# THE $H$-FUNCTION TRANSFORM

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Here we introduce a new integral transform whose kernel is the $H$-function. Since most of the important functions occurring in Applied Mathematics and Physics are special cases of the $H$-function, various integral transforms involving these functions as kernels follow as special cases of our transform. We mention some of them here and observe that a study of this transform gives general and useful results which serve as key formulae for several important integral transforms viz. Laplace transform, Hankel transform, Stieltjes transform and the various generalizations of these transforms. In the end we establish an inversion formula for the new transform and point out its special cases which are generalizations of results found recently.

## 1. Definition

We define the $H$-function transform by the equation

$$\phi(s) = s\int_0^\infty H_{p,q}^{m,n}\left[st \,\middle|\, \begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix}\right] f(t)\, dt \tag{1.1}$$

provided that the integral on the right hand side of (1.1) is absolutely convergent.

The $H$-function appearing in (1.1) was first introduced by Fox [2, p. 408] and is defined and represented as follows:

$$\begin{aligned} &H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix}\right] \\ &\qquad = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j \xi) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \xi)} x^\xi \, d\xi \end{aligned} \tag{1.2}$$

where $x$ may be real or complex but is not equal to zero and an empty product is interpreted as unity: $p, q, m, n$ are integers satisfying $1 \leqq m \leqq q$; $0 \leqq n \leqq p$; $\alpha_j (j = 1, \cdots, p)$, $\beta_j (j = 1, \cdots, q)$, are positive



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numbers and $a_j$ $(j = 1, \cdots, p)$, $b_j$ $(j = 1, \cdots, q)$, are complex numbers such that no pole of $\Gamma(b_h - \beta_h \xi)$ $(h = 1, \cdots, m)$, coincides with any pole of $\Gamma(1 - a_i + \alpha_i \xi)$ $(i = 1, \cdots, n)$, i.e.

$$\alpha_i(b_h + \nu) \neq \beta_h(a_i - \eta - 1) \tag{1.3}$$

$(\nu, \eta = 0, 1, \cdots; h = 1, \cdots, m; i = 1, \cdots, n)$.

Further the contour $L$ runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points

$$\xi = (b_h + \nu)/\beta_h \qquad h = 1, \cdots, m; \nu = 0, 1, \cdots \tag{1.4}$$

which are poles of $\Gamma(b_h - \beta_h \xi)$ $(h = 1, \cdots, m)$, lie to the right and the points

$$\xi = (a_i - \eta - 1)/\alpha_i, \quad i = 1, \cdots, n; \eta = 0, 1, \cdots \tag{1.5}$$

which are poles of $\Gamma(1 - a_i + \alpha_i \xi)$ $(i = 1, \cdots, n)$, lie to the left of $L$.

Such a contour is possible on account of (1.3).

In what follows $\{(a_p, \alpha_p)\}$ stands for the quantities

$$(a_1, \alpha_1), \cdots, (a_p, \alpha_p) \quad \text{and} \quad H_{p,q}^{m,n}[x] \quad \text{for} \quad H_{p,q}^{m,n}\left[x \middle| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_p, \beta_q)\} \end{matrix}\right]$$

## Asymptotic expansions for the $H$-function

Braaksma [1, p. 278] has shown that the $H$-function makes sense and defines an analytic function of $x$ in the following two cases:

$$\text{(i)} \qquad \delta > 0 \qquad (x \neq 0) \tag{1.6}$$

where $\delta$ stands for $\sum_{j=1}^{q} (\beta_j) - \sum_{j=1}^{p} (\alpha_j)$ throughout this section.

$$\text{(ii)} \qquad \delta = 0 \quad \text{and} \quad 0 < |x| < D^{-1} \tag{1.7}$$

where $D = \prod_{j=1}^{p} (\alpha_j)^{\alpha_j} \prod_{j=1}^{q} (\beta_j)^{-\beta_j}$.

From the equation (6.5) of the same paper we have

$$H_{p,q}^{m,n}[x] = 0(|x|^{\alpha}) \quad \text{for small } x, \tag{1.8}$$

where $\delta \geqq 0$ and $\alpha = \min \mathscr{R}(b_h/\beta_h)$ $(h = 1, \cdots, m)$.

