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The Convolution Sum $\sum_{m<n/16} \sigma(m)\sigma(n-16m)$

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Abstract. The convolution sum $\sum_{m<n/16} \sigma(m)\sigma(n-16m)$ is evaluated for all $n \in \mathbb{N}$. This evaluation is used to determine the number of representations of n by the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2+4x_5^2+4x_6^2+4x_7^2+4x_8^2$.

1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ denote the set of all integers. For $n \in \mathbb{N}$ and $k \in \mathbb{N}$ we set

$$\sigma_k(n) = \sum_{d|n} d^k,$$

where d runs through the positive integers dividing n. If $n \notin \mathbb{N}$, we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively. In [10] the third author showed that

$$(1.1)\quad \sum_{m<n/8} \sigma(m)\sigma(n-8m) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\Big(\frac{n}{2}\Big) + \frac{1}{16}\sigma_3\Big(\frac{n}{4}\Big) + \frac{1}{3}\sigma_3\Big(\frac{n}{8}\Big) + \Big(\frac{1}{24}-\frac{n}{32}\Big)\sigma(n) + \Big(\frac{1}{24}-\frac{n}{4}\Big)\sigma\Big(\frac{n}{8}\Big) - \frac{1}{64}c_8(n),$$

where $c_8(n)$ ($n \in \mathbb{N}$) are integers defined by

$$(1.2)\quad q\prod_{n=1}^{\infty}\big(1-q^{2n}\big)^4\big(1-q^{4n}\big)^4 := \sum_{n=1}^{\infty} c_8(n)q^n, \quad q \in \mathbb{C}, |q|<1.$$

We note that $c_8(1) = 1$ and $c_8(2n) = 0$, $n \in \mathbb{N}$. In this paper we evaluate the convolution sum

$$\sum_{m<n/16} \sigma(m)\sigma(n-16m)$$

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for all $n \in \mathbb{N}$. In order to do this, it is convenient to define, for $k \in \{1,2,3,4,5,6,7\}$ and $q \in \mathbb{C}$, $|q| < 1$,

$$A_k(q) := \prod_{n=1}^{\infty} (1+q^n)^{24-4k}(1-q^n)^8(1-q^{4n-2})^{16-2k}. \tag{1.3}$$

It is easy to show that

$$A_k(-q) = A_{8-k}(q). \tag{1.4}$$

Next we define rational numbers $c_{16}(n)$ $(n \in \mathbb{N})$ by

$$\sum_{n=1}^{\infty} c_{16}(n)q^n := \frac{1}{32}A_1(q) + \frac{3}{112}A_2(q) + \frac{1}{224}A_3(q) - \frac{1}{32}A_5(q) - \frac{3}{112}A_6(q) - \frac{1}{224}A_7(q), \tag{1.5}$$

so that $c_{16}(1) = 1$, $c_{16}(2) = 12/7$, $c_{16}(3) = 4/7$, $c_{16}(4) = 0$, $c_{16}(5) = -2$, $c_{16}(6) = -48/7$, *etc.* In Section 2 we show that

$$7c_{16}(n) \in \mathbb{Z},\ n \in \mathbb{N}, \tag{1.6}$$

and prove the following result.

Theorem 1.1 *For all $n \in \mathbb{N}$ we have*

$$\sum_{m<n/16} \sigma(m)\sigma(n-16m) = \frac{1}{768}\sigma_3(n) + \frac{1}{256}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{64}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{8}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{16}\right) + \left(\frac{1}{24} - \frac{n}{64}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{16}\right) - \frac{7}{256}c_{16}(n).$$

As an application of Theorem 1.1, we prove the following result in Section 3.

