1

The Geometry of Special Relativity

1.1 Introduction

1.1.1 Classical Physical Systems

A classical[1] physical system consists of three parts:

1. **Four-dimensional spacetime:** the *arena* of classical physics. We label a point in spacetime (an "event") by its coordinates:

$$x^\mu = (x^0, x^i) = (ct, \mathbf{x}), \tag{1.1}$$

where x^0 represents the time (we'll use units such that $c = 1$)[2] and $\mathbf{x}$ the position. Greek indices near the middle of the alphabet $(\lambda, \mu, \nu, \ldots)$ run from 0 to 3; Roman indices near the middle $(i, j, k, \ldots)$ run from 1 to 3.

2. **Particles and fields:** the *entities* of classical physics.

   (a) **Particles:** A particle is a structureless point object. Its *location*, $\mathbf{x}(t)$, as a function of time, tells you everything there is to say about it (beyond fixed properties such as mass and charge).[3] In 4-vector notation we represent the particle's trajectory (its **world line**) by $x^\mu(s)$, where s is any parameter used to denote points along the curve ($f(s)$ would do just as well, for any monotonic function f):

[1] In this book "classical" means "pre-quantum"; it *includes* special relativity.
[2] It's easy to reinsert the c's, when necessary, by dimensional analysis.
[3] We could treat point objects with spin, but let's keep things simple; in this course "particle" means spin 0.

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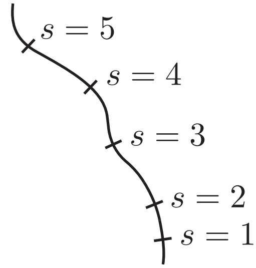

(b) **Fields:** A field is a function of position and time:

$$\varphi^{\alpha}(x). \tag{1.2}$$

Here α labels the components: one of them, if the field is temperature; six of them, in the case of electromagnetism. (In expressions like this x stands for the four components of x^{μ}.)

3. **Dynamics:** the *laws of motion*.

1.1.2 Symmetries

A **symmetry** is an operation that leaves an object or a system unchanged (**invariant**). A square, for example, is invariant under rotations (about a perpendicular axis through its center) by 90°, or 180°, or 270°, or reflections (in either diagonal, or a bisector of two opposite sides). Of particular importance to us are invariances of the laws of motion,[4] transformations that carry one possible motion into another. We stipulate that an invariance must have a well-defined inverse.[5]

Mathematically, the invariances of a system form a **group**.

Definition: A group, G, is a set of elements $(a, b, c, \ldots)$ and a law of "multiplication," with the following properties:

1. It is **closed**: if a and b are in G, so is their product, ab.
2. It is **associative**: $a(bc) = (ab)c$.
3. It contains a (unique) unit element, 1, such that $1a = a1 = a$ for every element a.
4. Each element a has a (unique) inverse, a^{-1}, such that $a^{-1}a = aa^{-1} = 1$.

[4] The ancient Greeks thought symmetries pertain to the actual motion: celestial objects ought to move on circles, because a circle is the most perfect (symmetrical) shape. But since the time of Newton we have understood that the more significant invariances apply to the *equations of motion*, and hence to the collection of all *possible* motions—the set of *solutions* to the equations of motion. The sun's gravitational field is spherically symmetric, but planetary orbits don't directly exhibit that symmetry—they're *elliptical*.

[5] This restriction eliminates trivial possibilities such as mapping all points on a particle trajectory onto some fixed point (sitting still at one point being—usually—a solution to the equations of motion). It is necessary in order to ensure that the invariances form a group.

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For example, the real numbers (except 0), with multiplication defined in the usual way, constitute an **Abelian** (commutative: $ab = ba$) group. Another group is the set of permutations of three objects (this group is *not* Abelian). We are interested here in the group of invariances of classical physics; "multiplication" in this context means application of two transformations in succession.

Example 1.1

Imagine a quantum mechanical system with nondegenerate energy levels. The state of the system at time $t = 0$ can be expanded in terms of the energy eigenstates:

$$|\psi(0)\rangle = \sum a_n |n\rangle, \tag{1.3}$$

and at any later time

$$|\psi(t)\rangle = \sum a_n e^{-iE_n t/\hbar} |n\rangle. \tag{1.4}$$

But the *phase* of $|n\rangle$ is arbitrary; physical predictions are unaffected by the transformation

$$|n\rangle \to e^{i\theta_n}|n\rangle, \tag{1.5}$$

for any collection of real numbers θ_n (independent of position and time). This is a huge invariance group, with an infinite number of parameters (if there are infinitely many eigenstates). But for the most part it is a *useless* invariance, which does not help us to solve the equations of motion.

So there exist trivial, accidental, or otherwise inconsequential invariances. One particularly *useful* class consists of the *geometrical* invariances of space and time: translations, rotations, dilations[6] (stretching), and so on. *Question:* What is the group of geometrical invariances of classical physics—the geometrical transformations that leave the laws of classical physics unchanged? A geometrical transformation is a change of coordinates:

$$x^\mu \to x'^\mu = y^\mu(x). \tag{1.6}$$

In the case of a particle trajectory,

$$x^\mu(s) \to y^\mu(x(s)). \tag{1.7}$$

Fields are more complicated, because not only do the *components* mix (if it's a vector field, and we're performing a rotation, the $\hat{\mathbf{x}}$ component will pick up $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ terms), but the *argument* (x) must be expressed in terms of the new coordinates (y): schematically,

[6] Eds. Coleman calls them "dilatations." Presumably permute:dilate::permutation:dilatation. But most modern authors use "dilation," and "dilatation" seems unnecessarily awkward.

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$$\varphi^\alpha(x) \to [\varphi^\alpha(x)]' = F[\varphi^\beta(y^{-1}(x))], \tag{1.8}$$

where F is some function denoting the transformation (mixing) of the components (φ^β), and y^{-1} is the inverse of Eq. 1.6. In words, the new fields at point y are some functions of the old fields at the point x that got mapped into y.

1.2 Poincaré Invariance

1.2.1 Geometrical Symmetries of Classical Physics

We'll focus for the moment on the case of particles. If there were *no* laws of motion (i.e. if *every* particle motion were possible), then *any* geometrical transformation would be an invariance. We'll whittle down this (huge) group by invoking some *actual* laws of motion:

1. **Newton's first law.** The allowed motions for a *free* particle are straight lines in spacetime, so the invariance group must (at a minimum) *take straight lines into straight lines*. One way to characterize a straight line is

$$x^\mu(s) = v^\mu s + b^\mu, \text{ where } v^\mu = \frac{dx^\mu}{ds} \text{ and } b^\mu \text{ are constants,} \tag{1.9}$$

which is the general solution to the differential equation

$$\frac{d^2x^\mu}{ds^2} = 0. \tag{1.10}$$

But wait: we *could* have used a different parameterization (say, s^3 instead of s); then

$$x^\mu(s) = v^\mu s^3 + b^\mu. \tag{1.11}$$

So Eq. 1.10 is not a reliable way to characterize straight lines—it's *sufficient*, but not *necessary*. Maybe a straight line satisfying 1.10 is transformed into a straight line that *doesn't* satisfy 1.10. In point of fact this worry is misguided: an invariance that carries straight lines into straight lines *automatically* takes linearly parameterized straight lines 1.9 into linearly parameterized straight lines.

Proof: For transformations that carry straight lines into straight lines:

(a) **Intersecting (or nonintersecting) straight lines go into intersecting (nonintersecting) straight lines.** If intersecting lines transformed into nonintersecting lines, the transformation for the point of intersection would be ill defined, since it would have to go to two different points—one on each line. And because we

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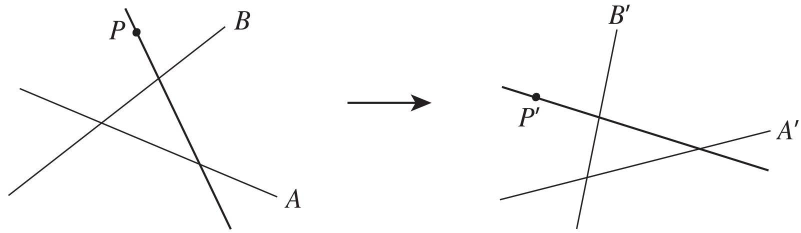
have stipulated that invariances have well-defined inverses, the same goes for nonintersecting to intersecting.