In the same paper Braaksma has considered the behaviour of the $H$-functions for large $x$ at full length. There he has considered different sets of conditions of the convergence of the integral given by (1.2). We shall, however, for lack of space, restrict ourselves throughout this paper to the case when the parameters of the $H$-function satisfy the following conditions of validity.

$$\text{(i)} \qquad A = \sum_{1}^{n} (\alpha_i) - \sum_{n+1}^{p} (\alpha_j) + \sum_{1}^{m} (\beta_j) - \sum_{m+1}^{q} (\beta_j) > 0. \tag{1.9}$$

$$\text{(ii)} \qquad |\arg x| < \tfrac{1}{2} A \pi. \tag{1.10}$$

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(The inequalities (1.9) and (1.10) given above have been denoted in Braaksma's paper by (2.13) and (2.15) respectively).

Also from the equation (2.16) of the same paper we get

$$(1.11)\qquad H^{m,n}_{p,q}[x] = 0(|x|^{\beta}) \qquad \text{for large } x,$$

provided that $\delta > 0$, the conditions given by (1.9) and (1.10) are satisfied and $\beta$ stands for

$$\max \mathscr{R}\left(\frac{a_i-1}{\alpha_i}\right) \qquad (i = 1, \cdots, n).$$

Finally from the equations (2.36), (2.43) and (4.12) of the same paper we infer that if $n = 0$, the $H$-function vanishes exponentially for large $x$ in certain cases, we have

$$(1.12)\qquad H^{m,0}_{p,q}[x] \sim 0(\exp\{-\delta x^{1/\delta} D^{1/\delta}\} x^{1/\delta(\mu+\frac{1}{2})})$$

where $\mu$ and $D$ stand for the quantities

$$\sum_{j=1}^{q}(b_j) - \sum_{j=1}^{p}(a_j) + \tfrac{1}{2}p - \tfrac{1}{2}q \quad \text{and} \quad \prod_{j=1}^{p}(\alpha_j)^{\alpha_j}\prod_{j=1}^{q}(\beta_j)^{-\beta_j}$$

respectively and the following conditions are satisfied:

(i) $\sum_{1}^{m}(\beta_j) - \sum_{m+1}^{q}(\beta_j) - \sum_{1}^{p}(\alpha_j) > 0$

(ii) $|\arg x| < \tfrac{1}{2}\pi\{\sum_{1}^{m}(\beta_j) - \sum_{m+1}^{q}(\beta_j) - \sum_{1}^{p}(\alpha_j)\}$

(iii) $\delta = \sum_{1}^{q}(\beta_j) - \sum_{1}^{p}(\alpha_j) > 0.$

REMARK. The following formula for the $H$-function can be proved by an obvious change of the variable in the integral of (1.2)

(1.13)

$$H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right] = H^{n,m}_{q,p}\left[\frac{1}{x} \,\middle|\, \begin{matrix}(1-b_1,\beta_1),\cdots,(1-b_q,\beta_q)\\(1-a_1,\alpha_1),\cdots,(1-a_p,\alpha_p)\end{matrix}\right]$$

Now on account of (1.13), we can transform the $H$-function with $\delta < 0$, and arg $(x)$ to one with $\delta > 0$ and arg $(1/x)$. Hence the asymptotic expansions for the case $\delta < 0$ can also be obtained from the results mentioned above by interchanging the role of $x$ at $x = 0$ and at $x = \infty$.

## Certain formulae for the *H*-function

By an obvious change of variable in the integral of (1.2) we get [4]

$$(1.14)\qquad x^{\sigma} H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}\{(\alpha_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right] = H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}\{(a_p+\sigma\alpha_p,\alpha_p)\}\\\{(b_q+\sigma\beta_q,\beta_q)\}\end{matrix}\right]$$

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Also from the definition (1.2) of the $H$-function and Fourier Mellin inversion theorem [3, p. 46] we have

$$\text{(1.15)}\quad \int_0^\infty x^{s-1} H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \cdots, (a_p, \alpha_p)\\ (b_1, \beta_1), \cdots, (b_q, \beta_q)\end{matrix}\right] dx = \frac{\prod_{j=1}^{m} \Gamma(b_j+\beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j+\alpha_j s)}$$

provided that integral on the L.H.S. of (1.15) is absolutely convergent.