Theorem 1.2 *For all $n \in \mathbb{N}$ the number $N(n)$ of $(x_1,\dots,x_8) \in \mathbb{Z}^8$ such that*

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 4x_5^2 + 4x_6^2 + 4x_7^2 + 4x_8^2$$

is given by

$$N(n) = \sigma_3(n) + 3\sigma_3\left(\frac{n}{2}\right) - 68\sigma_3\left(\frac{n}{4}\right) + 48\sigma_3\left(\frac{n}{8}\right) + 256\sigma_3\left(\frac{n}{16}\right) + 7c_{16}(n).$$

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2 Proof of Theorem 1.1

For $z \in \mathbb{C}$ with $|z| < 1$ we set

$$(2.1) \qquad w(z) = {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; z\Big) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \frac{z^n}{n!},$$

where ${}_2F_1$ is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol, see for example [4, p. 247], [8, p. 45]. Clearly $w(0) = 1$. The infinite series (2.1) diverges at $z = 1$ [4, p. 249], so that $w(1) = +\infty$. For $x \in \mathbb{R}$ with $0 \le x < 1$, we have

$$w(x) = 1 + \sum_{n=1}^{\infty} \frac{2n!^2}{n!^4 2^{4n}} x^n \ge 1,$$

so that

$$(2.2) \qquad w(x) \ne 0, \ 0 \le x < 1.$$

The derivative with respect to x of the function

$$y(x) := \pi \frac{w(1-x)}{w(x)}, \quad 0 < x < 1,$$

is [1, p. 87]

$$(2.3) \qquad y'(x) = \frac{-1}{x(1-x)w(x)^2}, \quad 0 < x < 1.$$

Thus, by (2.2) and (2.3), we have $y'(x) < 0, \ 0 < x < 1$. Hence, as x increases from 0 to 1, $y(x)$ strictly decreases from $y(0) = \pi \frac{w(1)}{w(0)} = +\infty$ to $y(1) = \pi \frac{w(0)}{w(1)} = 0$. Now restrict q so that $q \in \mathbb{R}$ and $0 < q < 1$. Thus $0 < -\log q < +\infty$. Hence there is a unique value of x between 0 and 1 such that $y(x) = -\log q$. The Eisenstein series $L(q)$ and $M(q)$ are defined by

$$(2.4) \qquad L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n,$$

$$(2.5) \qquad M(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n.$$

The discriminant function $\Delta(q)$ is defined by

$$\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

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It is proved in [2] that

$$L(q) = (1-5x)w^2 + 12x(1-x)w\frac{dw}{dx}, \tag{2.6}$$

$$M(q) = (1+14x+x^2)w^4, \tag{2.7}$$

$$\Delta(q) = \frac{x(1-x)^4 w^{12}}{2^4}. \tag{2.8}$$

Applying the duplication principle [2, p. 125],

$$q \longrightarrow q^2,\ x \longrightarrow \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2,\ w \longrightarrow \left(\frac{1+\sqrt{1-x}}{2}\right)w,$$

to (2.6), (2.7) and (2.8), we obtain

$$L(q^2) = (1-2x)w^2 + 6x(1-x)w\frac{dw}{dx}, \tag{2.9}$$

$$M(q^2) = (1-x+x^2)w^4, \tag{2.10}$$

$$\Delta(q^2) = \frac{x^2(1-x)^2 w^{12}}{2^8}. \tag{2.11}$$

Formulae (2.9) and (2.10) are given in [2, p. 122, p. 126]. Applying the duplication principle to (2.9), (2.10) and (2.11), we obtain

$$L(q^4) = \left(1-\frac{5}{4}x\right)w^2 + 3x(1-x)w\frac{dw}{dx}, \tag{2.12}$$

$$M(q^4) = \left(1-x+\frac{1}{16}x^2\right)w^4, \tag{2.13}$$

$$\Delta(q^4) = \frac{x^4(1-x)w^{12}}{2^{16}}. \tag{2.14}$$

Applying the duplication principle to (2.12) and (2.13), we have

$$L(q^8) = \left(\frac{5}{8} - \frac{11}{16}x + \frac{3}{8}\sqrt{1-x}\right)w^2 + \frac{3}{2}x(1-x)w\frac{dw}{dx}, \tag{2.15}$$

$$M(q^8) = \left(\frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \frac{15}{32}\sqrt{1-x} - \frac{15}{64}x\sqrt{1-x}\right)w^4. \tag{2.16}$$