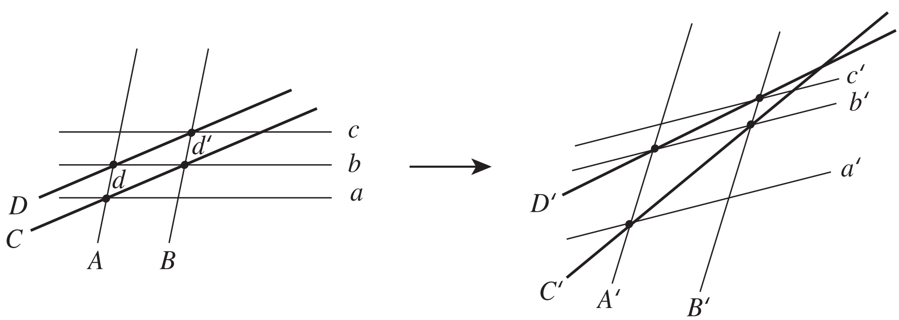
(b) **Planes go into planes.** Let P be a point in the plane defined by intersecting lines A and B (but not *on* either line), and draw a line from P intersecting A and B:



This line transforms into a line intersecting A' and B', so P' lies in the plane defined by A' and B'.

(c) **Parallel lines go into parallel lines.** This follows from (a) and (b).

(d) **Equidistant coplanar parallel lines go into equidistant coplanar parallel lines.** We know that coplanar parallel lines go into coplanar parallel lines, but could it be that equidistant ones (a, b, c) go into *non*equidistant ones (a', b', c')?



No: draw line A, and let the distance between its intersections with a and b be d. Now draw line B, parallel to A, and construct lines C and D, passing through the four intersections. By simple geometry, C and D are parallel (because a, b, and c are equidistant), and $d' = d$. However, unless a', b', and c' are *also* equidistant, C' and D' will *not* be parallel, violating (c).

So the transformation $x(s) \to y(x(s))$ takes equal intervals $(x(s_3) - x(s_2) = x(s_2) - x(s_1))$ into equal intervals $(y(s_3) - y(s_2) = y(s_2) - y(s_1))$, preserving the linear parameterization. QED

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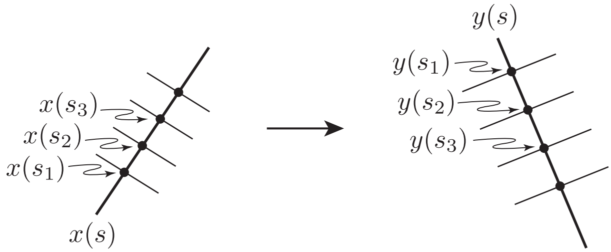

Under the transformation 1.6,

$$x^\mu \to y^\mu(x^\nu),$$

derivatives transform (by the chain rule)[7] as

$$\frac{dx^\mu}{ds} \to \frac{dy^\mu}{ds} = \frac{\partial y^\mu}{\partial x^\nu}\frac{dx^\nu}{ds}, \tag{1.12}$$

$$\frac{d^2x^\mu}{ds^2} \to \frac{d^2y^\mu}{ds^2} = \frac{\partial y^\mu}{\partial x^\nu}\frac{d^2x^\nu}{ds^2} + \frac{\partial^2 y^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}. \tag{1.13}$$

Because all straight lines ($d^2x^\mu/ds^2 = 0$) must transform into straight lines ($d^2y^\mu/ds^2 = 0$), it follows that invariances consistent with Newton's first law satisfy

$$\frac{\partial^2 y^\mu}{\partial x^\nu\,\partial x^\lambda} = 0 \tag{1.14}$$

(for all μ, ν, λ). The general solution is a *linear* function of x:

$$y^\mu = M^\mu_{\ \nu}\, x^\nu + b^\mu, \tag{1.15}$$

where the 16 elements of $M^\mu_{\ \nu}$ and the 4 components of b^μ are constants. (As a 4×4 matrix, $\det M \neq 0$, since $y(x)$ must have an inverse.) Newton's first law has reduced the geometrical invariances to a 20-parameter group, the **inhomogeneous affine group** (in four dimensions); with $b^\mu = 0$ it becomes the **homogeneous affine group**.

2. **Constancy of the velocity of light.** In empty space, light travels in straight lines, and according to special relativity the speed of light (in vacuum) is a universal constant, independent of the velocity of the source or the observer. If a light signal travels from point $\mathbf{x}$ to point $\mathbf{x}'$, departing at time t and arriving at time t', then

[7] We use the **Einstein summation convention**, whereby repeated indices are summed. Thus the third term in Eq. 1.12 carries an implicit summation sign, $\sum_{\nu=0}^{3}$.

$$c(t' - t) = |\mathbf{x}' - \mathbf{x}|, \tag{1.16}$$

or (setting $c = 1$)

$$(t' - t)^2 = (\mathbf{x}' - \mathbf{x})^2 = \sum_{i=1}^{3} [(x^i)' - x^i]^2. \tag{1.17}$$

Introducing the **metric tensor**[8]

$$g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.18}$$

we have

$$(x' - x)^\mu g_{\mu\nu} (x' - x)^\nu = 0. \tag{1.19}$$

The constancy of the speed of light means that if x and x' satisfy Eq. 1.19, then so too must the transformed coordinates y and y'. What does this tell us about M and b? Well,

$$x^\mu \to y^\mu = M^\mu_{\ \nu} x^\nu + a^\mu \ \Rightarrow \ (y' - y)^\mu = M^\mu_{\ \nu} (x' - x)^\nu, \tag{1.20}$$

so

$$M^\mu_{\ \kappa} (x' - x)^\kappa g_{\mu\nu} M^\nu_{\ \sigma} (x' - x)^\sigma = 0, \tag{1.21}$$

or

$$(x' - x)^\kappa \left[M^\mu_{\ \kappa} \, g_{\mu\nu} \, M^\nu_{\ \sigma} \right] (x' - x)^\sigma = 0. \tag{1.22}$$

This must hold for any x and x' satisfying Eq. 1.19. It follows that[9]

$$M^\mu_{\ \kappa} \, g_{\mu\nu} \, M^\nu_{\ \sigma} = \lambda g_{\kappa\sigma} \tag{1.23}$$

for some constant λ; or, in matrix notation,[10]

$$M^T g M = \lambda g. \tag{1.24}$$

8 Some authors use the other **signature** $(-, +, +, +)$; it doesn't matter, as long as you are consistent.

9 Although 1.23 obviously *guarantees* 1.22, it is not so clear that it is *required* by 1.22. But remember that this must hold for *any* x and x' satisfying 1.19, and from this it is not hard to show that 1.23 is in fact *necessary*.

10 Reading left to right, the first index (whether up or down) is the *row*, and the second (up or down) is the column. The significance of upness and downness will appear in due course. The superscript T denotes the transpose: $(M^T)^\mu_{\ \kappa} = M_\kappa^{\ \mu}$.

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What sorts of transformations remain, after invoking Newton's first law and the constancy of the speed of light? We can factor the matrix M as follows:

$$M = M_1 M_2, \text{ where } M_1 = |\det M|^{1/4} I \text{ and } M_2 = \frac{M}{|\det M|^{1/4}} \tag{1.25}$$

(I is the unit matrix). Thus any M is the product of a pure **dilation** M_1,

$$M_1 = \alpha I, \text{ so } M_1^T g M_1 = \alpha^2 g \text{ and hence } \lambda_1 = \alpha^2, \tag{1.26}$$

and a dilation-free term M_2 with determinant ± 1, for which

$$(\det M_2)(\det g)(\det M_2) = \lambda_2^4 (\det g) \;\Rightarrow\; \lambda_2^4 = 1 \;\Rightarrow \lambda_2 = \pm 1. \tag{1.27}$$

Actually, the negative sign is impossible,[11] so (in view of Eq. 1.26) $\lambda = \lambda_1 \lambda_2$ is in fact always positive.

3. **Eliminating dilations.** *Question:* Is our universe invariant under dilations? Imagine performing the Cavendish experiment to measure the gravitational force between two point masses:

$$F = G \frac{m_1 m_2}{r^2}, \tag{1.28}$$

giving an acceleration to m_1 in the amount

$$a_1 = G \frac{m_2}{r^2}. \tag{1.29}$$

Under a dilation (change of scale),

$$r \to \lambda r, \quad t \to \lambda t, \quad a \to \lambda^{-1} a. \tag{1.30}$$

So if dilation doesn't affect G or m_2, then a_1 goes like λ^{-1} but Gm_2/r^2 goes like λ^{-2}. Since G is a universal constant, it can't depend on λ, and since there is no mass continuum (no electron, for example, with slightly larger or smaller mass), mass cannot depend continuously on λ. *Conclusion:* Our universe is *not* invariant under dilations.[12]

[11] This follows from **Sylvester's law of inertia**; see, for instance, S. MacLane and G. Birkhoff, *A Survey of Modern Algebra*, 3rd ed., Macmillan (1965) p. 254. In essence, if $M^T g M = -g$ then $Q \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = -(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2$, so there is a 3-dimensional subspace ($x^0 = 0$) in which $Q < 0$, and another 3-dimensional subspace ($y^0 = 0$) in which $Q > 0$. But the entire space has only four dimensions, so this is impossible.