## 2. Special cases of the Integral transform defined by (1.1)

(i) *Generalized Hankel-Transform*

If we put $m = 1$, $n = p = 0$, $q = 2$, $b_1 = \nu$, $\beta_1 = 1$, $b_2 = -\lambda+\mu\nu$ and $\beta_2 = \mu$ in (1.1) it reduces to the Generalized Hankel-Transform defined by Ram Kumar [6, p. 191] viz

$$\text{(2.1)}\quad \phi(s) = s \int_0^\infty (sx)^\nu J_\lambda^\mu(sx) f(x)\, dx$$

on account of the formula [1, p. 279]

$$\text{(2.2)}\quad J_\lambda^\mu(x) = H_{0,2}^{1,0}[x|(0, 1), (-\lambda, \mu)],$$

where $J_\lambda^\mu$ is the Bessel-Maitland function defined by

$$\text{(2.3)}\quad J_\lambda^\mu(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!\,\Gamma(1+\lambda+\mu r)} \qquad (\mu > 0).$$

(2.1) in turn is an obvious generalization of the well known Hankel transform.

(ii) *Gauss's hypergeometric function transform*

If we put $m = 1$, $n = p = q = 2$, $a_1 = 1-\lambda$, $a_2 = 1-\mu$, $b_1 = 0$, $b_2 = 1-\nu$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ in (1.1) and replace $s$ by $(1/s)$ therein, we get by virtue of [4] the formula

$$\text{(2.4)}\quad \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\nu)}\, {}_2F_1(\lambda, \mu; \nu; -x) = H_{2,2}^{1,2}\left[x \,\middle|\, \begin{matrix}(1-\lambda, 1), (1-\mu, 1)\\ (0, 1), (1-\nu, 1)\end{matrix}\right],$$

the following transform studied by Rajendra Swaroop [5, p. 107]

$$\text{(2.5)}\quad \phi(s; \lambda, \mu, \nu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\nu)} \frac{1}{s} \int_0^\infty 2F_1\left((\lambda, \mu; \nu; -\frac{x}{s}\right) f(x)\, dx.$$

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(2.5) reduces to generalized Stieltjes transform given below; if we put $\mu = \nu$ in it, we have

$$\phi_1(s, \lambda) = \frac{s^{1-\lambda}}{\Gamma(\lambda)} [\phi(s); \lambda, \mu, \nu]_{\mu=\nu} = \int_0^\infty \frac{f(x)dx}{(s+x)^\lambda}. \tag{2.6}$$

(iii) *Meijer's G-function transform*

If we replace $m$ by $m+1$, $n$ by zero, $p$ by $m$ and $q$ by $m+1$ in (1.1), and further put $\alpha$'s and $\beta$'s equal to one therein and take $a_1 = \eta_1+\alpha_1, \cdots,$ $a_m = \eta_m+\alpha_m$; $b_1 = \eta_1, \cdots, b_m = \eta_m$, $b_{m+1} = e$, it reduces to the following transform introduced by Bhise [7, p. 57] by virtue of [1, p. 241]

$$\phi(s) = s\int_0^\infty G^{m+1,0}_{m,m+1}\left[sx\middle|\begin{matrix}\eta_1+\alpha_1, \cdots, \eta_m+\alpha_m\\ \eta_1, \cdots, \eta_m, e\end{matrix}\right] f(x)\,dx. \tag{2.7}$$

(2.7) itself is a general transform and reduces to well known Integral transforms viz. Laplace transform, Meijer transforms, Varma transform etc. as pointed out by Bhise [7].

## 3. Inversion formula

If $y^{k-1}f(y) \in L(0, \infty)$, $f(y)$ is of bounded variation in the neighbourhood of the point $y = t$, and

$$\phi(s) = s\int_0^\infty H^{m,n}_{p,q}\left[st\middle|\begin{matrix}(a_1, \alpha_1), \cdots, (a_p, \alpha_p)\\ (b_1, \beta_1), \cdots, (b_q, \beta_q)\end{matrix}\right] f(t)\,dt,$$

