Can. J. Math., Vol. XXXIV, No. 6, 1982, pp. 1255–1275

TOPOLOGICAL EXTENSION PROPERTIES AND PROJECTIVE COVERS

HARUTO OHTA

Introduction. All spaces considered in this paper are assumed to be (Hausdorff) completely regular, and all maps are continuous. Let $\mathscr{P}$ be a topological property of spaces. We shall identify $\mathscr{P}$ with the class of spaces having $\mathscr{P}$. A space having $\mathscr{P}$ is called a *$\mathscr{P}$-space*, and a subspace of a $\mathscr{P}$-space is called a *$\mathscr{P}$-regular space*. The class of $\mathscr{P}$-regular spaces is denoted by $R(\mathscr{P})$. Following [**37**], we call a closed hereditary, productive, topological property $\mathscr{P}$ such that each $\mathscr{P}$-regular space has a $\mathscr{P}$-regular compactification a *topological extension property*, or simply, an *extension property*. In this paper, we restrict our attention to extension properties $\mathscr{P}$ satisfying the following axioms:

(A_1) The two-point discrete space has $\mathscr{P}$.

(A_2) If each $\mathscr{P}$-regular space of nonmeasurable cardinal has $\mathscr{P}$, then $\mathscr{P} = R(\mathscr{P})$.

The existence of an extension property which fails to satisfy (A_2) is equivalent to the existence of measurable cardinal (see 5.4). If $\mathscr{P}$ is an extension property, then each $\mathscr{P}$-regular space X is a dense subspace of a $\mathscr{P}$-space $\mathscr{P}X$ such that every map from X to a $\mathscr{P}$-space admits a continuous extension over $\mathscr{P}X$ (cf. [**14**]). The space $\mathscr{P}X$ is called the *maximal $\mathscr{P}$-extension* of X. For example, if $\mathscr{P}$ is compactness or realcompactness, then $\mathscr{P}$ is an extension property and $\mathscr{P}X$ is the Stone-Čech compactification or the Hewitt realcompactification, respectively. A space is called *extremally disconnected* if the closure of every open set is open. It is known ([**17**], [**32**]) that for each space X there exist an extremally disconnected space EX and a perfect irreducible map (i.e., a perfect map which takes proper closed subsets onto proper subsets) k_X from EX onto X. The space EX is unique up to homeomorphism, and is called the *projective cover* (or the *absolute*) of X.

In this paper, we consider the problem under what conditions, both on $\mathscr{P}$ and on X, $\mathscr{P}(EX) = E(\mathscr{P}X)$. This problem was raised by Woods in [**38**], and the special case when $\mathscr{P}$ is realcompactness has been settled by Hardy and Woods in [**12**]. We obtain, for all extension properties $\mathscr{P}$ contained in the class $\mathscr{A}\mathscr{R}$ of almost realcompact spaces, several common necessary and sufficient conditions on X for the equality to hold, and also prove that the equality holds for every $\mathscr{P}$-regular space X if and only if

Received September 8, 1981 and in revised form March 24, 1982.

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

either every $\mathscr{P}$-space is compact or $\mathscr{P}$ is hereditary. Here, an almost realcompact space is a space which is the image of a realcompact space under a perfect map (cf. [**10**]). Some of our results are generalizations of Hardy and Woods's one.

