. Then Ω is a finite tree (of the same type as Σ_s), and there is no path in $\mathrm{mod}\, A$ from $P_A(x)$ to a module in $\mathcal{D}_s$. This implies that Ω contains a subtree of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$. Therefore the vector space category $\mathcal{X}_1 = \mathrm{Hom}_{E_1}(N_1, \mathrm{mod}\, E_1)$ contains a k-linear full subcategory given by a poset of type $(1,2)$. We show now that $t = 2$. Indeed, suppose that $t \geq 3$. Consider arrows $a_2 \longleftarrow x \longrightarrow a_3$ in Q_D with a_i in Q_{E_i}, $i = 2,3$. Denote by F the full subcategory of D given by the objects of E_1, a_2 and a_3. Then we get a convex subcategory $G = F[N_1 \oplus S(a_2) \oplus S(a_3)]$ of A. The corresponding vector space category $\mathrm{Hom}_F\big(N_1 \oplus S(a_2) \oplus S(a_3), \mathrm{mod}\, F\big)$ contains a k-linear full subcategory given by a poset of type $(1,1,1,2)$. By Nazarova's criterion (1.2), G is not tame. Hence A is not tame. This contradiction shows that $t = 2$.

We consider now carefully the structure of E_2. Let L be the support of N_2. We claim that L is a convex line in Q_A. Suppose first that L contains two incomparable objects c and d, with respect to the path order in Q_A. Observe that then $\mathrm{Hom}_{E_2}(N_2, \mathrm{mod}\, E_2)$ contains two orthogonal objects $\mathrm{Hom}_{E_2}\big(N_2, I_{E_2}(c)\big)$ and $\mathrm{Hom}_{E_2}\big(N_2, I_{E_2}(d)\big)$ with $\mathrm{End}_{E_2}\big(I_{E_2}(c)\big) \xrightarrow{\sim} k$ and $\mathrm{End}_{E_2}\big(I_{E_2}(d)\big) \xrightarrow{\sim} k$. Then using the poset of type $(1,2)$ associated to $\mathcal{X}_1$, we obtain a full subcategory $\mathcal{Y}$ of $\mathcal{K}$ of one of the following types:

(i) $\mathcal{Y}$ is the k-linear category of the poset $(1,1,1,2)$;

(ii) $\mathcal{Y}$ is given by two objects $\mathrm{Hom}_E(M, Y_1)$, $\mathrm{Hom}_E(M, Y_2)$ with trivial endomorphism rings, $\dim_k \mathrm{Hom}_E(M, Y_1) = 1$ and $\dim_k \mathrm{Hom}_E(M, Y_2) = 2$;

(iii) $\mathcal{Y}$ is given by an object $\mathrm{Hom}_E(M, Y)$ with trivial endomorphism ring and $\dim_k \mathrm{Hom}_E(M, Y) \geq 3$.

In any case we obtain a contradiction with the tameness of A, by (1.2) and [17, (2.4)]. Therefore, the vertices in Q_L are linearly ordered. Since the convex hull of L in Q_A yields a strongly simply connected category, L is a convex line in Q_A.

Let K be the biggest branch containing L and which is a convex subcategory of E_2. Then $N_2 = P_K(b)$, for the root b of K, and there is a maximal sectional path $N_2 = V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_t$ in Γ_K. Consider the convex subcategory D of A formed by x and the vertices in K and E_1. Then there is a splitting $\mathrm{ind}\, D = \mathcal{P}^0 \vee \hat{\mathcal{D}}_s \vee \mathcal{J}^0$, where $\mathcal{P}^0$ is the set of predecessors of V_t in Γ_K and $\mathcal{J}^0$ is formed by the proper successors of modules in $\Sigma_s \cup \{V_1, \ldots, V_t\}$. By (2.2), there is a sequence of convex subcategories of A, $D = D_0 \subset D_1 \subset \cdots \subset D_q = A$ such that $D_{i+1} = D_i[M_i]$ or $D_{i+1} = [M_i]D_i$ for some indecomposable D_i-module M_i. We show that for each i there is a splitting $\mathrm{ind}\, D_i = \mathcal{P}^i \vee \hat{\mathcal{D}}_s \vee \mathcal{J}^i$ satisfying that:

(α) The path $N_2 = V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_t$ is sectional in Γ_{D_i} and $\mathcal{P}^i$ is formed by all predecessors of V_t;

(β) $\mathcal{J}^i$ consists of the proper successors of modules in $\Sigma_s \cup \{V_1, \ldots, V_t\}$;

(γ) $\mathrm{Hom}_{D_i}(\mathcal{J}^i \vee \hat{\mathcal{D}}_s, \mathcal{P}^i) = 0$ and every map $X \longrightarrow Y$ with $X \in \mathcal{P}^i$ and $Y \in \mathcal{J}^i$ factors through a direct sum of modules V_j;

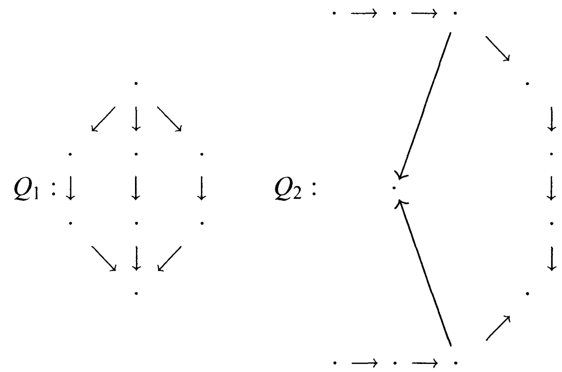
(δ) if $X \in \mathcal{P}^i$ and $\mathrm{Hom}_{D_i}(X, V_j) \neq 0$ for some $1 \leq j \leq q$, then $X|_D \in \mathrm{add}(\mathcal{P}^0)$.