Applying the duplication principle again, this time to (2.15) and (2.16), we obtain

$$\begin{aligned} L(q^{16}) = \Big(&\frac{11}{32} - \frac{23}{64}x + \frac{9}{32}\sqrt{1-x} + \frac{3}{16}(1-x)^{1/4} \\ &+ \frac{3}{16}(1-x)^{3/4}\Big)w^2 + \frac{3}{4}x(1-x)w\frac{dw}{dx}, \end{aligned} \tag{2.17}$$

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$$(2.18)\qquad M(q^{16}) = \Big(\frac{137}{512} - \frac{137}{512}x + \frac{1}{4096}x^2 + \frac{135}{512}\sqrt{1-x} - \frac{135}{1024}x\sqrt{1-x} + \frac{15}{64}(1-x)^{1/4} - \frac{105}{512}x(1-x)^{1/4} + \frac{15}{64}(1-x)^{3/4} - \frac{15}{512}x(1-x)^{3/4}\Big)\, w^4.$$

From (2.7), (2.10) and (2.13) we obtain

$$(2.19)\qquad w^4 = \frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4),$$

$$(2.20)\qquad xw^4 = \frac{1}{15}M(q) - \frac{1}{15}M(q^2),$$

$$(2.21)\qquad x^2w^4 = \frac{16}{15}M(q^2) - \frac{16}{15}M(q^4).$$

Next, for $k \in \{1,2,3,4,5,6,7\}$, we see from (2.8), (2.11) and (2.14) that

$$\Delta(q)^{\frac{k-4}{6}}\Delta(q^2)^{\frac{20-3k}{12}}\Delta(q^4)^{\frac{k-8}{12}} = (1-x)^{k/4}w^4 = A^k w^4,$$

where for convenience we have set

$$(2.22)\qquad A = (1-x)^{1/4}.$$

Note that $x = 1 - A^4$. Further

$$\begin{aligned}\Delta(q)^{\frac{k-4}{6}}&\Delta(q^2)^{\frac{20-3k}{12}}\Delta(q^4)^{\frac{k-8}{12}}\\ &= \prod_{n=1}^{\infty}(1-q^n)^{4k-16}(1-q^{2n})^{40-6k}(1-q^{4n})^{2k-16}\\ &= \prod_{n=1}^{\infty}(1-q^n)^{4k-16}(1-q^{2n})^{24-4k}(1-q^{4n-2})^{16-2k}\\ &= \prod_{n=1}^{\infty}(1-q^n)^{4k-16}(1-q^n)^{24-4k}(1+q^n)^{24-4k}(1-q^{4n-2})^{16-2k}\\ &= \prod_{n=1}^{\infty}(1+q^n)^{24-4k}(1-q^n)^{8}(1-q^{4n-2})^{16-2k}.\end{aligned}$$

Hence

$$(2.23)\qquad A_k(q) = A^k w^4 = \Delta(q)^{\frac{k-4}{6}}\Delta(q^2)^{\frac{20-3k}{12}}\Delta(q^4)^{\frac{k-8}{12}}.$$

From (2.6) and (2.17) we obtain

$$(2.24)\qquad L(q) - 16L(q^{16}) = \Big(-\frac{9}{2} + \frac{3}{4}x - 3A - \frac{9}{2}A^2 - 3A^3\Big)\, w^2.$$

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Squaring (2.24), we obtain

$$(2.25)\quad (L(q)-16L(q^{16}))^2 = \frac{117}{2}w^4 - 45xw^4 + \frac{9}{16}x^2w^4 + \frac{45}{2}Aw^4 + \frac{171}{4}A^2w^4 + \frac{99}{2}A^3w^4 + \frac{63}{2}A^5w^4 + \frac{63}{4}A^6w^4 + \frac{9}{2}A^7w^4.$$

