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ON SUPER EFFICIENCY IN SET-VALUED OPTIMISATION IN LOCALLY CONVEX SPACES

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The set-valued optimisation problem with constraints is considered in the sense of super efficiency in locally convex linear topological spaces. Under the assumption of nearly cone-subconvexlikeness, by applying the separation theorem for convex sets, Kuhn-Tucker and Lagrange necessary conditions for the set-valued optimisation problem to attain its super efficient solutions are obtained. Also, Kuhn-Tucker and Lagrange sufficient conditions are derived. Finally two kinds of unconstrained programs equivalent to set-valued optimisation problems are established.

1. INTRODUCTION

It is of value in both theory and algorithmic solution to transform a constrained optimisation problem into an unconstrained one. For a vector-valued optimisation problem, Chen and Rong [**1**] characterised the Benson proper efficiency in terms of Lagrange multiplier. For vector optimisation of set-valued maps, Li [**2**] obtained Kuhn-Tucker optimality conditions in the sense of weak efficiency.

On the other hand, various concepts of proper efficiency have been introduced. Borwein and Zhung [**3**] introduced the concept of super efficiency in normed vector spaces. Recently, Zheng [**4, 5**] introduced a new kind of efficiency, termed super efficiency, in locally convex topological vector spaces.

This paper is concerned with the optimality conditions for a set-valued optimisation problem in the sense of super efficiency in locally convex spaces. In Section 2, we give basic concepts and related results. In Section 3, under the assumption of nearly cone-subconvexlikeness, by applying the separation theorem for convex sets, the Kuhn-Tucker necessary condition for set-valued optimisation problems is obtained. Also, the Kuhn-Tucker sufficient condition is presented. In Section 4, Lagrange sufficient and necessary conditions are derived. Finally two kinds of unconstrained programs equivalent to set-valued optimisation programs are established.

Our methods are essentially different from those in [**1, 2**].

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2. Basic concepts and related results

Throughout the paper, let X be a linear topological space; Y and Z be locally convex linear topological spaces; C and D be closed convex pointed cones in Y and Z respectively; and finally let Y^* and Z^* be the topological dual spaces of Y and Z, respectively. If M is a nonempty subset of Y, the closure, interior and generated cone of M are denoted by $\operatorname{cl} M$, $\operatorname{int} M$ and $\operatorname{cone} M$, respectively. The dual cone of D is defined by $D^* = \{f \in Z^* : f(d) \geqslant 0, \forall d \in D\}$. A nonempty convex subset B of C is called a base of C if $0 \overline{\in} \operatorname{cl} B$ and $C = \operatorname{cone} B = \bigcup_{\lambda \geqslant 0} \lambda B = \bigcup_{\lambda \geqslant 0} \{\lambda x : x \in B\}$. Write $B^{st} = \{f \in Y^* : \exists t > 0 \text{ such that } f(b) \geqslant t, \forall b \in B\}$, $C^* = \{f \in Y^* : f(x) \geqslant 0, \forall x \in C\}$.

Definition 2.1: (See [4].) Suppose $\emptyset \neq M \subset Y$, and $y \in M$. Then y is called a super efficient point of M, written as $y \in SE(M, C)$, if, for any neighbourhood V of 0 in Y, there exists a neighbourhood U of 0 such that

$$\operatorname{cl}\bigl(\operatorname{cone}(M - y)\bigr) \cap (U - C) \subset V.$$

Remark 2.1. (See [4].) $y \in SE(M, C)$ if and only if for each neighbourhood V of 0 in Y, there exists a neighbourhood U of 0 such that

$$\operatorname{cone}(M - y) \cap (U - C) \subset V.$$

Definition 2.2: (See [6].) Let $E \subset X$, a set-valued map $F : E \to 2^Y$ is said to be nearly C-subconvexlike on E if $\operatorname{cl}\bigl(\operatorname{cone}(F(E) + C)\bigr)$ is convex.

Definition 2.2: (See [6].) Let $E \subset X$, a set-valued map $F : E \to 2^Y$ is said to be nearly C-convexlikeness and C-subconvexlikeness.

By $L(Z, Y)$ we denote the set of all continuous linear operators from Z into Y. Write $L^+(Z, Y) = \{T \in L(Z, Y) : T(D) \subset C\}$. Suppose $F : X \to 2^Y$, $G : X \to 2^Z$. Then $(F, G) : X \to 2^{Y \times Z}$ is defined by $(F, G)(x) = F(x) \times G(x)$, and $F + TG : X \to 2^Y$ is defined by $(F + TG)(x) = F(x) + T\bigl(G(x)\bigr)$.