For $i = 0$, this is clear. Assume we have (α)–(δ) for i. We consider the two possible situations for $i+1$. Assume $D_{i+1} = D_i[M_i]$. We prove that $\mathrm{Hom}_{D_i}(M_i, V_j) = 0$, for all j. Otherwise, assume $\mathrm{Hom}_{D_i}(M_i, V_j) \neq 0$. By ($\alpha$) and ($\gamma$), $M_i \in \mathcal{P}^i$; by (δ), $M_i|_D \in \mathrm{add}(\mathcal{P}^0)$. By the induction hypothesis, the conditions of (2.4) are satisfied for the splitting $\mathrm{ind}\, K = \mathcal{P}' \vee \mathcal{V} \vee \mathcal{J}'$, where $\mathcal{P}'$ is the set of proper predecessors of modules V_i $(1 \leq i \leq t)$, $\mathcal{V}$ is the set $\{V_i = i = 1, \ldots, t\}$ and $\mathcal{J}'$ are the proper successors of modules $V_i (1 \leq i \leq t)$ in $\mathrm{ind}\, K$. Therefore, there exists an indecomposable K-module R such that $K[R]$ is a convex subcategory of A and $R \in \mathcal{P}^0$. Then (4.2) applies: either $K[R]$ is a branch bigger than K, a contradiction; or there are indecomposable modules Y_1, Y_2 over $K[M]$ which together with X_1 yield a k-linear subcategory of $\mathrm{Hom}_{E_1 \times K[R]}(N_1 \oplus N_2, \mathrm{mod}\, E_1 \times K[R])$ given by a poset of type $(1,1,1,2)$. Since this poset is of wild type, then D_{i+1} is wild, a contradiction. Therefore $\mathrm{Hom}_{D_i}(M_i, V_j) = 0$, for all j. If $M_i \in \mathcal{P}^i$, by (γ) we also have $\mathrm{Hom}_{D_i}(M_i, \hat{\mathcal{D}}_s \vee \mathcal{J}^i) = 0$. Hence $\mathrm{ind}\, D_{i+1} = \mathcal{P}^{i+1} \vee \hat{\mathcal{D}}_s \vee \mathcal{J}^i$ with $\mathcal{P}^{i+1}$ given by $\mathcal{P}^i$ and all indecomposable D_{i+1}-modules $X = (V, X_0, \gamma)$ with $V \neq 0, X_0 \in \mathrm{mod}\, D_i$. Clearly this splitting satisfies (α)–(δ). If $M_i \in \mathcal{J}^i \vee \mathcal{D}_s$, we get $\mathrm{ind}\, D_{i+1} = \mathcal{P}^i \vee \hat{\mathcal{D}}_s \vee \mathcal{J}^{i+1}$ with $\mathcal{J}^{i+1}$ given by $\mathcal{J}^i$ and all indecomposable D_{i+1}-modules $X = (V, X_0, \gamma)$ with $V \neq 0$ (recall that $\hat{\mathcal{D}}_s$ was already closed under predecessors in $\mathrm{mod}\, A$). This splitting also satisfies (α)–(γ). Assume now that $D_{i+1} = [M_i]D_i$. In this case we show that $\mathrm{Hom}_{D_i}(V_j, M_i) = 0$ for all j. Indeed, this follows as dual to the situation above using the dual statements of (2.4) and (4.2). Therefore, if $M_i \in \mathcal{P}^i$, then $\mathrm{ind}\, D_{i+1} = \mathcal{P}^{i+1} \vee \hat{\mathcal{D}}_s \vee \mathcal{J}^i$; if $M_i \in \mathcal{J}^i$, then $\mathrm{ind}\, D_{i+1} = \mathcal{P}^i \vee \hat{\mathcal{D}}_s \vee \mathcal{J}^{i+1}$, satisfying in both cases (α)–(δ).

Consider the category $\mathcal{P}^q$ obtained in the final step of this process. If $\mathcal{P}^q \neq \mathcal{P}^0$, we may assume that $\mathcal{P}^j = \mathcal{P}^0, j \geq 0$ and $M_j \in \mathcal{P}^j$. Call y the vertex in $Q_{D_{j+1}}$ such that $\mathrm{rad}\, P_{D_{j+1}}(y) = M_j$ (resp. $I_{D_{j+1}}(y)/S_{D_{j+1}}(y) = M_j$) if $D_{j+1} = D_j[M_j]$ (resp. $D_{j+1} = [M_j]D_j$). Then $I_{D_{j+1}}(y) \in \mathcal{P}^{j+1}$. Indeed, in both cases $\mathrm{ind}\, D_{j+1} = \mathcal{P}^{j+1} \vee \hat{\mathcal{D}}_s \vee \mathcal{J}^j$ and $I_{D_{j+1}}(y) \notin \mathrm{ind}\, D_j$. Hence $I_A(y) \in \mathcal{P}^q$. Then for an indecomposable sincere A-module X, we get a path $P_A(x) \longrightarrow X \longrightarrow I_A(y)$, which is impossible since $P_A(x) \in \mathcal{J}^q$. Hence $\mathcal{P}^q = \mathcal{P}^0$, showing that $\mathcal{P}^0$ is a convex subquiver of Γ_A. Moreover, the sincerity of A implies that there are no injective modules in $\mathcal{P}^0$. Also, there are only finitely many predecessors of modules V_i $(1 \leq i \leq t)$ and they are of the form $\tau_A^q V_i, q \geq 0$.