Appealing to (2.19), (2.20), (2.21) and (2.25), we obtain

$$(2.26)\quad (L(q)-16L(q^{16}))^2 = \frac{9}{10}M(q) - \frac{21}{5}M(q^2) + \frac{309}{5}M(q^4) + \frac{45}{2}Aw^4 + \frac{171}{4}A^2w^4 + \frac{99}{2}A^3w^4 + \frac{63}{2}A^5w^4 + \frac{63}{4}A^6w^4 + \frac{9}{2}A^7w^4.$$

From (2.16), (2.19), (2.20), (2.21) and (2.22), we deduce

$$(2.27)\quad M(q^8) = -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{64}A^2w^4 + \frac{15}{64}A^6w^4.$$

From (2.18), (2.19), (2.20), (2.21), (2.22), (2.23) we obtain

$$(2.28)\quad M(q^{16}) = -\frac{9}{512}M(q^2) + \frac{73}{256}M(q^4) + \frac{15}{512}Aw^4 + \frac{135}{1024}A^2w^4 + \frac{105}{512}A^3w^4 + \frac{105}{512}A^5w^4 + \frac{135}{1024}A^6w^4 + \frac{15}{512}A^7w^4.$$

From (1.5) and (2.23) we have

$$(2.29)\quad \sum_{n=1}^{\infty} c_{16}(n)q^n = \frac{1}{32}Aw^4 + \frac{3}{112}A^2w^4 + \frac{1}{224}A^3w^4 - \frac{1}{32}A^5w^4 - \frac{3}{112}A^6w^4 - \frac{1}{224}A^7w^4.$$

Then, from (2.27), (2.28) and (2.29), we deduce

$$(2.30)\quad \frac{24}{5}M(q^8) - \frac{1152}{5}M(q^{16}) - 504\sum_{n=1}^{\infty} c_{16}(n)q^n = \frac{45}{2}Aw^4 + \frac{171}{4}A^2w^4 + \frac{99}{2}A^3w^4 + \frac{63}{2}A^5w^4 + \frac{63}{4}A^6w^4 + \frac{9}{2}A^7w^4.$$

We see from (2.26) and (2.30) that

$$(2.31)\quad \big(L(q)-16L(q^{16})\big)^2 = \frac{9}{10}M(q) - \frac{21}{5}M(q^2) + \frac{309}{5}M(q^4) + \frac{24}{5}M(q^8) - \frac{1152}{5}M(q^{16}) - 504\sum_{n=1}^{\infty} c_{16}(n)q^n.$$

Then

$$576\sum_{n=1}^{\infty}\sum_{m<n/16}\sigma(m)\sigma(n-16m)q^n = \Big(24\sum_{l=1}^{\infty}\sigma(l)q^l\Big)\Big(24\sum_{m=1}^{\infty}\sigma(m)q^{16m}\Big)$$
$$= 1 - L(q) - L(q^{16}) + L(q)L(q^{16}),$$

so that

$$\text{(2.32)}\quad \sum_{n=1}^{\infty}\sum_{m<n/16}\sigma(m)\sigma(n-16m)q^n = \frac{1}{576} - \frac{1}{576}L(q) - \frac{1}{576}L(q^{16})$$
$$+ \frac{1}{18432}L(q)^2 + \frac{1}{72}L(q^{16})^2 - \frac{1}{18432}\left(L(q) - 16L(q^{16})\right)^2.$$

The following result is classical

$$\text{(2.33)}\qquad L(q)^2 = 1 + \sum_{n=1}^{\infty}(240\sigma_3(n) - 288n\sigma(n))q^n,$$

see for example [5, 6]. Letting $q \to q^{16}$ in (2.4) and (2.33), we obtain

$$\text{(2.34)}\qquad L(q^{16}) = 1 - 24\sum_{n=1}^{\infty}\sigma\Big(\frac{n}{16}\Big)q^n,$$

$$\text{(2.35)}\qquad L(q^{16})^2 = 1 + \sum_{n=1}^{\infty}\Big(240\sigma_3\Big(\frac{n}{16}\Big) - 18n\sigma\Big(\frac{n}{16}\Big)\Big)q^n.$$