Consider the following vector optimisation problem with set-valued maps:

$$\text{set} - \text{valued optimisation problem} \quad \min \ F(x) \quad \text{such that } \ G(x) \cap (-D) \neq \emptyset, x \in X.$$

The feasible set of set-valued optimisation problem is denoted by A, that is,

$$A = \{x \in X : G(x) \cap (-D) \neq \emptyset\}.$$

Definition 2.3: $x_0 \in A$ is called a super efficient solution of the set-valued optimisation problem if $F(x_0) \cap SE(F(A), C) \neq \emptyset$; (x_0, y_0) is called a super efficient element of the set-valued optimisation problem if $x_0 \in A$ and $y_0 \in F(x_0) \cap SE(F(A), C)$.

Lemma 2.1. (See [7].) *Suppose C has a bounded base B, $\emptyset \neq M \subset Y$, then*

$$SE(M,C) = SE(M+C,C).$$

Lemma 2.2. (See [8].) *Suppose C has a bounded base B, $\emptyset \neq M \subset Y$, $y^* \in M$. If there exists $g \in \operatorname{int} C^*$ such that $g(y^*) = \min\{g(y) : y \in M\}$, then $y^* \in SE(M,C)$.*

Remark 2.2. From the proof of [8, Theorem 3.1], we see the above Lemma does not need the assumption that M is C-convex.

3. Kuhn-Tucker optimality conditions

Let $\emptyset \neq S \subset Y$, $\overline{y} \in Y$, $\varphi \in Y^*$. For convenience, let $\varphi(S) \geqslant \varphi(\overline{y})$ stand for $\varphi(y) \geqslant \varphi(\overline{y})$, $\forall y \in S$.

In the same way as in the proof of [2, Lemma 1.1], we can verify the following Lemma.

Lemma 3.1. *Let $\varphi \in D^* \setminus \{0_{Z^*}\}$ and $d \in \operatorname{int} D$, then $\varphi(d) > 0$.*

Lemma 3.2. (See [9, Proposition 2.1(h)].) *If B is a bounded base for C, then $B^{st} = \operatorname{int} C^*$.*

Theorem 3.1. *Suppose C has a bounded base, (x_0, y_0) is a super efficient element of set-valued optimisation problem, $\bigl(F(x) - y_0, G(x)\bigr)$ is nearly $C \times D$-subconvexlike on X and there exists an $\overline{x} \in X$ such that $G(\overline{x}) \cap (-\operatorname{int} D) \neq \emptyset$, then there exist $s^* \in \operatorname{int} C^*$, $k^* \in D^*$ such that*

$$\inf_{x \in X}\Bigl(s^*\bigl(F(x)\bigr) + k^*\bigl(G(x)\bigr)\Bigr) = s^*(y_0), \tag{1}$$

and

$$\inf k^*\bigl(G(x_0)\bigr) = 0, \tag{2}$$

where $s^*\bigl(F(x)\bigr) = \bigcup\limits_{y \in F(x)} s^*(y)$, $k^*\bigl(G(x)\bigr) = \bigcup\limits_{z \in G(x)} k^*(z)$.

Proof: Since (x_0, y_0) is a super efficient element of the set-valued optimisation problem, we get

$$y_0 \in F(x_0) \cap SE\bigl(F(A), C\bigr).$$

From Lemma 2.1, one has $y_0 \in SE\bigl(F(A) + C, C\bigr)$. Suppose B is a bounded base of C, then there is an open convex balanced neighbourhood V of 0 such that

$$(-B) \cap V = \emptyset. \tag{3}$$

From $y_0 \in SE\bigl(F(A) + C, C\bigr)$, it follows that there is an open convex neighbourhood U of 0 such that $U \subset V/2$ and $\operatorname{cone}\bigl(F(A) + C - y_0\bigr) \cap (U - C) \subset V/2$. Thus

$$\operatorname{cone}\bigl(F(A) + C - y_0\bigr) \cap (U - B) \subset V/2 \cap (U - B). \tag{4}$$