Define Σ_{s+1} as the full subquiver of $\mathcal{C}$ formed by all vertices of $\tau_A^{-r_s}\Sigma_s$, $P_A(x)$ and the modules $V_1, \ldots, V_\ell$. Then Σ_{s+1} is a tree and the full translation subquiver $\mathcal{D}_{s+1}$ of $\mathcal{C}$ formed by all modules of the form $\tau_A^q X$, $q \geq 0, X \in \Sigma_{s+1}$, is closed under predecessors in $\mathcal{C}$. All conditions (a)–(e) are satisfied for Σ_{s+1}. We continue this process until we get Σ_m a sectional tree such that $\mathcal{D}_m$ contains all the projectives modules of $\mathcal{C}$. Therefore the sequence $\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_m$ satisfies (a)–(f) and A is a tilted algebra. This completes the proof of the proposition. ∎

4.3. The fully commutative algebras given by the quivers below satisfy the conditions of Proposition 4.1



These algebras are domestic with two one-parameter families of indecomposable modules. By [12], every tame sincere tilted algebra is domestic; by [15], these algebras admit at most two one-parameter families of indecomposable modules.

5. **The coil case.** In this section we will prove a result which covers a situation complementary to the Proposition 4.1.

5.1. PROPOSITION. *Let A be a tame, sincere, strongly simply connected algebra. Assume that A contains a convex subcategory B satisfying the following conditions:*

i) B is either a representation-infinite tilted algebra of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$ with a complete slice in the preinjective component and some projective outside the preprojective component or B is a tubular algebra;

ii) there exists a convex subcategory of A of the form $[N]B$ for some indecomposable B-module N.

Then A is a coil algebra.

The proof will be given at the end of the section after proving some technical results. We use freely the notation and results introduced in section 3.

5.2. LEMMA. *Let B be a tubular extension of a tame concealed algebra C. Assume that B is tubular or tilted of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$. Let $\mathcal{C}_1, \mathcal{C}_2$ be two tubular components of Γ_B containing projective modules and let $M_i \in \mathcal{C}_i^{\gamma}$, $i = 1, 2$. If M_1 is not copivoting and the situation (1) of (3.4) occurs, then $[M_2]([M_1]B)$ is not tame.*

PROOF. Assume that $[M_1]B$ is tame.

There is an indecomposable C-module N_1 such that $\mathrm{Hom}_C(X, N_1) = \mathrm{Hom}_B(X, M_1)$ for every C-module X. Then N_1 is a non-simple regular C-module. Since $[N_1]C$ is tame then C is of type $\tilde{\mathbf{D}}_m$, N_1 has regular length 2 and lies on a tube of rank $m-2$ in Γ_C. By (3.4), there is a module Y (in fact there are infinitely many!) in the preprojective component of Γ_C such that $\dim_k \mathrm{Hom}_C(Y, N_1) = 2$. We produce a family $(X_\lambda)_{\lambda \in k}$ of pairwise non-isomorphic indecomposable $[N_1]C$-modules of the following form: choose two linearly independent elements ρ_1, ρ_2 in $\mathrm{Hom}_C(Y, N_1)$, then $X_\lambda = \big(Y, k, \gamma_\lambda\colon Y \otimes DN_1 \longrightarrow k, y \otimes f \mapsto f(\rho_1 + \lambda\rho_2)y\big)$. Moreover, this family is orthogonal, that is $\mathrm{Hom}(X_\lambda, X_\mu) = 0$ for $\lambda \neq \mu$.

Since N_1 lies on the tube of rank $m-2$, then B is of tubular type $\big(2, 2+p_1, (m-2)+p_2\big)$ with $p_1, p_2 \geq 1$. Therefore M_2 belongs to a coil obtained by ray insertions in a tube of rank 2 in Γ_C.

Let N_2 be an indecomposable C-module such that $\mathrm{Hom}_C(X, N_2) = \mathrm{Hom}_B(X, M_2)$ for every C-module X. Then one checks that $\mathrm{Hom}_C(Y, N_2) \neq 0$. Therefore $\mathrm{Hom}_{[M_1]B}(X_\lambda, M_2) \neq 0$ for every $\lambda \in k$. In particular, $\mathrm{Hom}_{[M_1]B}(\mathrm{mod}[M_1]B, M_2)$ contains a poset of type $(1, 1, 1, 1, 1)$. Hence $[M_2]([M_1]B)$ is not tame. ∎

5.3. LEMMA. *Let B be a tubular extension of a tame concealed algebra C. Assume that B is tubular or tilted of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$. Let $\mathcal{C}_i$ $(1 \leq i \leq s)$ be the tubular components of Γ_B containing preinjective modules. Let $M_i \in \mathcal{C}_i^{\frown}$, $i = 1, \ldots, s$ and assume the iterated coextension ${}_{i=1}^{s}[M_i]B$ is tame. Then for some j, M_j is copivoting.*

PROOF. Assume that M_1 is not copivoting, By (3.4), C is concealed of type $\tilde{\mathbf{D}}_m$, $\mathcal{C}_1$ has at least 4 projective modules and $s \geq 2$. Assume first that we get the situation (1) of (3.4). By (5.2), $[M_2]([M_1]B)$ is not tame, a contradiction. We can assume that we have the situation (2) of (3.4).