[12] This still leaves open the possibility of invariance under *combined* dilations and Lorentz transformations. We'll eliminate that option in Section 1.2.6.

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From these three overall constraints ((1) Newton's first law, (2) the constancy of the speed of light, and (3) the absence of scale invariance)[13] it follows that the maximum[14] possible geometrical invariance of classical physics is[15]

$$\boxed{x^\mu \to y^\mu = \Lambda^\mu{}_\nu x^\nu + b^\mu \quad \text{with} \quad \Lambda^T g \Lambda = g.} \tag{1.31}$$

The group of all such transformations (all possible Λ's and all b's) is called the **Poincaré group** (or the **inhomogeneous Lorentz group**). The subgroup $b^\mu = 0$ is the (homogeneous) **Lorentz group**.

1.2.2 Active and Passive Transformations

I have been thinking of the geometrical invariances as *active* transformations, in which the system is *physically moved* to a new location or orientation (or, in the case of dilations, shrunk or expanded). But one can achieve the same effect (formally) by a *passive* transformation, changing the *coordinates* in the reverse sense, while leaving the system itself fixed. (In criminal circles these are known as the "alibi" and "alias" strategies.) There is a kind of duality here, summarized in the following table:

	Active ($x \to \Lambda x + b$) "alibi"	**Passive** ($x \to \Lambda^{-1} x - b$) "alias"
Acts on	position/motion	coordinates
Invariance	solutions $\to$ solutions	form invariance of the equations of motion

[13] Caveats: (1) assumes that Newton's first law holds even for inaccessibly high velocities, (2) assumes that the speed of light is independent of the motion of the source even for unattainably high source speeds, and (3) assumes that the law of universal gravitation holds (at least approximately) for all speeds and separation distances. These assumptions are all, of course, open to potential experimental falsification.

[14] I have shown that the geometrical symmetry group is no *bigger* than the Poincaré group, but I have not proved that it couldn't be *smaller*; in principle, some new physical law might reduce it even further. Einstein postulated that there is in fact *no* further reduction; we call this assumption **Lorentz invariance**.

[15] Eds. We have used M for the *generic* linear transformation (1.15); Λ denotes a **Lorentz transformation**, satisfying $\Lambda^T g \Lambda = g$.

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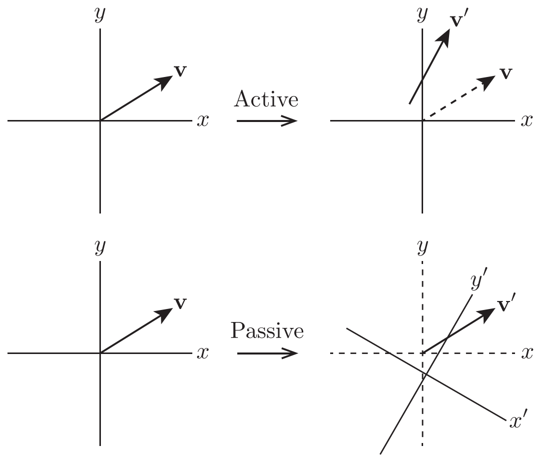


Einstein preferred the alias viewpoint, but the two perspectives are, for the most part, equivalent.

1.2.3 Minkowski Space

Notice that Lorentz transformations

$$\Lambda^T g \Lambda = g \tag{1.32}$$

are *not* (in general) **rotations** in four dimensions. Those would be generated by **orthogonal** matrices,

$$R^T R = I, \tag{1.33}$$

and preserve the (positive definite) quadratic form

$$x^T x = x^T I x = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{1.34}$$

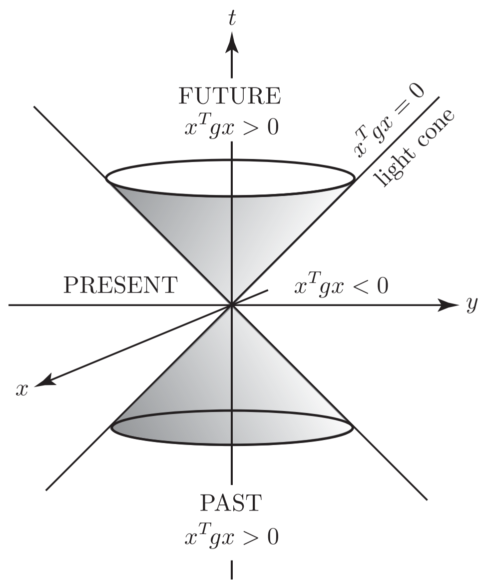
(that is, if $y = Rx$, then $y^T y = x^T x$). By contrast, Lorentz transformations preserve the indefinite quadratic form

$$x^T g x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{1.35}$$

(i.e. if $y = \Lambda x$ then $y^T g y = x^T g x$). If this quantity is positive, then x is said to be **time-like**; if it is negative, x is **space-like**; if it is zero, x is **light-like**. More generally, if $y = \Lambda x$ and $z = \Lambda w$ then $y^T g z = (\Lambda x)^T g (\Lambda w) = x^T \Lambda^T g \Lambda w = x^T g w$. This invariant quantity is the **scalar** (or **dot**) **product** of x and w; if $x^T g w = 0$, x and w are **orthogonal**.[16]

[16] Don't confuse orthogonal *vectors* with orthogonal *matrices* (1.33)—same word, different meanings.

Minkowski space (the 4-dimensional spacetime of special relativity) separates into distinct regions, as illustrated in the following figure (which you must imagine includes an undrawable z-coordinate, so the cones are really hyper-cones):



If you are sitting at the origin ($x = y = z = t = 0$), your **future** is the locus of all points in spacetime that you can influence; your **past** is the locus of all points that can have influenced you. As for the **present**, you cannot affect anything there, and it cannot affect you—to do so would require a signal propagating faster than the speed of light.

1.2.4 Topological Structure of the Lorentz Group

Orthogonal transformations in three dimensions fall into two disjoint "components":

(1) **Rotations:** determinant $+1$ (3 parameters; e.g. the Euler angles),
(2) **Reflections:** determinant -1 (3 parameters; e.g. the Euler angles).

A 3-dimensional rotation can be represented by a point within a sphere of radius π (with antipodal surface points identified): the axis of rotation defines the direction to the point, and the angle of rotation tells you its radial coordinate. Thus the north pole would specify a rotation by 180° about the north–south axis (which has the same effect as a rotation by 180° about the south–north axis). All rotations are

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continuously connected (to one another, and to the identity), but reflections are *not* continuously connected to rotations—they are represented by a separate sphere.

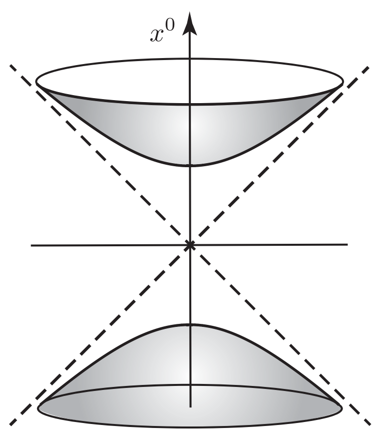
Question: How many parameters characterize a (homogeneous) Lorentz transformation, $\Lambda^T g \Lambda = g$ (at most 16, of course, since Λ is a 4×4 matrix, but presumably fewer than that)? And how many disjoint components does the Lorentz group possess? If we know how four basis vectors (for instance, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$) transform under Λ, then we know how *any* vector transforms. Let's start with $(1, 0, 0, 0)$; it gets transformed (under Λ) to some vector (x^0, x^1, x^2, x^3), with

$$x^T g x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 1 - 0 - 0 - 0 = 1, \tag{1.36}$$

so

$$x^0 = \pm\sqrt{1 + (x^1)^2 + (x^2)^2 + (x^3)^2}. \tag{1.37}$$

This defines a hyperboloid of two sheets:



Every x^1, x^2, and x^3 is possible, but once they are specified, x^0 is determined, up to an overall sign, so there are three continuous parameters, and one discrete choice.