It follows from (3) and $U \subset V/2$ that

$$\frac{1}{2}V \cap (U - B) \subset \frac{1}{2}V \cap \Big(\frac{1}{2}V - B\Big) = \emptyset,$$

which together with (4) leads to

$$\text{(5)} \qquad \operatorname{cone}\big(F(A) + C - y_0\big) \cap (U - B) = \emptyset.$$

Let $\varphi(x) = \big(F(x) - y_0, G(x)\big)$. In what follows, we prove

$$\text{(6)} \qquad \Big(\operatorname{cone}\big(\varphi(X) + C \times D\big)\Big) \cap \big((U - B) \times (-\operatorname{int} D)\big) = \emptyset,$$

where $\varphi(X) = \bigcup_{x \in X} \big(F(x) - y_0, G(x)\big)$. The proof is by contradiction. Otherwise, from $0_Z \overline{\in} - \operatorname{int} D$ we conclude that there exist $\lambda > 0$ and $\widehat{x} \in X$ such that

$$\Big(\lambda\big(F(\widehat{x}) + C - y_0, G(\widehat{x}) + D\big)\Big) \cap \big((U - B) \times (-\operatorname{int} D)\big) \neq \emptyset.$$

So,

$$\text{(7)} \qquad \Big(\lambda\big(F(\widehat{x}) + C - y_0\big)\Big) \cap (U - B) \neq \emptyset,$$

and

$$\text{(8)} \qquad \Big(\lambda\big(G(\widehat{x}) + D\big)\Big) \cap (-\operatorname{int} D) \neq \emptyset.$$

From (8) we get $(G(\widehat{x}) + D) \cap (-\operatorname{int} D) \neq \emptyset$, and since D is closed convex pointed cone, it follows that $G(\widehat{x}) \cap (-\operatorname{int} D) \neq \emptyset$, which gives $\widehat{x} \in A$. This along with (7) leads to

$$\Big(\lambda\big(F(A) + C - y_0\big)\Big) \cap (U - B) \neq \emptyset,$$

which contradicts (5). Hence (6) holds.

Since $U - B$ and $-\operatorname{int} D$ are open, it follows from (6) that

$$\operatorname{cl}\Big(\operatorname{cone}\big(\varphi(X) + C \times D\big)\Big) \cap \big((U - B) \times (-\operatorname{int} D)\big) = \emptyset.$$

By hypothesis that $\varphi(x)$ is nearly $C \times D$-subconvexlike on X, we conclude that $\operatorname{cl}\Big(\operatorname{cone}\big(\varphi(X) + C \times D\big)\Big)$ is convex. From the separation theorem for convex sets, there exists $(s^*, k^*) \in Y^* \times Z^* \setminus \big\{(0_{Y^*}, 0_{Z^*})\big\}$ such that

$$\text{(9)} \qquad (s^*, k^*)\Big(\operatorname{cl}\Big(\operatorname{cone}\big(\varphi(X) + C \times D\big)\Big)\Big) \geqslant s^*(U - B) + k^*(-\operatorname{int} D).$$

Since $\operatorname{cl}\Big(\operatorname{cone}\big(\varphi(X) + C \times D\big)\Big)$ is a cone on which (s^*, k^*) is bounded below, we derive

$$\text{(10)} \qquad (s^*, k^*)\Big(\operatorname{cl}\Big(\operatorname{cone}\big(\varphi(X) + C \times D\big)\Big)\Big) \geqslant 0.$$

It follows from $(0_Y, 0_Z) \in \operatorname{cl}\Big(\operatorname{cone}\big(\varphi(X) + C \times D\big)\Big)$ and (9) that

$$0 \geqslant s^*(U - B) + k^*(-\operatorname{int} D). \tag{11}$$

From $(0_Y, 0_Z) \in C \times D$ and (10) we get $(s^*, k^*)\big(\varphi(X)\big) \geqslant 0$. Thus $(s^*, k^*)\big(\varphi(x)\big) \geqslant 0$, $\forall x \in X$. In other words, $s^*(F(x) - y_0) + k^*\big(G(x)\big) \geqslant 0$, hence

$$s^*\big(F(x)\big) + k^*\big(G(x)\big) \geqslant s^*(y_0). \tag{12}$$

Firstly, we prove $k^* \in D^*$. In fact, according to (11),

$$k^*(\operatorname{int} D) \geqslant s^*(U - B). \tag{13}$$

Since $\operatorname{int} D$ is a cone and on which k^* is bounded below, one derives

$$k^*(\operatorname{int} D) \geqslant 0. \tag{14}$$

Since D is a closed convex cone, $D = \operatorname{cl}(\operatorname{int} D)$. For any $d \in D$, there exists a net $\{d_\sigma\} \subset \operatorname{int} D$ such that $d = \lim d_\sigma$. Then $k^*(d) = \lim k^*(d_\sigma) \geqslant 0$. Therefore $k^*(D) \geqslant 0$, $k^* \in D^*$.