If M_2 is not copivoting, as before, $\mathcal{C}_2$ has at least 4 projective modules and the tubular type of B is neither Dynkin nor Euclidean, a contradiction. Hence M_2 is copivoting. ∎

5.4. From now on we keep the following notation and hypothesis.

Let A be a tame strongly simply connected algebra. Let B be a convex subcategory of A which is a tubular extension of a tame concealed algebra C. We assume that B is either tubular or tilted of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$ with a complete slice in the preinjective component and some projective module outside the preprojective component. Obviously, we may assume that B is a maximal convex subcategory of A satisfying these properties. Let D be a maximal coil enlargement of B which is convex in A. Then as in (3.3), there is a unique maximal tubular coextension D^- of C which is a convex subcategory of A such that D is obtained from D^- by a sequence of admissible operations of types ad 1), ad 2) or ad 3). Then $\Gamma_D = \mathcal{P}_\infty \vee \mathcal{C} \vee \mathcal{J}_0$ where $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ is the family of coils obtained from the tubular family $\mathcal{T} = (T_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ of Γ_C by admissible operations. Moreover, $\mathcal{P}_\infty$ (resp. $\mathcal{J}_0$) is formed by D^--modules (resp. D^+-modules).

In fact, we may restate the assertion of Proposition 5.1:

CLAIM. Assume A is sincere and satisfies (i) and (ii) in (5.1). Then $A = D$.

The proof is given in (5.9). In the paragraph below we analyze carefully the structure of D and its enlargements.

5.5. Let $\mathcal{C}_\lambda$ be a coil in Γ_D. Let $\mathcal{C}_\lambda^{\frown}$ be the full convex subquiver of $\mathcal{C}_\lambda$ formed by the non-directing modules in $\mathcal{C}_\lambda$.

We define the *left border of* $\mathcal{C}_\lambda^{\frown}$ (resp. *right border of* $\mathcal{C}_\lambda^{\frown}$) as the set $\ell(\mathcal{C}_\lambda)$ (resp. $r(\mathcal{C}_\lambda)$) of modules $X \in \mathcal{C}_\lambda^{\frown}$ such that there is an irreducible map $Y \longrightarrow X$ (resp $X \longrightarrow Y$) with Y directing.

The *left directing part* of $\mathcal{C}_\lambda$ (resp. *right directing part* of $\mathcal{C}_\lambda$) is the set $d^-(\mathcal{C}_\lambda)$ (resp. $d^+(\mathcal{C}_\lambda)$) of directing modules $M \in \mathcal{C}_\lambda$ which are predecessors (resp. successors) of modules in $\ell(\mathcal{C}_\lambda)$ (resp. $r(\mathcal{C}_\lambda)$). The following properties are clear:

(a) $\mathrm{Hom}_D\big(\mathcal{C}_\lambda^{\frown}, d^-(\mathcal{C}_\lambda)\big) = 0 = \mathrm{Hom}_D\big(d^+(\mathcal{C}_\lambda), \mathcal{C}_\lambda^{\frown}\big), 0 = \mathrm{Hom}_D\big(d^+(\mathcal{C}_\lambda), d^-(\mathcal{C}_\lambda)\big)$.

(b) Any morphism $0 \neq f\colon Y \longrightarrow X$ with $Y \in d^-(\mathcal{C}_\lambda)$ and $X \in \mathcal{C}_\lambda^{\frown} \vee d^+(\mathcal{C}_\lambda)$ factorizes through $\mathrm{add}\big(\ell(\mathcal{C}_\lambda)\big)$. Moreover, $\mathrm{Im} f \in \mathrm{add}\big(\ell(\mathcal{C}_\lambda)\big)$. Dually for the right border.

(c) Let M be an indecomposable module in $\mathcal{C}_\lambda$ such that $\mathrm{Hom}_D(M, \mathcal{C}_\lambda^{\frown}) = 0$. Then $M \in d^+(\mathcal{C}_\lambda)$.