Putting (2.4), (2.31), (2.33) and (2.34) into (2.32), we obtain, on equating the coefficients of q^n, the asserted formula of Theorem 1.1. ■

By Theorem 1.1 we have

$$7c_{16}(n) = \frac{1}{3}\sigma_3(n) + \sigma_3\Big(\frac{n}{2}\Big) + 4\sigma_3\Big(\frac{n}{4}\Big) + 16\sigma_3\Big(\frac{n}{8}\Big) + \frac{256}{3}\sigma_3\Big(\frac{n}{16}\Big)$$
$$+ \Big(\frac{32}{3} - 4n\Big)\sigma(n) + \Big(\frac{32}{3} - 64n\Big)\sigma\Big(\frac{n}{16}\Big)$$
$$- 256\sum_{m<n/16}\sigma(m)\sigma(n-16m)$$
$$\equiv \frac{1}{3}\sigma_3(n) - \frac{1}{3}\sigma(n) + \frac{1}{3}\sigma_3\Big(\frac{n}{16}\Big) - \frac{1}{3}\sigma\Big(\frac{n}{16}\Big) \pmod 1.$$

As $\sigma_3(k) \equiv \sigma(k) \pmod 3$, $k \in \mathbb{N}$, we deduce (1.6). Replacing n by $2n$ in Theorem 1.1, and making use of the elementary identities

$$\text{(2.36)}\qquad \sigma(2k) = 3\sigma(k) - 2\sigma\Big(\frac{k}{2}\Big), \quad k \in \mathbb{N},$$

$$\text{(2.37)}\qquad \sigma_3(2k) = 9\sigma_3(k) - 8\sigma_3\Big(\frac{k}{2}\Big), \quad k \in \mathbb{N},$$

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we obtain

$$(2.38)\quad \sum_{m<n/8} \sigma(m)\sigma(2n-16m) = \frac{9}{768}\sigma_3(n) - \frac{5}{768}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{64}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{8}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{16}\right) + \left(\frac{1}{8} - \frac{3n}{32}\right)\sigma(n) - \left(\frac{1}{12} - \frac{n}{16}\right)\sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{2}\right)\sigma\left(\frac{n}{8}\right) - \frac{7}{256}c_{16}(2n).$$

Next, by (2.36) we have

$$\sum_{m<n/8} \sigma(m)\sigma(2n-16m) = 3\sum_{m<n/8} \sigma(m)\sigma(n-8m) - 2\sum_{m<n/8} \sigma(m)\sigma\left(\frac{n}{2} - 4m\right).$$

By [7, Theorem 4], we have

$$(2.39)\quad \sum_{m<n/4} \sigma(m)\sigma(n-4m) = \frac{1}{48}\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{4}\right).$$

Appealing to (1.1), (1.2), and (2.39) with n replaced by $n/2$, we obtain

$$(2.40)\quad \sum_{m<n/8} \sigma(m)\sigma(2n-16m) = \frac{1}{64}\sigma_3(n) + \frac{1}{192}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) + \left(\frac{1}{8} - \frac{3}{32}n\right)\sigma(n) - \left(\frac{1}{12} - \frac{n}{16}\right)\sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{2}\right)\sigma\left(\frac{n}{8}\right) - \frac{3}{64}c_8(n).$$

Equating (2.38) and (2.40), we obtain

Corollary 2.1 $c_{16}(2n) = \frac{12}{7}c_8(n),\ n \in \mathbb{N}$.

By Corollary 2.1, we see that $c_{16}(4n) = \frac{12}{7}c_8(2n) = 0$, $n \in \mathbb{N}$. Appealing to (1.4)