Secondly, we verify $s^* \neq 0_{Y^*}$. Otherwise, $s^* = 0_{Y^*}$ and $k^* \neq 0_{Z^*}$. Hence $k^* \in D^* \setminus \{0_{Z^*}\}$. From (12) it follows that

$$k^*\big(G(x)\big) \geqslant 0, \forall x \in X. \tag{15}$$

On the other hand, by hypothesis, there is a $p \in G(\overline{x}) \cap (-\operatorname{int} D)$, from Lemma 3.1 we have $k^*(p) < 0$, which contradicts (15).

Thirdly, we check $s^* \in \operatorname{int} C^*$. Let $k_0 \in \operatorname{int} D$, $\lambda > 0$. From (13), $k^*(\lambda k_0) \geqslant s^*(U - B)$. Letting $\lambda \to 0^+$, one has $0 \geqslant s^*(U - B)$. Hence

$$s^*(B) \geqslant s^*(U). \tag{16}$$

Since $s^* \neq 0_{Y^*}$, there exists $u_0 \in U$ such that $s^*(u_0) = t > 0$. [Otherwise, for any $u \in U$, $s^*(u) \leqslant 0$. Then for any $y \in Y$, by the absorption of U, there is a $\lambda_0 > 0$ such that $\lambda_0 y \in U$, hence, $s^*(\lambda_0 y) \leqslant 0$, $\lambda_0 s^*(y) \leqslant 0$, $s^*(y) \leqslant 0$, which together with $s^* \in Y^*$ gives $s^* = 0_{Y^*}$, a contradiction.] Choosing $u_0 \in U$ in (16), we get $s^*(B) \geqslant t$, that is, $s^* \in B^{st}$. Invoking Lemma 3.2, we get $s^* \in \operatorname{int} C^*$.

Fourthly, we show the validity of (2). Substituting $x = x_0$ into (12), from $y_0 \in F(x_0)$, we see that

$$k^*\big(G(x_0)\big) \geqslant 0. \tag{17}$$

On the other hand, since $x_0 \in A$, we know there exists $p \in G(x_0) \cap (-D)$, hence $k^*(p) \leqslant 0$. This together with (17) leads to $k^*(p) = 0$. Therefore,

$$0 \in k^*\big(G(x_0)\big). \tag{18}$$

Combining (17) and (18), we obtain (2).

Finally, to end the proof, we show (1) is true. From (12), we get

$$\inf_{x \in X} \Big(s^*\big(F(x)\big) + k^*\big(G(x)\big) \Big) \geqslant s^*(y_0). \tag{19}$$

On the other hand, from $y_0 \in F(x_0)$ and (18), one derives $s^*(y_0) \in s^*\big(F(x_0)\big) + k^*\big(G(x_0)\big)$, which together with (19) leads to that (1) holds.

The proof is completed. ∎

Theorem 3.2. *Suppose C has a bounded base and the following conditions hold for the set-valued optimisation problem:*

(i) $x_0 \in A$;

(ii) *there exist $y_0 \in F(x_0)$, $s^* \in \operatorname{int} C^*$, $k^* \in D^*$ such that*

$$\inf_{x \in X} \Big(s^*\big(F(x)\big) + k^*\big(G(x)\big) \Big) = s^*(y_0).$$

Then (x_0, y_0) is a super efficient element of set-valued optimisation problem.