LEMMA. *Let $M \in \mathcal{C}_\lambda$ be such that* $\mathrm{Hom}_D(M, \mathcal{C}_\lambda^{\frown}) = 0$. *Let $E = D[M]$. Then there is a component $\mathcal{C}'_\lambda$ of Γ_E containing $\mathcal{C}_\lambda^{\frown}$ as a full subquiver such that the modules of $\mathcal{C}'_\lambda$ are divided in three parts, $\mathcal{C}'_\lambda = \mathcal{C}_\lambda^{\frown} \vee d^-(\mathcal{C}_\lambda) \vee d^+(\mathcal{C}'_\lambda)$, where $d^+(\mathcal{C}'_\lambda)$ are those modules in $\mathcal{C}'_\lambda$ which are successors of modules in $r(\mathcal{C}_\lambda)$. Moreover, the following conditions are satisfied: a)*$\mathrm{Hom}\big(\mathcal{C}_\lambda^{\frown}, d^-(\mathcal{C}_\lambda)\big) = 0 = \mathrm{Hom}\big(d^+(\mathcal{C}'_\lambda), \mathcal{C}_\lambda^{\frown}\big)$, $0 = \mathrm{Hom}\big(d^+(\mathcal{C}'_\lambda), d^-(\mathcal{C}_\lambda)\big)$. *b)Any morphism $0 \neq f\colon Y \longrightarrow X$ with $Y \in d^-(\mathcal{C}_\lambda)$ and $X \in \mathcal{C}_\lambda^{\frown} \vee d^+(\mathcal{C}'_\lambda)$ factorizes through* $\mathrm{add}\big(\ell(\mathcal{C}_\lambda)\big)$. *The dual* (b^*) *also holds.* ∎

We say that the component $\mathcal{C}'_\lambda$ is an *altered coil.*

5.6. LEMMA. *Let A be a tame, sincere, strongly simply connected algebra and B be a convex subcategory of A satisfying (i) and (ii) of (5.1). Then there is a unique tubular component of Γ_B containing projective modules.*

PROOF. Assume T_1 and T_2 are tubular components of Γ_B containing projective modules. Clearly we may assume that B is a maximal convex subcategory of A which is a tubular extension of C. Let D be the maximal coil enlargement of B. As in (5.5), $\Gamma_D = \mathcal{P}_\infty \vee \mathcal{C} \vee \mathcal{J}_0$ where $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ is a family of coils. We may assume that T_1 is contained in $\mathcal{C}_1$.

By (2.2), there is a sequence of convex subcategories of A, $D = D_0 \subset D_1 \subset \cdots \subset D_s = A$ such that $D_{i+1} = D_i[M_i]$ or $D_{i+1}[M_i]D_i$ for an indecomposable D_i-module M_i.

First, assume that there is no indecomposable module $M \in \mathcal{C}_1$ such that $[M]B$ is a convex subcategory of A. We consider the splitting $\mathrm{ind}\, D = \mathcal{P}^0 \vee \mathcal{J}^0$ such that $\mathcal{P}^0 = \mathcal{P}_\infty \vee \big(d^-(\mathcal{C}_\lambda) \vee \mathcal{C}_\lambda^{\frown}\big)_{\lambda \neq 1}$ and $I^0 = \mathcal{C}_1 \vee \big(d^+(\mathcal{C}_\lambda)\big)_{\lambda \neq 1} \vee I_0$ and $\mathrm{Hom}_D(I^0, \mathcal{P}^0) = 0$. We construct inductively a splitting $\mathrm{ind}\, D_i = \mathcal{P}^i \vee I^i$ with $\mathrm{Hom}_{D_i}(\mathcal{J}^i, \mathcal{P}^i) = 0$ such that $\mathcal{P}^s$ (resp. $\mathcal{J}^s$) contains an injective (resp. projective) module. This will contradict the sincerity of A, proving the result in this case. Assume that for all $i \leq j$ we have shown that $\Gamma_{D_i} = \mathcal{P}^i_\infty \vee \mathcal{C}^i \vee \mathcal{J}^i_0$ with $\mathcal{C}^i = (\mathcal{C}^i_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ a family of altered coils with $\mathcal{C}_\lambda^{i\frown} = \mathcal{C}_\lambda^{\frown}$, $\mathrm{Hom}_{D_i}(\mathcal{C}^i \vee \mathcal{J}^i_0, \mathcal{P}^i_\infty) = 0 = \mathrm{Hom}_{D_i}(\mathcal{J}^i_0, \mathcal{C}^i)$. First observe that this yields a splitting $\mathrm{ind}\, D_i = \mathcal{P}^i \vee \mathcal{J}^i$ with $\mathrm{Hom}_{D_i}(\mathcal{J}^i, \mathcal{P}^i) = 0$ as desired, for $i \leq j$. Indeed, set $\mathcal{P}^i = \mathcal{P}^i_\infty \vee \big(d^-(\mathcal{C}^i_\lambda) \vee \mathcal{C}_\lambda^{\frown}\big)_{\lambda \neq 1}$ and $\mathcal{J}^i = \mathcal{C}^i_1 \vee \big(d^+(\mathcal{C}^i_\lambda)\big)_{\lambda \neq 1} \vee \mathcal{J}^i_0$. Now, we prove that $\Gamma_{D_{j+1}}$ has a similar structure.

If $D_{j+1} = D_j[M_j]$, it is enough to show that

(α) if $M_j \in \mathcal{P}^j$, then $\mathrm{Hom}_{D_j}(M_j, \mathcal{C}_\lambda^{\frown} \vee \mathcal{J}^j) = 0$ for all $\lambda \in \mathbf{P}_1(k)$;

(β) if $M_j \in \mathcal{C}^j_1$, then $M_j \in d^-(\mathcal{C}^j_1)$ with $\mathrm{Hom}_{D_j}\big(M_j, \ell(\mathcal{C}_1)\big) = 0$ or $M_j \in d^+(\mathcal{C}^j_1)$.