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and (1.5), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} c_{16}(2n)q^{2n} &= \frac{1}{2}\Big(\sum_{n=1}^{\infty} c_{16}(n)q^n + \sum_{n=1}^{\infty} c_{16}(n)(-q)^n\Big)\\ &= \frac{1}{2}\Big(\frac{1}{32}A_1(q) + \frac{3}{112}A_2(q) + \frac{1}{224}A_3(q)\\ &\qquad - \frac{1}{32}A_5(q) - \frac{3}{112}A_6(q) - \frac{1}{224}A_7(q)\\ &\qquad + \frac{1}{32}A_7(q) + \frac{3}{112}A_6(q) + \frac{1}{224}A_5(q)\\ &\qquad - \frac{1}{32}A_3(q) - \frac{3}{112}A_2(q) - \frac{1}{224}A_1(q)\Big)\\ &= \frac{3}{224}(A_1(q) - A_3(q) - A_5(q) + A_7(q)).\end{aligned}$$

Hence, by Corollary 2.1, we have

$$A_1(q) - A_3(q) - A_5(q) + A_7(q) = 128\sum_{n=1}^{\infty} c_8(n)q^{2n}. \tag{2.41}$$

Using (1.2) and (1.3) in (2.41), we obtain the following identity.

Corollary 2.2

$$\begin{aligned}&\prod_{n=1}^{\infty}(1+q^n)^{20}(1-q^n)^8(1-q^{4n-2})^{14} - \prod_{n=1}^{\infty}(1+q^n)^{12}(1-q^n)^8(1-q^{4n-2})^{10}\\ &\quad - \prod_{n=1}^{\infty}(1+q^n)^4(1-q^n)^8(1-q^{4n-2})^6 + \prod_{n=1}^{\infty}(1+q^n)^{-4}(1-q^n)^8(1-q^{4n-2})^2\\ &\qquad = 128q^2\prod_{n=1}^{\infty}(1-q^{4n})^4(1-q^{8n})^4.\end{aligned}$$

From [7, Lemma] we have

$$\begin{aligned}\sum_{m<n/16}\sigma(m)\sigma(n-16m) &= -\frac{1}{24}\sigma_3(n) + \frac{1}{4}\sigma_3\Big(\frac{n}{16}\Big) + \frac{1}{24}\sigma(n) - \frac{n}{4}\sigma\Big(\frac{n}{16}\Big)\\ &\quad + \frac{1}{4}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x\equiv -y \pmod{16}}} ab + \frac{1}{4}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x\equiv y \pmod{16}}} ab.\end{aligned}$$

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Combining this result with Theorem 1.1, we obtain the following finite formula for $c_{16}(n)$ ($n \in \mathbb{N}$).

Corollary 2.3

$$c_{16}(n) = \frac{11}{7}\sigma_3(n) + \frac{1}{7}\sigma_3\Big(\frac{n}{2}\Big) + \frac{4}{7}\sigma_3\Big(\frac{n}{4}\Big) + \frac{16}{7}\sigma_3\Big(\frac{n}{8}\Big) + \frac{64}{21}\sigma_3\Big(\frac{n}{16}\Big) - \frac{4}{7}n\sigma(n) + \frac{32}{21}\sigma\Big(\frac{n}{16}\Big) - \frac{64}{7}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x\equiv -y \pmod{16}}} ab - \frac{64}{7}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ x\equiv y \pmod{16}}} ab.$$

3 Proof of Theorem 1.2

For $l \in \mathbb{N}_0$ we set $r_4(l) = \operatorname{card}\{(x_1,x_2,x_3,x_4) \in \mathbb{Z}^4 \mid x_1^2+x_2^2+x_3^2+x_4^2 = l\}$, so that $r_4(0) = 1$. It is a classical result of Jacobi (see for example [9]) that

$$r_4(n) = 8\sum_{\substack{d\mid n\\ 4\nmid d}} d = 8\sigma(n) - 32\sigma\Big(\frac{n}{4}\Big), \quad n \in \mathbb{N}. \tag{3.1}$$

Then the number $N(n)$ of representations of $n \in \mathbb{N}$ by the quadratic form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 4x_5^2 + 4x_6^2 + 4x_7^2 + 4x_8^2$$

is given by

$$N(n) = \sum_{\substack{l,m\in\mathbb{N}_0\\ l+4m=n}} r_4(l)r_4(m) = r_4(n) + r_4\Big(\frac{n}{4}\Big) + \sum_{\substack{l,m\in\mathbb{N}\\ m<n/4}} r_4(m)r_4(n-4m).$$