then

$$\begin{aligned}&\tfrac{1}{2}\{f(t+0)+f(t-0)\}\\ &= \frac{1}{2\pi i}\lim_{\gamma\to\infty}\int_{c-i\gamma}^{c+i\gamma} \frac{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j k-\beta_j)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j k+\alpha_j)}{\prod_{j=1}^{m}\Gamma(b_j-\beta_j k+\beta_j)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j k-\alpha_j} t^{-k}F(k)dk\end{aligned} \tag{3.1}$$

where

$$F(k) = \int_0^\infty s^{-k-1}\phi(s)\,ds \tag{3.2}$$

provided that

$$A = \sum_1^n (\alpha_j) - \sum_{n+1}^p (\alpha_j) + \sum_1^m (\beta_j) - \sum_{m+1}^q (\beta_j) > 0,$$

$$|\arg s| < \tfrac{1}{2}A\pi,\ R\left(\frac{b_j}{\beta_j} - k+1\right) > 0 \qquad (j = 1, \cdots, m),$$

$$R\left(\frac{\alpha_h - 1}{\alpha_h} - k+1\right) < 0 \qquad (h = 1, \cdots, n),$$

and the $H$-function transform of $|f(t)|$ exists.

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PROOF. From (1.1) we have

$$(3.3)\quad \int_0^\infty s^{-k-1}\phi(s)ds = \int_0^\infty s^{-k}\left\{\int_0^\infty H_{p,q}^{m,n}\left[st\,\middle|\,\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right]f(t)dt\right\}ds.$$

On inverting the order of integration in (3.3) we get

(3.4)

$$\int_0^\infty s^{-k-1}\phi(s)ds = \int_0^\infty t^k f(t)\left\{\int_0^\infty (st)^{-k}H_{p,q}^{m,n}\left[st\,\middle|\,\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right]ds\right\}dt.$$

Evaluating the $s$-integral in (3.4) with the help of (1.15) we get

$$(3.5)\quad \begin{aligned} F(k) &\equiv \int_0^\infty s^{-k-1}\phi(s)ds \\ &= \frac{\prod_{j=1}^m \Gamma(b_j-\beta_j k+\beta_j)\prod_{j=1}^n\Gamma(1-a_j+\alpha_j k-\alpha_j)}{\prod_{j=m+1}^q \Gamma(1-b_j+\beta_j k-\beta_j)\prod_{j=n+1}^p \Gamma(a_j-\alpha_j k+\alpha_j)}\int_0^\infty t^{k-1}f(t)\,dt,\end{aligned}$$

provided that

$$A = \sum_1^n(\alpha_j) - \sum_{n+1}^p(\alpha_j) + \sum_1^m(\beta_j) - \sum_{m+1}^q(\beta_j) > 0,$$

$$|\arg s| < \tfrac{1}{2}A\pi,\ R\left(\frac{b_j}{\beta_j}-k+1\right) > 0 \qquad (j = 1,\cdots,m),$$

$$R\left(\frac{a_h-1}{\alpha_h}-k+1\right) < 0 \qquad (h = 1,\cdots,n).$$

Applying the Mellin transform [3, p. 46], in (3.5) we get the required result.

All that remains now is to justify the inversion of the order of integration in (3.3). We see that the $t$-integral is absolutely convergent if the $H$-function transform of $|f(t)|$ exists, the $s$-integral is convergent if

$$A = \sum_1^n(\alpha_j) - \sum_{n+1}^p(\alpha_j) + \sum_1^m(\beta_j) - \sum_{m+1}^q(\beta_j) > 0,$$

$$|\arg s| < \tfrac{1}{2}A\pi,\ R\left(\frac{b_j}{\beta_j}-k+1\right) > 0 \qquad (j = 1,\cdots,m),$$

$$R\left(\frac{a_h-1}{\alpha_h}-k+1\right) < 0 \qquad (h = 1,\cdots,n),$$

and the resulting integral is absolutely convergent if $t^{k-1}f(t)\in L(0,\infty)$. Hence the interchange of the order of integration in (3.3) is justified by virtue of De La Vallée Poussin's well known theorem.

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SPECIAL CASES. (i) If we specialize the parameters in (1.1) so that the $H$-function transform is transformed to Gauss's hypergeometric function transform, the inversion formula given above yields a result of Swaroop [5, p. 108].

(ii) Again if in the above inversion formula we reduce the transform (1.1) to the $G$-function transform defined by (2.7), we obtain Bhise's result [7, p. 59].

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