In Section 1, we review known results and define some symbols. Section 2 is devoted to a study of extension properties contained in $\mathscr{A}\mathscr{R}$. Main theorems are proved in Section 3. In particular, we give ten conditions on a $\mathscr{P}$-regular space X each of which is equivalent to the equality $\mathscr{P}(EX) = E(\mathscr{P}X)$ provided that $\mathscr{C}_{\mathscr{P}} \neq \mathscr{P} \subset \mathscr{A}\mathscr{R}$, where $\mathscr{C}_{\mathscr{P}}$ is the class of $\mathscr{P}$-regular compact spaces. It is also shown that, conversely, if those conditions and the equality are equivalent to each other, then either $\mathscr{C}_{\mathscr{P}} \neq \mathscr{P} \subset \mathscr{A}\mathscr{R}$ or $\mathscr{P} = R(\mathscr{P})$. Our theory is closely related to various interesting problems about extension properties; for example, the preservation of properties of the maximal $\mathscr{P}$-extension under maps, the problem of when $\mathscr{P}(X \times Y) = \mathscr{P}X \times \mathscr{P}Y$ for $\mathscr{P}$-regular spaces X and Y, and a classification of extension properties. These applications are discussed in Section 4. Section 5 contains a sequence of examples to which preceding sections refer. For details and examples of extension properties see [**37**], [**13**] and [**14**], and for projective covers see [**38**], [**17**] and [**32**]. The terminology and notation will be used as in [**8**].

1. Preliminaries. Let $\mathscr{O}$ be an extension property such that $\mathscr{O}$-regularity is complete regularity and $\mathscr{P}$ an extension property. Then

$\mathscr{O}_{\mathscr{P}}$ denotes the class of $\mathscr{P}$-regular $\mathscr{O}$-space,

$\mathscr{A}\mathscr{P}$ denotes the class of $\mathscr{P}$-regular spaces that are the images under a perfect map of some $\mathscr{P}$-space,

$\mathscr{E}\mathscr{P}$ denotes the class of extremally disconnected $\mathscr{P}$-spaces,

$\mathscr{P}^*$ denotes the class of $\mathscr{P}$-regular spaces X for which $\mathscr{P}(EX) = E(\mathscr{P}X)$.

Both $\mathscr{O}_{\mathscr{P}}$ and $\mathscr{A}\mathscr{P}$ are known ([**37**]) to be extension properties. We always use $\mathscr{C}$ and $\mathscr{R}$ to denote compactness and realcompactness, respectively. Following [**11**] and [**37**], we use βX, $\beta_{\mathscr{P}}X$ and υX for $\mathscr{C}X$, $\mathscr{C}_{\mathscr{P}}X$ and $\mathscr{R}X$, respectively. A subspace Y of a space X is said to be *$\mathscr{P}$-embedded* in X if each map from Y to a $\mathscr{P}$-space admits a continuous extension over X. The maximal $\mathscr{P}$-extension $\mathscr{P}X$ of a $\mathscr{P}$-regular space X is the unique $\mathscr{P}$-space in which X is dense and $\mathscr{P}$-embedded ([**14**]), and the continuous extension over $\mathscr{P}X$ of a map $f: X \to Y$ with $Y \in \mathscr{P}$ is denoted by $\mathscr{P}f: \mathscr{P}X \to Y$. In case $\mathscr{P} = \mathscr{C}$ ($\mathscr{P} = \mathscr{C}_{\mathscr{P}}$), we use βf ($\beta_{\mathscr{P}} f$) for $\mathscr{P}f$. We list basic facts about extension properties; (a) and (b) are simple generalizations of results in [**14**] and appear in [**37**].

1.1. THEOREM. *Let $\mathscr{P}$ be an extension property and $X \in R(\mathscr{P})$.*

(a) *$\mathscr{P}X$ is the intersection of all subspaces of $\beta_{\mathscr{P}}X$ that contain X and have $\mathscr{P}$, so $X \subset \mathscr{P}X \subset \beta_{\mathscr{P}}X$* ([**37**, 1.3]).

(b) *If f is a perfect map from X onto a $\mathscr{P}$-space Y, then X has $\mathscr{P}$* ([**37**, 1.2]).

(c) *Each* 0*-dimensional space is* $\mathscr{P}$*-regular* ([**37**, 1.4]).

(d) *Either* $\mathscr{P}$ *is contained in the class of countably compact spaces or the countable discrete space has* $\mathscr{P}$ ([**37**, 2.9]).