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Assume that $M_j \in \mathcal{P}^j$ and $\mathrm{Hom}_{D_j}(M_j, N) \neq 0$ for some $N \in \mathcal{C}_\lambda^{j\curvearrowright} = \mathcal{C}_\lambda^{\curvearrowright}$. The induction hypothesis implies that the conditions of (2.4) are satisfied for D with the splitting $\mathrm{ind}\, D = \mathcal{P}_\infty \vee \mathcal{C} \vee \mathcal{J}_0$. Consequently we get an indecomposable D-module R such that $D[R]$ is convex in A and $R \in \mathcal{P}_\infty \vee \mathcal{C}^\curvearrowright$. If R belongs to $\mathcal{P}_\infty \subset \mathrm{ind}\, D^-$, by (1.6), $D^-[R]$ is a tubular coextension of C, contradicting the maximality of D^-. If $R \in \mathcal{C}^\curvearrowright$, by (2.3) we get a convex extension $B[R']$ of B in A. By (3.4), R' is pivoting and $B[R']$ is a tubular extension of C, again a contradiction. Since $(\mathcal{C}_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ weakly separates $\mathcal{P}^j$ from $\mathcal{J}^j$ (1.6), the case $\mathrm{Hom}_{D_j}(M_j, \mathcal{J}^j) \neq 0$ reduces to the situation $\mathrm{Hom}_{D_j}(M_j, \mathcal{C}_\lambda^\curvearrowright) \neq 0$ just considered. This shows (α). The proof of (β) is similar. The case $D_{j+1} = [M_j]D_j$ is dual. Finally, observe that there is a projective module in $\mathcal{J}^s$ (belonging to $\mathcal{C}_1^\curvearrowright$). By (5.1ii), there is a convex subcategory of A of the form $[N]B$. Consider $\Gamma_B = \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \mathcal{J}_0^B$ as in (1.6). If $N \in \mathcal{J}_0^B$, then $[N]B$ is a tubular extension of C (1.6), a contradiction to the maximality of B. Therefore $N \in \mathcal{P}_0^B \vee \mathcal{T}_0^B$. By hypothesis $N \notin \mathcal{C}_1$, considering the structure of $\Gamma_{D_s} = \Gamma_A$, we get that $N \in \mathcal{P}^0$. Let y in Q_A be minimal with $I_A(y)|_B = N$. Hence, if y in D_i, then $I_{D_i}(y) \in \mathcal{P}^i$. Thus $I_A(y) \in \mathcal{P}^s$. This finishes the proof in the first case.

Now, let $T_1, \ldots, T_s$ be the tubular components of Γ_B containing projectives. Let $N_i \in \mathcal{C}_i$ be B-modules such that $[N_i]B$ is a convex subcategory of A. If for every $N \in \mathcal{C}_1$ such that $[N]B$ is a convex subcategory of A, we have $N \in d^-(\mathcal{C}_1)$, then we may repeat the above argument to get a contradiction to the sincerity of A. Therefore we may assume that $N_i \in \mathcal{C}_i^\curvearrowright$. Then (5.3) assures that some N_j (say $j = 1$) is copivoting. Therefore (1.6) implies that $[N_1]D^-$ a tubular coextension of C, a contradiction to the maximality of D^-. We are done. ∎

5.7. LEMMA. *Under the hypothesis of (5.1), D^- is tubular or tilted of type $\tilde{\mathbf{E}}_q$, $6 \leq q \leq 8$ with a complete slice in the preprojective component.*

PROOF. Since D^- is a tame tubular coextension of C, we have only to exclude the case that D^- is tilted of type $\tilde{\mathbf{D}}_m$.

Assume D^- is tilted of type $\tilde{\mathbf{D}}_m$, then C is of type $\tilde{\mathbf{D}}_s$ with $s \leq m$. Consider first the case where B is tubular of type $(2, 2, 2, 2)$. Then $s = 4$. First observe that $D^- = C$. Indeed, otherwise there is a coray module N in C such that $[N]C$ is convex in D^-. Let T_{λ_0} be the tube of Γ_C which was inserted to form B; let T_{λ_1} be the tube of Γ_C where N lies. Since D^- in not tubular, then $\lambda_0 \neq \lambda_1$. By (2.2), we may find a sequence of convex subcategories of A, $C = B_0 \subset B_1 = B \subset B_2 = [N]B \subset \cdots \subset B_t = A$ such that $B_{i+1} = B_i[M_i]$ or $B_{i+1} = [M_i]B_i$ for an indecomposable B_i-module M_i, $i = 0, \ldots, t-1$. The quiver Γ_{B_2} may be described as:

$$\mathcal{P}_2 \vee \mathcal{T}_0' \vee (\mathcal{T}_\gamma)_{\gamma \in \mathbf{Q}^+} \vee \mathcal{T}_\infty \vee I_1$$

where $\mathcal{T}_\gamma, \mathcal{T}_\infty$ are tubular families of B-modules and $\mathcal{T}_0' = (T_\lambda')_\lambda$ is a tubular family such that T_λ' is a stable tube for $\lambda \neq \lambda_0, \lambda_1$; T_{λ_0}' is a ray-inserted tube and T_{λ_1}' is a coray-inserted tube and I_1 is the preinjective component of Γ_{B_1}.