Similarly, $(0, 1, 0, 0)$ transforms to (y^0, y^1, y^2, y^3), with

$$y^T g y = (y^0)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2 = 0 - 1 - 0 - 0 = -1, \tag{1.38}$$

or

$$(y^0)^2 + 1 = (y^1)^2 + (y^2)^2 + (y^3)^2. \tag{1.39}$$

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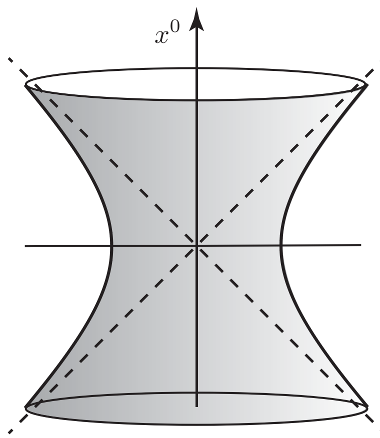
This time it's a hyperboloid of *one* sheet:



where y^0 determines the radius of the (y^1, y^2, y^3) sphere, leaving two angular variables; again, three parameters. However,

$$y^T g x = y^0 x^0 - y^1 x^1 - y^2 x^2 - y^3 x^3 = 0 \cdot 1 - 1 \cdot 0 - 0 \cdot 0 - 0 \cdot 0 = 0 \quad (1.40)$$

eliminates one variable (we could solve, for instance, for y^3), so there remain two (continuous) parameters, and no new discrete choices.

The same goes for $(0, 0, 1, 0) \to (z^0, z^1, z^2, z^3)$, except that there are now *two* orthogonality constraints ($z^T g x = z^T g y = 0$), leaving just one additional free parameter. Finally, for $(0, 0, 0, 1) \to (w^0, w^1, w^2, w^3)$ we have

$$(w^0)^2 - (w^1)^2 - (w^2)^2 - (w^3)^2 = -1, \quad (1.41)$$

$$w^0 x^0 - w^1 x^1 - w^2 x^2 - w^3 x^3 = 0, \quad (1.42)$$

$$w^0 y^0 - w^1 y^1 - w^2 y^2 - w^3 y^3 = 0, \quad (1.43)$$

$$w^0 z^0 - w^1 z^1 - w^2 z^2 - w^3 z^3 = 0. \quad (1.44)$$

From the last three we can solve for w^1, w^2, and w^3 (in terms of w^0 and the other parameters); then 1.41 determines w^0 up to a sign. So there are no new (continuous) parameters, but one discrete sign choice.

Conclusion:

The Lorentz group has four disjoint components, and six continuous parameters.

We classify the four disjoint components of the Lorentz group according to whether the sign of the time stays the same (**orthochronous**) or changes (**nonorthochronous**), and whether the spatial part turns a right-handed coordinate

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system into another right-handed system or (by including a reflection) into a left-handed system. An example of a right-handed orthochronous transformation is the identity:

$$\Lambda = I \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.45}$$

an example of a left-handed orthochronous transformation is **parity**:

$$\Lambda = P \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.46}$$

an example of a right-handed nonorthochronous transformation is **time reversal**:

$$\Lambda = T \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.47}$$

and an example of a left-handed nonorthochronous transformation is the product PT:

$$\Lambda = PT = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.48}$$

Notice (from Eq. 1.32) that

$$\det(\Lambda) = +1\ (\textbf{proper})\text{ or } -1\ (\textbf{improper}). \tag{1.49}$$

I'll use the symbol $\mathcal{L}_+^\uparrow$ to denote the proper (subscript $+$) orthochronous (superscript $\uparrow$) sector; $-$ for improper and $\downarrow$ for nonorthochronous:

Symbol	$\mathcal{L}_+^\uparrow$	$\mathcal{L}_-^\uparrow$	$\mathcal{L}_-^\downarrow$	$\mathcal{L}_+^\downarrow$
Name	right-handed orthochronous	left-handed orthochronous	right-handed nonorthochronous	left-handed nonorthochronous
Example	I	P	T	PT
$\det(\Lambda)$	1 (proper)	-1 (improper)	-1 (improper)	1 (proper)

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Notice that $\mathcal{L}_+^\uparrow$ contains the identity (and all transformations continuously connected to the identity); I'll call it $\mathcal{L}_0$. Suppose Λ is in $\mathcal{L}_-^\uparrow$. Since $P^2 = 1$, I can write $\Lambda = P(P\Lambda)$, and $P\Lambda$ is in $\mathcal{L}_0$. So any element of $\mathcal{L}_-^\uparrow$ can be expressed as the product of P with an element of $\mathcal{L}_0$. The same goes for the other two sectors:

$$\mathcal{L}_-^\uparrow = P\mathcal{L}_0, \quad \mathcal{L}_-^\downarrow = T\mathcal{L}_0, \quad \mathcal{L}_+^\downarrow = PT\mathcal{L}_0. \tag{1.50}$$

Thus the discrete invariances P and T, together with $\mathcal{L}_0$, generate the entire group. For the most part we will confine our attention to $\mathcal{L}_0$, the "connected part" of the (homogeneous) Lorentz group.

1.2.5 Rotations and Boosts

Within $\mathcal{L}_0$, two kinds of transformations are of special interest: spatial **rotations**, and **boosts**. The former act only on the spatial coordinates:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_3 & \\ 0 & & & \end{pmatrix}, \tag{1.51}$$

where R_3 is a 3×3 orthogonal matrix. The rotations constitute a 3-parameter subgroup (the ordinary 3-dimensional rotation group, SO(3)). Boosts comprise another 3-parameter set. A boost (or "pure" Lorentz transformation)[17] acts in a plane that includes the time axis; a boost in the xt plane transforms a unit vector in the t direction as follows:

$$B : \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}, \tag{1.52}$$

with $a^2 - b^2 = 1$ and $a > 0$ (orthochronous), so we can write

$$a = \cosh\phi, \quad b = \sinh\phi \tag{1.53}$$

for some (real) number ϕ. Meanwhile, it takes a unit vector in the x direction,

$$B : \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} c \\ d \\ 0 \\ 0 \end{pmatrix}. \tag{1.54}$$

[17] Eds. Coleman calls them "accelerations," but "boost" is the standard term.

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The transformed vectors must be orthogonal, because the original vectors were:

$$c\ \cosh\phi - d\ \sinh\phi = 0 \ \Rightarrow\ \frac{c}{\sinh\phi} = \frac{d}{\cosh\phi} \equiv \alpha, \tag{1.55}$$

and their “lengths” are preserved ($c^2 - d^2 = -1$), so $\alpha = \pm 1$. Which sign do we want? Evidently

$$B = \begin{pmatrix} \cosh\phi & \pm\sinh\phi & 0 & 0 \\ \sinh\phi & \pm\cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.56}$$

but $\det(B) = +1$, so we need the plus sign.

Under the boost B, a particle at rest, whose trajectory in spacetime (using t as the parameter) is $x = 0$, $t = s$, is transformed to $x' = \sinh\phi\, s$, $t' = \cosh\phi\, s$, and its velocity is

$$v = \frac{x'}{t'} = \tanh\phi. \tag{1.57}$$

Notice that whereas ϕ (the **rapidity**) can be any (real) number, $\tanh\phi$ is always between -1 and $+1$: the limiting speed achievable by a boost is c (which is 1 in our units). Note also that a boost looks very much like a rotation, except that (and this is crucial) the circular functions (sine and cosine) are replaced by hyperbolics (sinh and cosh). Equation 1.56 can be cast in a more familiar form by solving 1.57 for $\cosh\phi$ and $\sinh\phi$ in terms of v:

$$B = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2}}. \tag{1.58}$$

Every transformation in $\mathcal{L}_0$ can be expressed as the product (in either order) of a rotation and a boost:

$$\Lambda = BR = R'B'. \tag{1.59}$$

Proof: Since Λ carries the vector (1,0,0,0) into a vector of “length” 1,

$$\Lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh\phi \\ \hat{e}_x\ \sinh\phi \\ \hat{e}_y\ \sinh\phi \\ \hat{e}_z\ \sinh\phi \end{pmatrix} \tag{1.60}$$

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(for some real ϕ), where $\hat{\mathbf{e}}$ is a unit 3-vector. We might as well choose our axes so $\hat{\mathbf{e}}$ is in the x direction. Then

$$\Lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh\phi \\ \sinh\phi \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \Lambda = \begin{pmatrix} \cosh\phi & ? & ? & ? \\ \sinh\phi & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{pmatrix}. \tag{1.61}$$