Appealing to (3.1), we obtain

$$N(n) = 8\sigma(n) - 32\sigma\Big(\frac{n}{4}\Big) + 8\sigma\Big(\frac{n}{4}\Big) - 32\sigma\Big(\frac{n}{16}\Big) + \sum_{m<n/4}\Big(8\sigma(m) - 32\sigma\Big(\frac{m}{4}\Big)\Big)\Big(8\sigma(n-4m) - 32\sigma\Big(\frac{n}{4}-m\Big)\Big).$$

Thus

$$N(n) - \Big(8\sigma(n) - 24\sigma\Big(\frac{n}{4}\Big) - 32\sigma\Big(\frac{n}{16}\Big)\Big) = 64\sum_{m<n/4}\sigma(m)\sigma(n-4m) - 256\sum_{m<n/4}\sigma\Big(\frac{m}{4}\Big)\sigma(n-4m) - 256\sum_{m<n/4}\sigma(m)\sigma\Big(\frac{n}{4}-m\Big) + 1024\sum_{m<n/4}\sigma\Big(\frac{m}{4}\Big)\sigma\Big(\frac{n}{4}-m\Big).$$

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The value of the first sum is given in (2.39). The second sum is

$$\sum_{m<n/4} \sigma\Big(\frac{m}{4}\Big)\sigma(n-4m) = \sum_{m<n/16} \sigma(m)\sigma(n-16m),$$

whose value is given in Theorem 1.1. The third sum is

$$\sum_{m<n/4} \sigma(m)\sigma\Big(\frac{n}{4}-m\Big) = \frac{5}{12}\sigma_3\Big(\frac{n}{4}\Big) + \Big(\frac{1}{12}-\frac{n}{8}\Big)\sigma\Big(\frac{n}{4}\Big),$$

see [3, 7]. The fourth sum is (by (2.39))

$$\begin{aligned}\sum_{m<n/4} \sigma\Big(\frac{m}{4}\Big)\sigma\Big(\frac{n}{4}-m\Big) &= \sum_{m<n/16} \sigma(m)\sigma\Big(\frac{n}{4}-4m\Big)\\ &= \frac{1}{48}\sigma_3\Big(\frac{n}{4}\Big) + \frac{1}{16}\sigma_3\Big(\frac{n}{8}\Big) + \frac{1}{3}\sigma_3\Big(\frac{n}{16}\Big)\\ &\quad + \Big(\frac{1}{24}-\frac{n}{64}\Big)\sigma\Big(\frac{n}{4}\Big) + \Big(\frac{1}{24}-\frac{n}{16}\Big)\sigma\Big(\frac{n}{16}\Big).\end{aligned}$$

Putting these results together we obtain the assertion of Theorem 1.2. ■

The values of $N(n)$ for $n = 1, 2, \ldots, 20$ are given in the following table.

n	$N(n)$	$\sigma_3(n)$	$c_{16}(n)$	$\sigma_3(n)+3\sigma_3(n/2)-68\sigma_3(n/4)$ $+48\sigma_3(n/8)+256\sigma_3(n/16)+7c_{16}(n)$
1	8	1	1	8
2	24	9	12/7	24
3	32	28	4/7	32
4	32	73	0	32
5	112	126	−2	112
6	288	252	−48/7	288
7	320	344	−24/7	320
8	240	585	0	240
9	680	757	−11	680
10	1488	1134	−24/7	1488
11	1376	1332	44/7	1376
12	896	2044	0	896
13	2352	2198	22	2352
14	4416	3096	288/7	4416
15	3520	3528	−8/7	3520
16	2160	4681	0	2160
17	5264	4914	50	5264
18	8952	6813	−132/7	8952
19	6816	6860	−44/7	6816
20	4032	9198	0	4032

We conclude by remarking that the table suggests that $c_{16}(4k+1) \in \mathbb{Z}$ for $k \in \mathbb{N}_0$. We have verified this for $0 \le k \le 100$.

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