Proof:

$$\begin{aligned} s^*(y_0) &= \inf_{x \in X} \Big(s^*\big(F(x)\big) + k^*\big(G(x)\big) \Big) \leqslant \inf_{x \in A} \Big(s^*\big(F(x)\big) + k^*\big(G(x)\big) \Big) \\ &\leqslant \inf_{x \in A} \Big(s^*\big(F(x)\big) + k^*\big(G(x) \cap (-D)\big) \Big). \end{aligned}$$

From $k^*\big(G(x) \cap (-D)\big) \leqslant 0$, we get $s^*(y_0) \leqslant \inf_{x \in A} s^*\big(F(x)\big)$, hence $s^*(y_0) = \min\big\{ s^*(y) : y \in F(A) \big\}$, by Lemma 2.2 , one has $y_0 \in SE(F(A), C)$, which together with $y_0 \in F(x_0)$ gives that (x_0, y_0) is a super efficient element of set-valued optimisation problem. ∎

From Theorem 3.1 and Theorem 3.2 it is easy to obtain the following Corollary.

Corollary 3.1. *Suppose C has a bounded base, $x_0 \in A$, $y_0 \in F(x_0)$, there is $\overline{x} \in X$ such that $G(\overline{x}) \cap (-\operatorname{int} D) \neq \emptyset$, and $\big(F(x) - y_0, G(x)\big)$ is nearly $C \times D$-subconvexlike on X. Then (x_0, y_0) is a super efficient element of set-valued optimisation problem if and only if there exist $s^* \in \operatorname{int} C^*$, $k^* \in D^*$ such that*

$$\inf_{x \in X} \Big(s^*\big(F(x)\big) + k^*\big(G(x)\big) \Big) = s^*(y_0).$$

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4. Lagrange optimality conditions

Theorem 4.1. *Suppose C has a bounded base, (x_0, y_0) is a super efficient element of set-valued optimisation problem, $(F(x)-y_0, G(x))$ is nearly $C\times D$-subconvexlike on X and there exists an $\overline{x}\in X$ such that $G(\overline{x})\cap(-\operatorname{int} D)\neq\emptyset$. Then there exists $\overline{T}\in L^+(Z,Y)$ such that $\overline{T}\big(G(x_0)\cap(-D)\big)=\{0_Y\}$ and (x_0,y_0) is a super efficient element of the following unconstrained optimisation problem:*

$$\min_{x\in X}\psi(x)=F(x)+\overline{T}\big(G(x)\big).$$

Proof: From the proof of Theorem 3.1, we see that there exist $s^*\in\operatorname{int}C^*$, $k^*\in D^*$ such that (12) holds, namely,

$$s^*(y)+k^*(z)\geqslant s^*(y_0),\forall y\in F(x),z\in G(x),x\in X. \tag{20}$$

For any $\overline{z}\in G(x_0)\cap(-D)$, substituting $x=x_0, y=y_0, z=\overline{z}$ into (20) we get

$$k^*(\overline{z})\geqslant 0.$$

On the other hand, from $\overline{z}\in -D$ and $k^*\in D^*$ it follows that $k^*(\overline{z})\leqslant 0$. Thus

$$k^*(\overline{z})=0,\forall\overline{z}\in G(x_0)\cap(-D). \tag{21}$$

Since $s^*\in\operatorname{int}C^*$, we can choose $c\in C$ such that $s^*(c)=1$. Define a linear operator $\overline{T}:Z\to Y$ by

$$\overline{T}(z)=k^*(z)c,\forall z\in Z. \tag{22}$$

Thus $\overline{T}(D)=k^*(D)c\subset C$, this implies

$$\overline{T}\in L^+(Z,Y).$$

In view of (21) and (22), we deduce that $\overline{T}(\overline{z})=0_Y$, $\forall\overline{z}\in G(x_0)\cap(-D)$, that is,

$$\overline{T}\big(G(x_0)\cap(-D)\big)=\{0_Y\}. \tag{23}$$

On the other hand, $\forall x\in X$,

$$\begin{aligned} s^*\big(\psi(x)\big)&=s^*\Big(F(x)+\overline{T}\big(G(x)\big)\Big)\\ &=s^*\Big(F(x)+k^*\big(G(x)\big)c\Big)\\ &=s^*\big(F(x)\big)+k^*\big(G(x)\big)s^*(c)\\ &=s^*\big(F(x)\big)+k^*\big(G(x)\big). \end{aligned}$$