Let B be the *boost* (1.56) that has the same effect on (1,0,0,0):

$$B = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.62}$$

Define

$$L \equiv B^{-1}\Lambda = \begin{pmatrix} 1 & a & b & c \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{pmatrix}. \tag{1.63}$$

Actually, the top row has to be (1,0,0,0), because L is itself a Lorentz transformation, and hence satisfies $L^T g L = g$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & ? & ? & ? \\ b & ? & ? & ? \\ c & ? & ? & ? \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & a & b & c \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{pmatrix} \tag{1.64}$$

$$= \begin{pmatrix} 1 & a & b & c \\ a & ? & ? & ? \\ b & ? & ? & ? \\ c & ? & ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow a = b = c = 0.$$

So L is in fact a rotation:

$$R = B^{-1}\Lambda \quad \Rightarrow \quad \Lambda = BR. \quad \text{QED} \tag{1.65}$$

1.2.6 Simultaneous Dilations and Lorentz Transformations

In Section 1.2.1 we eliminated pure dilations (λI) as elements of the geometrical invariance group of classical physics, but what about dilations combined with

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Lorentz transformations: $\lambda\Lambda$? Suppose the group contained both $\lambda_1\Lambda$ and $\lambda_2\Lambda$ for one and the same Λ; in that case it would also contain

$$(\lambda_1\Lambda)(\lambda_2\Lambda)^{-1} = (\lambda_1\lambda_2^{-1})I, \tag{1.66}$$

which would be a pure dilation unless $\lambda_1 = \lambda_2$. So there cannot be two *different* λ's for a given Λ; λ must be uniquely determined by Λ—every Lorentz transformation Λ carries a particular dilation λ. Is that possible? Because every Lorentz transformation is the product of a rotation and a boost ($\Lambda = RB$), and rotations are certainly in the invariance group, so too is

$$R^{-1}(\lambda RB) = \lambda B, \tag{1.67}$$

so (by the same argument as before) λ depends only on the boost, not on the rotation: $\lambda(\mathbf{v})$. In fact, by rotational invariance, it can only depend on the *magnitude* of $\mathbf{v}$: $\lambda(v^2)$. Now $B(\mathbf{v})B(-\mathbf{v}) = I$, so the invariance group must also contain

$$I = (\lambda(\mathbf{v})B(\mathbf{v}))\,(\lambda(\mathbf{v})B(\mathbf{v}))^{-1} = \lambda(v^2)B(\mathbf{v})\lambda(v^2)B(-\mathbf{v}) = \left(\lambda(v^2)\right)^2, \tag{1.68}$$

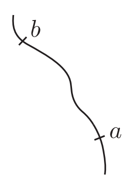
and hence $\lambda(v^2) = \pm 1$. But $\lambda(0) = 1$ (the identity), and we assume continuity, so $\lambda(v^2) = 1$: dilations are not allowed *even if* they are tied to Lorentz transformations.

1.3 Time Dilation and Lorentz Contraction

1.3.1 Arc Length and Proper Time

In Minkowski space the analog to arc length along a world line $x^\mu(s)$ is

$$\tau(b) - \tau(a) = \int_a^b ds \left(g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)^{1/2} = \int_a^b \sqrt{dt^2 - dx^2 - dy^2 - dz^2}. \tag{1.69}$$



It is independent of parameterization (unchanged if we use $s' = f(s)$ in place of s), and it is Lorentz invariant. (The square root is OK, since world lines are everywhere time-like.) If the particle is at rest, the integral is just the elapsed time. If the particle is moving at constant velocity, it can be brought to rest by a Lorentz transformation, and since arc length is invariant, it is still the time elapsed on the particle's own watch—its **proper time**, τ. But what about a particle that speeds up or slows

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down? The curved path in spacetime can be approximated by short line segments, each corresponding to a different **comoving** observer traveling at constant velocity. When it is necessary to change directions (to "pass the baton" to the next comoving observer), the two observers are at the same spacetime point, and can synchronize their clocks unambiguously. Thus the sum of the observers' time intervals (which is to say, the total arc length) still corresponds to the elapsed time on the particle's own clock. *Conclusion:* Arc length is proper time, *even when the particle accelerates*. Proper time is the "natural" parameterization for a world line; it is defined (up to an additive constant) by

$$\boxed{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1.} \tag{1.70}$$

Problem 1.1

Alice and Bob both travel from spacetime point a to spacetime point b. Alice goes by the straight line path (in Minkowski space); Bob wanders around—his world line is curved. *Question:* Which trip takes longer, according to each traveler's own watch?

1.3.2 Time Dilation

Suppose observer $\mathcal{O}$ is at rest at the origin. Observer $\mathcal{O}'$ starts out at the origin, at $t = t' = 0$, and moves away (say, in the x direction) at constant speed $v = dx/dt$. When $\mathcal{O}$'s clock reads time t, what time t' does $\mathcal{O}$ observe on the (moving) clock carried by $\mathcal{O}'$? That is, what is the elapsed (proper) time on the $\mathcal{O}'$ clock? From Eq. 1.69,

$$t' = \tau = \int\sqrt{1-\left(\frac{dx}{dt}\right)^2-\left(\frac{dy}{dt}\right)^2-\left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{1-v^2}\,t = \frac{1}{\gamma}t. \tag{1.71}$$

Notice that t' is *smaller* than t; the *moving clock runs slow*. This is known as **time dilation**.

Incidentally, if $\mathcal{O}$ observes $\mathcal{O}'$ going at velocity $\mathbf{v}$, then $\mathcal{O}'$ observes $\mathcal{O}$ going at velocity $-\mathbf{v}$. That is to say,[18]

$$(B(\mathbf{v}))^{-1} = B(-\mathbf{v}). \tag{1.72}$$

[18] Eds. Actually, Coleman already used this in Eq. 1.68.

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Proof: Choose axes such that the motion is along the x direction. From Eq. 1.56,

$$B(v) = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.73}$$

where $v = \tanh\phi$, and hence $-v = -\tanh\phi = \tanh(-\phi)$. Then

$$B(-v) = B(-\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.74}$$

Therefore

$$\begin{aligned} B(-v)B(v) &= \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (\cosh^2\phi - \sinh^2\phi) & 0 & 0 & 0 \\ 0 & (\cosh^2\phi - \sinh^2\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I. \quad \text{QED} \end{aligned}$$

1.3.3 Lorentz Contraction

Two stripes (A and B) are painted across a road, a distance d apart. A car, going at constant speed v, takes time t to go from A to B:

$$d = vt. \tag{1.75}$$

Now examine the same process from the perspective of an observer in the car (using clocks and meter sticks moving with the car):

$$d' = v't'. \tag{1.76}$$

The speeds are the same (as we saw in Section 1.3.2), but the time interval on the moving clock is reduced (it's running slow, Eq. 1.71), so

$$d' = \sqrt{1 - v^2}\, d = \frac{1}{\gamma} d. \tag{1.77}$$

Lengths shrink, for a moving observer, by just the factor necessary to compensate for time dilation. This is called Lorentz–Fitzgerald contraction, or **Lorentz contraction**, for short.

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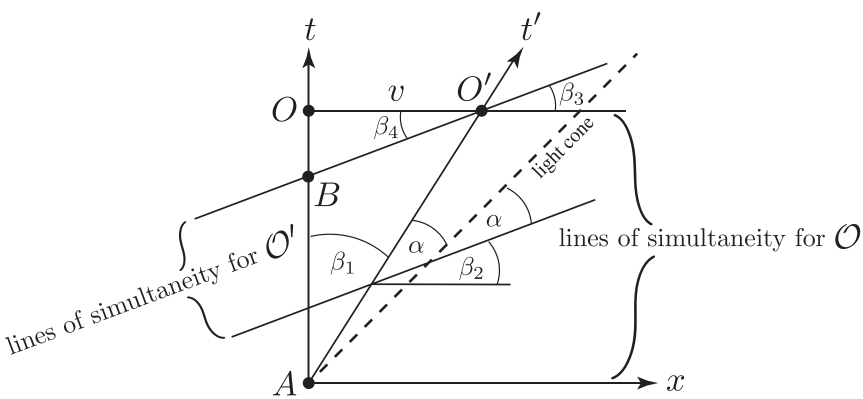
Lorentz contraction only affects dimensions *parallel to the direction of motion*; lengths *perpendicular* to the motion are not contracted.[19] *Beware:* An **observation** is what you get *after* correcting for the time the signal took to get to you; what you **observe**, therefore, is not at all the same as what you *see* (or hear). Time dilation and Lorentz contraction pertain to what you *observe*, and relativity is almost always talking about *observations*.