Now, the splitting of $\mathrm{ind}\, C = \mathcal{P} \vee \mathcal{J}$, where $\mathcal{P} = \mathcal{P}_0 \vee (T_\lambda)_{\lambda \neq \lambda_0}$, $\mathcal{J} = T_{\lambda_0} \vee \mathcal{J}_0$ with $\mathcal{P}_0$ (resp. $\mathcal{J}_0$) the preprojective (resp. preinjective) component of Γ_C, satisfies the hypothesis (1)–(4) of (1.3). Therefore A is not sincere, a contradiction.

Hence $D^- = C$. By hypothesis (5.1ii), there is a convex subcategory $[N]B$ of A for an indecomposable B-module N. Since $[N]B$ is tame and N is not a regular C-module, then either N belongs to T'_{λ_0} (as above) or N is a preprojective module. The latter case would produce a splitting situation as before (Use the splitting ind $\mathcal{C} = \mathcal{P} \vee \mathcal{J}$ with the above notation). We may assume that N is in T'_{λ_0}. An application of the dual of (2.3) (with $B := C$, $\mathcal{P}$ the preprojective modules...) yields a convex subcategory $[R]C$ of A with R an indecomposable regular C-module. This contradicts that $D^- = C$.

Now, we assume that B is not of tubular type $(2,2,2,2)$. Since C is of type $(2,2,s-2)$, there are two tubes T_1, T_2 in Γ_B containing projective modules. By hypothesis (5.1ii) and Lemma 5.6, A is not sincere. Therefore the result follows. ∎

5.8. *Proof of Proposition 5.1.* Let B be a maximal tubular extension of the tame concealed algebra C and D a maximal coil enlargement of B which are convex in A. By hypothesis and (5.7), both B and D^- are either tubular or tilted algebras of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$. Therefore by (5.6) (and its proof), there is a unique tube in Γ_B (resp. Γ_{D^-}) containing projective (resp. injective) modules. Hence $\Gamma_D = \mathcal{P}_\infty \vee \mathcal{C} \vee I_0$, where $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ is a family of coils such that for $\lambda \neq \lambda_0$, $\mathcal{C}_\lambda$ is a stable tube and $\mathcal{C}_{\lambda_0}$ contains both projective and injective modules (the sincerity of A and (5.6) imply the $\mathcal{C}_{\lambda_0}$ is the unique coil containing projective or injective modules). Moreover, $\mathcal{P}_\infty$ (resp. I_0) is formed by D^--modules (resp. B-modules).

Consider a family $D = D_0 \subset D_1 \subset \cdots \subset D_s = A$ of convex subcategories of A such that either $D_{i+1} = D_i[M_i]$ or $D_{i+1} = [M_i]D_i$ for some indecomposable D_i-module M_i. We want to prove that $s = 0$. Assume $s \geq 1$. Without loss of generality (since B and D^- satisfy dual conditions), we may assume that $D_1 = [M_1]D$. In case $M_1 \in I_0$, then M_1 is a B-module. Since $[M_1]B$ is tame, by (1.6), it is a tubular extension of C, contradicting the maximality of B. In case M_1 lies on $\mathcal{C}$ and $\mathrm{Hom}_D(N, M_1) \neq 0$ for some $N \in \mathcal{C}_\lambda^{\curvearrowright}$, $\lambda \in \mathbf{P}_1(k)$, then (3.4) and (5.6) imply that M_1 is copivoting. Then $D_1 = [M_1]D$ is a coil enlargement of C, contradicting the maximality of D.

Finally, assume that M_1 lies on $\mathcal{P}_\infty$ or $M_1 \in \mathcal{C}_{\lambda_1}$ with $\mathrm{Hom}_D(\mathcal{C}_{\lambda_1}^{\curvearrowright}, M_1) = 0$. Then $\Gamma_{D_1} = \mathcal{P}'_\infty \vee (\mathcal{C}_\lambda^1)_\lambda \vee \mathcal{J}_0$ such that $\mathcal{C}_\lambda^1 = \mathcal{C}_\lambda$ for $\lambda \neq \lambda_1$ and $\mathcal{C}_{\lambda_1}^1$ is a coil with $\mathcal{C}_{\lambda_1}^{1\curvearrowright} = \mathcal{C}_{\lambda_1}^{\curvearrowright}$ and $M_1 \in d^-(\mathcal{C}_{\lambda_1})$. Then we find a splitting ind $D_1 = \mathcal{P}^1 \vee I^1$ with $\mathcal{P}^1 = \mathcal{P}_\infty^1 \vee d^-(\mathcal{C}_\lambda^1) \vee (\mathcal{C}_\lambda^{1\curvearrowright})_{\lambda \neq \lambda_1}$ and $I^1 = I^1 = \mathcal{C}_{\lambda_1}^\lambda \vee d^+(\mathcal{C}_\lambda^1) \vee I_0$. As in (5.6), we construct a splitting ind $D_i = \mathcal{P}^i \vee I^i$ such that $\mathrm{Hom}_{D_i}(\mathcal{J}^i, \mathcal{P}^i) = 0$, $\mathcal{P}^s$ contains an injective module and $\mathcal{J}^s$ a projective module. Of course, this contradicts the sincerity of A. This completes the proof that $A = D$ is a coil algebra. ∎