From (12) it follows that

$$s^*\big(\psi(X)\big)\geqslant s^*(y_0). \tag{24}$$

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From $y_0 \in F(x_0)$ and (23), one derives $y_0 \in F(x_0) + \overline{T}\big(G(x_0)\big) = \psi(x_0)$, which together with (24) gives that $s^*(y_0) = \min\{s^*(w) : w \in \psi(X)\}$. Invoking Lemma 2.2, we get $y_0 \in SE(\psi(X), C)$, Hence $y_0 \in \psi(x_0) \cap SE(\psi(X), C)$, therefore (x_0, y_0) is a super efficient element of the following unconstrained optimisation problem:

$$\min_{x \in X} \psi(x).$$

This finishes the proof. ∎

THEOREM 4.2. *Suppose C has a bounded base, $x_0 \in A$, $y_0 \in F(x_0)$. If there exists $\overline{T} \in L^+(Z, Y)$ such that $0_Y \in \overline{T}\big(G(x_0)\big)$ and (x_0, y_0) is a super efficient element of the following unconstrained programming:*

$$\min_{x \in X} \psi(x) = F(x) + \overline{T}\big(G(x)\big),$$

then (x_0, y_0) is a super efficient element of set-valued optimisation problem.

PROOF: $y_0 \in F(x_0) \subset F(x_0) + \overline{T}\big(G(x_0)\big) = \psi(x_0) \subset \psi(X)$, where $\psi(X) = \bigcup_{x \in X} \psi(x) = \bigcup_{x \in X} \Big(F(x) + \overline{T}\big(G(x)\big)\Big)$. Since (x_0, y_0) is a super efficient element of unconstrained set-valued optimisation problem, we have $y_0 \in SE\big(\psi(X), C\big)$, which together with Lemma 2.1 gives $y_0 \in SE\big(\psi(X) + C, C\big)$. By definition, for any neighbourhood V of 0 in Y, there exists a neighbourhood $U$of 0 such that

$$\operatorname{cone}(\psi(X) + C - y_0) \cap (U - C) \subset V. \tag{25}$$

On the other hand, we have the following relations:

$$\begin{aligned} x \in A \Rightarrow\ & G(x) \cap (-D) \neq \emptyset \\ \Rightarrow\ & \exists z_x \in G(x) \ \text{ such that } \ z_x \in -D \\ \Rightarrow\ & \overline{T}(z_x) \in -C \\ \Rightarrow\ & C - \overline{T}(z_x) \subset C + C \\ \Rightarrow\ & C \subset \overline{T}(z_x) + C \\ \Rightarrow\ & C \subset \overline{T}\big(G(x)\big) + C, \end{aligned}$$

where $\Rightarrow$ means implies. Thus,

$$\begin{aligned} F(A) + C - y_0 &= \bigcup_{x \in A} (F(x) + C - y_0) \\ &\subset \bigcup_{x \in A} \Big(F(x) + \overline{T}\big(G(x)\big) + C - y_0\Big) \\ &\subset \bigcup_{x \in X} \Big(F(x) + \overline{T}\big(G(x)\big) + C - y_0\Big) \\ &= \psi(X) + C - y_0. \end{aligned}$$

Therefore,

$$\operatorname{cone}(F(A) + C - y_0) \subset \operatorname{cone}(\psi(X) + C - y_0),$$

which together with (25) gives

$$\operatorname{cone}(F(A) + C - y_0) \cap (U - C) \subset V.$$

This implies $y_0 \in SE\big(F(A)+C, C\big)$. Invoking Lemma 2.1, we deduce $y_0 \in SE\big(F(A), C\big)$, hence $y_0 \in F(x_0) \cap SE\big(F(A), C\big)$. Thus (x_0, y_0) is a super efficient element of set-valued optimisation problem. ∎

The following Corollary is a direct consequence of Theorem 4.1 and Theorem 4.2.

Corollary 4.1. *Suppose C has a bounded base, $x_0 \in A$, $y_0 \in F(x_0)$, $\big(F(x) - y_0, G(x)\big)$ is nearly $C \times D$-subconvexlike on X and there exists an $\overline{x} \in X$ such that $G(\overline{x}) \cap (-\operatorname{int} D) \neq \emptyset$. Then (x_0, y_0) is a super efficient element of set-valued optimisation problem if and only if there exists $\overline{T} \in L^+(Z, Y)$ such that $0_Y \in \overline{T}\big(G(x_0)\big)$ and (x_0, y_0) is a super efficient element of the following unconstrained programming:*

$$\min_{x \in X} \psi(x) = F(x) + \overline{T}\big(G(x)\big).$$

Remark 4.1. Corollary 3.1 and Corollary 4.1 convert constrained optimisation set-valued optimisation problem into unconstrained programming.

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