1.4 Examples and Paradoxes

1.4.1 The Time Dilation Paradox

Time dilation raises an apparent paradox: If $\mathcal{O}$ says the $\mathcal{O}'$ clock is running slow, $\mathcal{O}'$ can say with equal justice that the $\mathcal{O}$ clock is running slow (and by the same factor). Who's right? They *both* are! It's a matter of **simultaneity**, which is different for the two observers. On a Minkowski diagram, *lines of simultaneity make the same angle with the light cone as does the time axis*. Thus, in the following figure, lines of simultaneity for $\mathcal{O}$ (which are, of course, horizontal) all make the same angle with the light cone (to wit, 45°) as the t-axis does, while lines of simultaneity for $\mathcal{O}'$ all make the same angle with the light cone as the t'-axis does (to wit, α).



First I'll resolve the paradox, then I'll confirm the rule. Suppose $AO = 1$, the time shown on the $\mathcal{O}$ clock. Then $\sqrt{1-v^2}$ is the time shown simultaneously (according to $\mathcal{O}$) on the $\mathcal{O}'$ clock; $\mathcal{O}$ reports that the $\mathcal{O}'$ clock is running slow. By simple geometry, $\beta_1 = \beta_2 = \beta_3 = \beta_4 \equiv \beta$ (note that $\alpha + \beta = 45°$), and so

$$\frac{OB}{v} = \frac{OO'}{AO} = \frac{v}{1} \;\Rightarrow\; OB = v^2 \;\Rightarrow\; AB = 1 - v^2. \tag{1.78}$$

[19] Eds. For a nice proof, see E. F. Taylor and J. A. Wheeler, *Spacetime Physics*, Freeman, San Francisco (1966), page 21.

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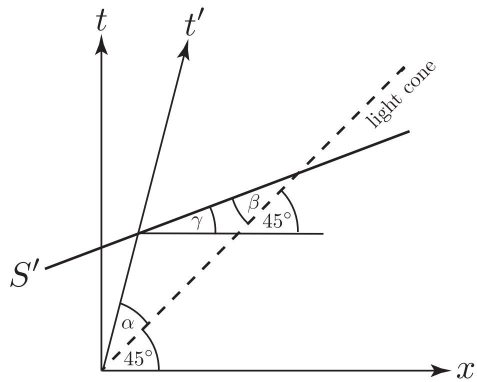
Therefore $\mathcal{O}'$ says that when the $\mathcal{O}'$ clock reads $\sqrt{1-v^2}$, the $\mathcal{O}$ clock reads $(1-v^2)$, and hence that the $\mathcal{O}$ clock is running slow (by the same factor, $\sqrt{1-v^2}$). Paradox resolved.

Now let's justify the rule for constructing lines of simultaneity. We want to show that S' is a line of simultaneity for $\mathcal{O}'$ if $\alpha = \beta$:



Proof: The equation for the t'-axis is $x' = 0$, or (using 1.73)

$$x' = \Lambda^1{}_0\, t + \Lambda^1{}_1\, x = (\sinh\phi)t + (\cosh\phi)x = 0, \quad t = -(\coth\phi)\, x.$$

The equation for S' is $t' = \text{constant}$ (if it is to be a line of simultaneity in $\mathcal{O}'$):

$$t' = \Lambda^0{}_0\, t + \Lambda^0{}_1\, x = (\cosh\phi)t + (\sinh\phi)x = \text{const}, \quad t = -(\tanh\phi)\, x + \text{const}.$$

So the slope of the t'-axis is

$$\tan(\alpha + 45^\circ) = -\coth\phi,$$

and the slope of S' is

$$\tan\gamma = -\tanh\phi.$$

Now, $\gamma + \beta + (180^\circ - 45^\circ) = 180^\circ$, so $\gamma = 45^\circ - \beta$, and hence

$$\tan\gamma = \tan(45^\circ - \beta) = \frac{\tan 45^\circ - \tan\beta}{1 + \tan 45^\circ \tan\beta} = \frac{1 - \tan\beta}{1 + \tan\beta} = -\tanh\phi$$

and

$$\tan(\alpha + 45^\circ) = \frac{\tan\alpha + 1}{1 - \tan\alpha} = -\coth\phi \;\Rightarrow\; \frac{1 - \tan\alpha}{1 + \tan\alpha} = -\tanh\phi.$$

So $\alpha = \beta$. QED

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1.4.2 The Twin Paradox

Alice boards a rocket ship, which accelerates uniformly in the x direction. After a certain time, it decelerates (at the same rate) back to $v = 0$. Then it reverses direction, and returns to earth in the same way. Because moving clocks run slow, not as much time will have elapsed on the rocket clock as on a stationary earth clock, so at their reunion Alice will have aged less than her twin brother Bob, who stays at home. Suppose the trip takes 40 years by her watch, and the acceleration is g. How many years has Bob aged in the process?

We need to determine the rocket's position (x) as a function of time (t). By "uniform acceleration" we mean that a passenger *on the rocket* experiences an unchanging acceleration, g. From *her* perspective she is at a fixed position (seat 23B, or (x_0, y_0, z_0)), and her watch reads proper time, τ. Thus her coordinates are $y^\mu = (\tau, x_0, y_0, z_0)$, and her velocity is

$$\dot{y}^\mu = (1, 0, 0, 0), \tag{1.79}$$

where the dot denotes differentiation with respect to proper time. Of course, she is not in an inertial reference frame; these coordinates refer to her **instantaneously comoving inertial frame**. You cannot get her acceleration by differentiating again, because it involves transfer to a new comoving frame. We'll get it instead by an indirect route.

It follows from 1.70 that

$$g_{\mu\nu}\ddot{x}^\mu \dot{x}^\nu + g_{\mu\nu}\dot{x}^\mu \ddot{x}^\nu = 0, \quad \text{so} \quad g_{\mu\nu}\dot{x}^\mu \ddot{x}^\nu = 0. \tag{1.80}$$

Thus **proper acceleration** ($\ddot{x}$) is always orthogonal to **proper velocity** ($\dot{x}$). So her acceleration must have the form

$$\ddot{y}^\mu = (0, g, 0, 0). \tag{1.81}$$

Thus

$$g_{\mu\nu}\ddot{x}^\mu \ddot{x}^\nu = g_{\mu\nu}\ddot{y}^\mu \ddot{y}^\nu = -g^2. \tag{1.82}$$

We'll take this as the (Lorentz-invariant) characterization of "uniform acceleration." Now let's examine her motion from the earth's perspective. Proper velocity always has "length" 1 (Eq. 1.70),

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 1; \tag{1.83}$$

for motion in the x direction we have

$$\dot{x}^\mu = (\cosh\varphi,\ \sinh\varphi, 0, 0) \quad \text{so} \quad \ddot{x}^\mu = \dot{\varphi}(\sinh\varphi,\ \cosh\varphi, 0, 0). \tag{1.84}$$

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Hence

$$g_{\mu\nu}\ddot{x}^\mu \ddot{x}^\nu = (\dot{\varphi})^2(\sinh^2\varphi - \cosh^2\varphi) = -(\dot{\varphi})^2. \tag{1.85}$$

For uniform acceleration, therefore, Eq. 1.82 says $\dot{\varphi} = g$, or $\varphi = g\tau$. Setting $x = t = 0$ at $\tau = 0$,

$$\dot{x}^0 = \frac{dt}{d\tau} = \cosh\varphi = \cosh(g\tau) \quad \Rightarrow \quad t = \frac{1}{g}\sinh(g\tau), \tag{1.86}$$

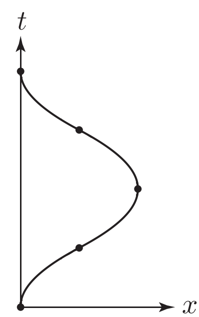
$$\dot{x}^1 = \frac{dx}{d\tau} = \sinh\varphi = \sinh(g\tau) \quad \Rightarrow \quad x = \frac{1}{g}\left(\cosh(g\tau) - 1\right). \tag{1.87}$$

These are parametric equations for a hyperbola; $gt = \sinh(g\tau)$ and $(gx + 1) = \cosh(g\tau)$, so

$$(gx+1)^2 - (gt)^2 = 1. \tag{1.88}$$

For this reason, uniform acceleration is known as **hyperbolic motion**.

In the figure below, each of the four segments is a hyperbolic arc (the scale on the x-axis is not the same as on the t-axis; with equal scales the slope would never be less than 1).