6. **Main theorem and remarks.**

6.1. *Proof of the Theorem.* By duality, we may assume that A admits a maximal convex subcategory B which is a tubular extension of a tame concealed algebra C, such that B is either tubular or a representation-infinite tilted algebra of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$ having a complete slice in its preinjective component. Therefore for any convex subcategory of A of the form $B[M]$, M is a preinjective B-module. Indeed, M is not preprojective since $B[M]$ is tame. If M belongs to a tubular component, then by (3.4), M is pivoting and therefore $B[M]$ is a tubular extension of C, contradicting the maximality of B. Moreover, observe that in case there is a convex subcategory $B[M]$ of A, then B is not a tubular algebra. Indeed otherwise the splitting lemma implies that A is not sincere (if $\operatorname{ind} B = \mathcal{P}^B \vee \mathcal{J}^B$ where $\mathcal{J}^B$ is formed by the preinjective modules, then $\operatorname{ind} B[M] = \mathcal{P}^B \vee \mathcal{J}'$ with $\operatorname{Hom}_{B[M]}(\mathcal{J}', \mathcal{P}^B) = 0$ and we may apply (1.3)).

If A admits no convex subcategory of the from $[N]B$, then (4.1) applies and A is tilted. Assume $[N]B$ is a convex subcategory of A. We still have two possibilities: if B admits a projective module outside the preprojective component of Γ_B, then (5.1) applies and A is a coil algebra. Otherwise, we consider the maximal coil enlargement D of B which is a convex subcategory of A. We get that D^- satisfies the hypothesis of (4.1) and therefore A is tilted. ∎

6.2. Let A be a strongly simply connected algebra. We denote by $n(A)$ the number of vertices of the quiver Q_A. By $c(A)$ we denote the number of convex subcategories of A which are tame concealed. Then A is representation-infinite if and only if $c(A) \geq 1$.

COROLLARY. *Let A be a tame, sincere, strongly simply connected algebra. Assume A admits a convex subcategory which is tubular or a representation-infinite tilted category of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$. Then*

a) if A is tilted, $c(A) \leq 2$ and $n(A) \leq 19$

b) if A is a coil algebra, $c(A) \leq 3$ and $n(A) \leq 13$.

PROOF. (a) Assume first that A is tilted. Then by [15], $c(A) \leq 2$. In case $c(A) = 2$, the possible algebras A with $n(A) \geq 20$ were classified in [16]. None of the algebras of that list contains a tilted category of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$. Hence $n(A) \leq 19$ in our case.

In case $c(A) = 1$, then A is a finite enlargement of a category B which is tilted of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$ with a complete slice in its preinjective component. Suppose that A is tilted of type Δ. Then Δ is a tree with at most 3 extremal points. Otherwise, there is a projective $P_A(a)$ whose radical $R = \operatorname{rad} P_A(a)$ is indecomposable with at least two irreducible arrows $N_i \xrightarrow{\alpha_i} R$ in Γ_A, and such that $\mathcal{S}(R \longrightarrow) \setminus \{P_A(a)\}$ is a tree containing some $\tilde{\mathbf{E}}_p$. Here $\Sigma = \mathcal{S}(R \longrightarrow)$ denotes the section in Γ_A starting at R (that is, Σ is the full subquiver of Γ_A formed by those vertices which receive a sectional path from R). We shall show that $n(A) \geq 15$ implies that $\big(\operatorname{Hom}_D(R, \operatorname{mod} D)\big)$ is representation-infinite, where D is the convex subcategory of A formed by the vertices in the support of modules in $\mathcal{S}(R \longrightarrow) \setminus \{P_A(a)\}$. This contradicts that A is a finite enlargement of B. Hence Δ has at most 3 terminal vertices. Applying [16, Theorem 3] we get $n(A) \leq 13$.

Indeed, if Σ has 4 or more terminal vertices, then we get a k-linear subcategory of $\big(\operatorname{Hom}_D(R, \operatorname{mod} D)\big)$ given by the poset $(1, 1, 1, 1)$. So, assume that Σ has only 3 terminal vertices. By similar arguments to [16, (3.3) to (3.7)], we get that $n(D) \leq 13$, contradicting that $15 \leq n(A) = n(D) + 1$.

(b) If A is a coil algebra, then A^+ (resp. A^-) is a tubular extension (resp. tubular coextension) of a tame concealed algebra C. Both A^+ and A^- are either tubular or tilted of type $\tilde{\mathbf{E}}_p$, $6 \leq p \leq 8$.

Observe that, since A is sincere, A^+ is tubular of type $(2,2,2,2)$ if and only if so is A^-. In this case $n(A) = 7$. If A is tubular or tilted of Euclidean type, then $n(A) \leq 10$. So, we may assume that A is neither tubular nor tilted of Euclidean type and that A^+ and A^- are not tubular of type $(2,2,2,2)$. If follows from (5.6) that C is of type $\tilde{\mathbf{E}}_q$, $6 \leq q \leq 8$. Then $n(A) \leq n(A^+) + n(A^-) - n(C) \leq 10 + 10 - 7 = 13$. Finally, if both A^+ and A^- are tubular, then $c(A) = 3$; otherwise $c(A) \leq 2$. ■

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