If we measure time in years and distance in light-years, the unit of acceleration would be

$$1\,\frac{\text{light-yr}}{\text{yr}^2} = \frac{c}{\text{yr}} \approx \frac{3\times 10^8\ \text{m/s}}{\pi \times 10^7\ \text{s}} \approx 9.5\ \text{m/s}^2, \tag{1.89}$$

which is pretty close to the acceleration of gravity on earth. That means we can take $g = 1$ in our units, for a reasonably comfortable ride. The acceleration segment lasts 10 years, by Alice's watch (the whole trip takes her 40 years). But her brother has aged

$$t = 4\sinh(10) \approx 2\,e^{10} = 44{,}000\ \text{yr} \tag{1.90}$$

(note that earth time goes up exponentially with rocket time).

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1.4.3 Doppler Shift

Consider a spaceship approaching you (at rest on the earth) with constant speed v. The spaceship sends out a light pulse every second (by the *spaceship* clock). At time $t = t' = 0$, when the rocket is a distance d away, the first signal is sent, so it gets to you at $t = d$. The second signal is sent when the *spaceship* clock reads $t' = 1 = t\sqrt{1 - v^2}$, so yours reads $t = 1/\sqrt{1 - v^2}$. At that time (in your system) the spaceship is at

$$d - \frac{v}{\sqrt{1 - v^2}} \tag{1.91}$$

(which is also the time it takes for the pulse to reach you). Thus the second signal leaves at $1/\sqrt{1 - v^2}$ (your time), and arrives at

$$t = \frac{1}{\sqrt{1 - v^2}} + d - \frac{v}{\sqrt{1 - v^2}}. \tag{1.92}$$

The interval between signals received is

$$(1) \quad \Delta t = \frac{1 - v}{\sqrt{1 - v^2}} = \sqrt{\frac{1 - v}{1 + v}}. \tag{1.93}$$

If the rocket is moving *away* from you, v has the opposite sign, and

$$(2) \quad \Delta t = \sqrt{\frac{1 + v}{1 - v}}. \tag{1.94}$$

In case (1) you *see* the spaceship clock running *faster* than yours (the Doppler effect swamps time dilation); in case (2) you *see* greater time dilation than expected (Doppler and dilation conspire). But always you *observe* time dilation: $t = t'/\sqrt{1 - v^2}$.

1.4.4 The Bandits and the Train

A row of bandits regularly fires on the daily train as it passes by at a speed close to c. One day a bandit calls in sick, so there is a gap in the line. The bandit chief points out that the train will be Lorentz contracted, and there will come an interval (when the foreshortened train is directly opposite the missing bandit) when no bullets hit the train. The engineer notes that from the perspective of a person on the train, it is the row of bandits that will be contracted, and the fact that one is missing will hardly be noticeable; the train will be fired upon without interruption.

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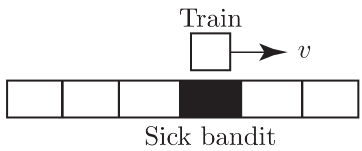
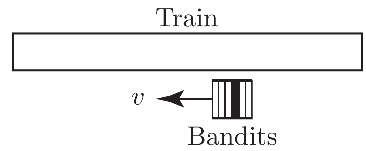
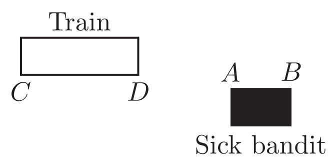


Chief's Perspective



Engineer's Perspective

Who is right? Does there occur a moment when no bullets hit the train? *Answer:* They *both* are—for their respective reference systems. The problem is conflicting notions of simultaneity:



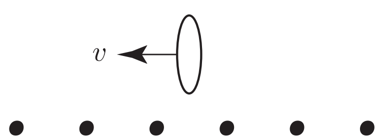
When D meets B,

- the chief says C is already past A (C meets A *before* D meets B),
- the engineer says C has not yet reached A (C meets A *after* D meets B).

There is no contradiction, just a different perspective concerning the sequence of two events.

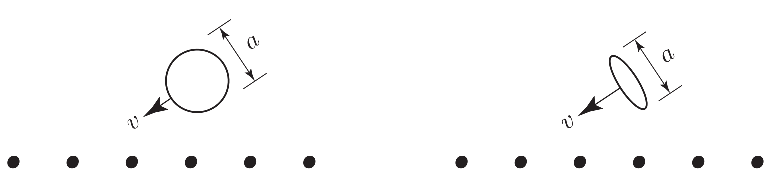
1.4.5 The Prisoner's Escape

A (spherical) prisoner proposes to escape by running so fast that Lorentz contraction will permit him to slip between the bars:



His cellmate retorts that in the escaping prisoner's reference frame it is the distance between the bars that will be Lorentz contracted, and it will be *even more* difficult for him to get out. Who is right? (This is a better problem than the train and bandit paradox, because they *cannot* both be correct: at the end of the day either he's a free man, or he's *not*.)

Answer: He does *not* escape. The trouble arises when he changes direction to slip through the bars:



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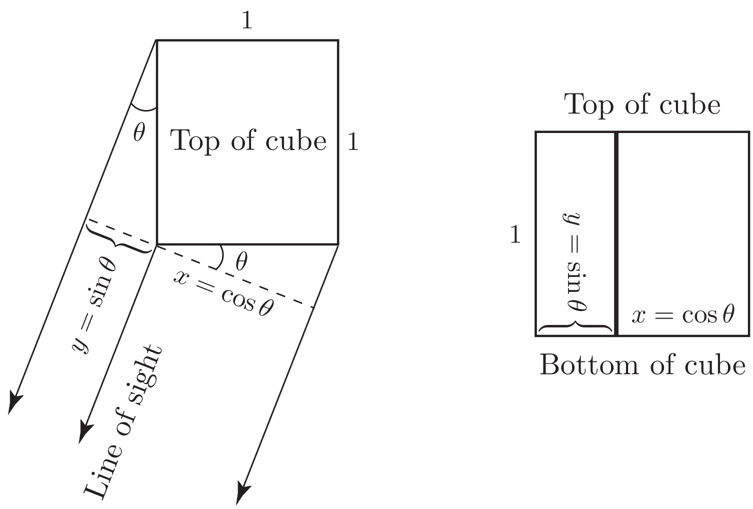
In the nonrelativistic case (left) there is no Lorentz contraction, the spacing between the bars is (presumably!) less than his diameter (a), and he cannot get through; in the relativistic case (right) he is indeed Lorentz contracted *along the direction of motion*—he becomes an ellipsoid (or a spheroid, or something), but the critical dimension a is unchanged, and it is no easier (nor more difficult) for him to slip through. The essential point is that *dimensions perpendicular to the direction of motion are not contracted.*

1.4.6 The Moving Cube

Imagine looking at a cube, of side 1, very far away (so there is no parallax—rays reaching your eye from all parts of the cube are parallel). The cube is at rest (so are you), and oriented so that you can see two faces. The view from above is shown in the first figure; what you see is shown in the second figure:



Question: What do you see if the cube is *moving*, to the left, at speed v? This is a rare case in which we deliberately ask what you *see*, not what you *observe*; what you *observe* is just a Lorentz-contracted version of the figure on the right, but what you *see* must take into account the fact that light from more distant parts of the cube takes longer to reach your eye. In the figure below I have drawn the top face of the cube, in your (stationary) reference frame, where it is Lorentz contracted along the direction of motion. At the same instant you receive the light from the left corner (O), you also get light from the right corner (A), but since the latter had to travel a greater distance (AB) it must have left somewhat earlier, when the cube was in the "old" position, as indicated.

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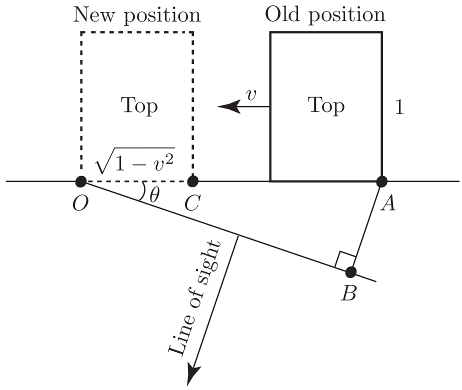


By simple trigonometry,

$$OA \equiv L, \quad AB = L\sin\theta, \quad AC = vt, \tag{1.95}$$

where $t = AB$ is the time it takes light to travel the "extra" distance. So

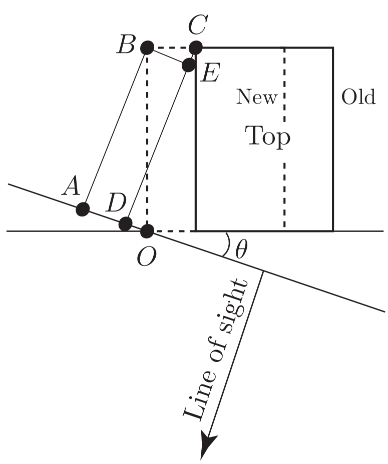
$$L = OC + AC = \sqrt{1-v^2} + v(L\sin\theta), \tag{1.96}$$

and hence

$$L = \frac{\sqrt{1-v^2}}{1 - v\sin\theta}, \quad OB = L\cos\theta = \frac{\sqrt{1-v^2}\,\cos\theta}{1 - v\sin\theta} = x. \tag{1.97}$$

This replaces x in the previous figure (notice that it reduces to $\cos\theta$ when $v \to 0$).

What about y? Again, the light from O arrives at the same time as the light from the back corner (C), which had to travel an extra distance CD, and therefore must have left earlier, when the cube was in the "old" position:



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We're looking for $y = OD$. All the acute angles are equal to θ, so

$$BC \equiv \ell, \quad CE = \ell \sin\theta, \quad BE = \ell \cos\theta, \quad AB = \cos\theta, \quad AO = \sin\theta. \tag{1.98}$$

The delay time is

$$CD = AB + CE = \ell \sin\theta + \cos\theta, \tag{1.99}$$

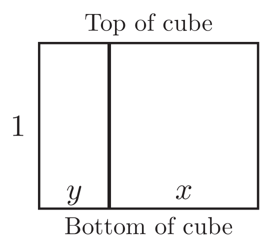
so ℓ (the distance the cube moves as light goes from C to D) is

$$\ell = v(\ell \sin\theta + \cos\theta) \quad \Rightarrow \quad \ell = \frac{v\cos\theta}{1 - v\sin\theta}. \tag{1.100}$$

Then (since $AD = BE$)

$$y = OD = AO - AD = \sin\theta - \frac{v\cos^2\theta}{1 - v\sin\theta} = \frac{\sin\theta - v}{1 - v\sin\theta}. \tag{1.101}$$

What you *see*, then, is



where x and y are given by Eqs. 1.97 and 1.101. Now comes a small miracle:

$$x^2 + y^2 = \frac{\sin^2\theta - 2v\sin\theta + v^2 + \cos^2\theta - v^2\cos^2\theta}{(1 - v\sin\theta)^2} = 1, \tag{1.102}$$

so we might as well define a new angle θ', such that $x = \cos\theta'$ and $y = \sin\theta'$, and what we *see* looks exactly like what we saw for a cube at rest, only rotated at a different angle. It doesn't look like a *contracted* cube at all, but rather like a *rotated* cube! We *observe* a contracted cube, but we *see* a rotated cube. What is more, we could construct any *other* object out of cubical "Lego," so the same conclusion holds quite generally.[20]

[20] Eds. In 1939 George Gamow published a famous book, *Mr. Tompkins in Wonderland*, in which he imagined a world where the speed of light was just a few mph, and he described seeing a Lorentz-contracted bicycle with a skinny rider, as it passed by. His blunder was corrected by many authors: he had failed to distinguish what you see from what you observe. R. Penrose, *Proc. Cambridge Phil. Soc.* **55**, 137 (1959); J. Terrell, *Phys. Rev.* **116**, 1041 (1959); V. F. Weisskopf, *Physics Today* **13**, 24 (September 1960); M. L. Boas, *Am. J. Phys.* **29**, 283 (1961).

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1.4.7 Tachyons

Could there exist particles that travel faster than light (**tachyons**)? They are perfectly consistent with Lorentz invariance. You could not convert an ordinary particle into a tachyon by Lorentz transformation, but one tachyon is connectable to another tachyon by Lorentz transformation (though it may be going backward in time). A tachyon's velocity would be space-like, and by Lorentz transformation the world line of a tachyon moving at constant velocity could be made to coincide with (say) the x-axis—it would take no time at all to get from point a to point b: the particle would be in two places at the same time (and everywhere in between). We would presumably like to exclude such outlandish behavior, but this is an *independent* assumption; it does not follow from Lorentz invariance alone.

Problem 1.2

Aberration. A spaceship moves with velocity $\mathbf{v}$ along its axis of symmetry. A star is at rest; the vector from the spaceship to the star makes an angle θ with this axis. What is the angle at which an observer on the spaceship sees the star?

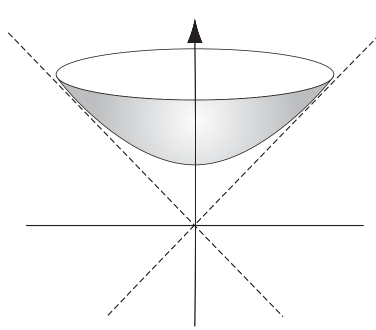
Problem 1.3

Milne's model of an expanding universe. Imagine a collection of noninteracting particles (galaxies, if you like). They all start out ($t = 0$) at the origin, but with "randomly" distributed velocities. If the distribution of velocities is Lorentz invariant, then we can take any particle to be the one "at rest." At a later time t, the faster-moving particles will be farther away. This is a primitive model for an expanding universe.

To construct a Lorentz-invariant velocity distribution, note that

$$g_{\mu\nu} v^\mu v^\nu = v^2 = 1 \tag{1.103}$$

(Eq. 1.70), where $v^\mu \equiv dx^\mu / d\tau$ is the proper velocity, τ is the proper time (for the particle in question), and v^0 is positive. Thus the velocities all lie on the forward hyperboloid:



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Let's express the number of particles in the volume d^4v (of velocity space) as[21]

$$N\,\delta(v^2-1)\,d^4v, \tag{1.104}$$

for some constant N. The delta function is manifestly Lorentz invariant, as is d^4v, so this defines a Lorentz-invariant distribution of velocities on the hyperboloid. What is the resulting distribution in the 3-velocity[22] $\mathbf{v} = d\mathbf{x}/d\tau$? Using the standard formula for the delta function of a function,

$$\delta(f(z)) = \sum_i \frac{1}{|f'(z_i)|}\delta(z-z_i), \tag{1.105}$$

where the sum is over the zeros of f $(f(z_i)=0)$,[23]

$$\begin{aligned}\delta((v^0)^2-\mathbf{v}^2-1) &= \frac{1}{2|\sqrt{\mathbf{v}^2+1}|}\delta(v^0-\sqrt{\mathbf{v}^2+1}) \\ &\quad + \frac{1}{2|-\sqrt{\mathbf{v}^2+1}|}\delta(v^0+\sqrt{\mathbf{v}^2+1}) \\ &= \frac{1}{2v^0}\,\delta(v^0-\sqrt{\mathbf{v}^2+1})\end{aligned} \tag{1.106}$$

(the other term vanishes because $v^0 > 0$). Thus the number of particles in the volume d^4v is

$$N\,\delta(v^2-1)\,d^4v = \frac{N}{2v^0}\delta\left(v^0-\sqrt{\mathbf{v}^2+1}\right)\,dv^0d^3\mathbf{v}. \tag{1.107}$$

Integrating over v^0 we get the number of particles with velocities in the range $d^3\mathbf{v}$:

$$\frac{N}{2\sqrt{\mathbf{v}^2+1}}\,d^3\mathbf{v}. \tag{1.108}$$

(a) Find the density of galaxies (the number per unit volume) as a function of position and time: $\rho(\mathbf{x},t)$.

(b) What is "Hubble's law" (speed $|\mathbf{v}|$ as a function of distance $|\mathbf{x}|$) in this universe?

(c) What would someone (at rest at the origin) actually *see* (as opposed to *observe*) at time t—what density of galaxies, as a function of distance, and what speed law?

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[21] Note that the Lorentz-invariant density is unique (up to the overall constant N). Given the density at one point, A, I can, by Lorentz transformation, carry that point to any other point B on the hyperboloid, and thus determine the density at B. There cannot exist two *different* Lorentz-invariant distributions.

[22] Note that this is *not* the ordinary 3-velocity $d\mathbf{x}/dt$, but the *proper* 3-velocity $\mathbf{v} = d\mathbf{x}/d\tau$.

[23] Ordinarily I would write the square of a 3-vector $(\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2)$ as v^2, but this notation has been preempted for the 4-vector $(v^2 = (v^0)^2 - \mathbf{v}\cdot\mathbf{v})$, so I'll write the square of the 3-vector using boldface: $\mathbf{v}^2$ (even though it's a scalar).