(e) $\mathscr{E}\mathscr{A}\mathscr{P} = \mathscr{E}\mathscr{P}$ ([**37**, 3.4]).

(f) *The discrete space of cardinality* $\mathfrak{m}$ *has* $\mathscr{P}$ *if and only if every topological sum of* $\mathfrak{m}$ *many* $\mathscr{P}$*-spaces has* $\mathscr{P}$ ([**13**, 7.18]).

It is known that 0-dimensionality is an extension property (cf. [**2**]). By (a) and (c), 0-dimensional compactness is the smallest extension property. If X is an extremally disconnected space, then so is βX, and hence it follows from (c) that $\beta X = \beta_{\mathscr{P}} X$ for each extension property $\mathscr{P}$. Therefore we use $\beta(EX)$ for $\beta_{\mathscr{P}}(EX)$, omitting $\mathscr{P}$; a similar remark applies to $\beta_{\mathscr{P}} f$. By an *extension* of a space X we mean a space that contains X as a dense subspace. The following properties of projective covers and perfect maps are well known.

1.2. THEOREM. *Let* $\mathscr{P}$ *be an extension property and* $X \in R(\mathscr{P})$.

(a) $\beta(EX) = E(\beta_{\mathscr{P}} X)$ *and* $\beta k_X = k_{\beta_{\mathscr{P}} X}$ (cf. [**38**, p. 328]).

(b) $EX \subset \mathscr{P}(EX) \subset E(\mathscr{P}X) \subset \beta(EX)$ *and* $E(\mathscr{P}X) = (\beta k_X)^{-1}[\mathscr{P}X]$ (cf. [**38**, p. 346]).

(c) *If* $f: X \to Y$ *is a perfect onto map, then there exists a perfect map* h *from* EY *onto a closed subspace of* X *such that* $k_Y = f \circ h$ (cf. [**32**, p. 309]).

(d) *A map* $f: X \to Y$ *is perfect if and only if, whenever* S *and* T *are extensions of* X *and* Y, *respectively, and* $F: S \to T$ *is a continuous extension of* f, *then* $F[S - X] \subset T - Y$ (cf. [**8**, 3.7.16]).

(e) *If the composition* $f \circ g$ *of maps* $f: X \to Y$ *and* $g: Y \to Z$ *is perfect, then* $g|f[X]$ *and* f *are perfect* (cf. [**8**, 3.7.10]).

Recall from [**37**] that two extension properties $\mathscr{P}$ and $\mathscr{Q}$ are *coregular* if $R(\mathscr{P}) = R(\mathscr{Q})$. For such extension properties $\mathscr{P}$ and $\mathscr{Q}$, let $\mathscr{P} \otimes \mathscr{Q}$ denote the class of all $\mathscr{P}$-regular spaces X such that $\mathscr{P}X = \mathscr{Q}X$.

1.3. THEOREM. *Let* $\mathscr{P}$ *and* $\mathscr{Q}$ *be coregular extension properties.*

(a) $\mathscr{A}\mathscr{P} = \mathscr{A}\mathscr{Q}$ *if and only if* $\mathscr{E}\mathscr{P} = \mathscr{E}\mathscr{Q}$.

(b) *If* $\mathscr{P} \subset \mathscr{Q}$ *and* $\mathscr{A}\mathscr{P} = \mathscr{A}\mathscr{Q}$, *then* $\mathscr{P}^* = \mathscr{Q}^* \cap (\mathscr{P} \otimes \mathscr{Q})$.

Proof. (a) Assume that $\mathscr{E}\mathscr{P} = \mathscr{E}\mathscr{Q}$. If $X \in \mathscr{A}\mathscr{P}$, then it follows from 1.1(b), 1.1(e) and our assumption that $EX \in \mathscr{E}\mathscr{A}\mathscr{P} = \mathscr{E}\mathscr{P} = \mathscr{E}\mathscr{Q}$, so $X \in \mathscr{A}\mathscr{Q}$. The proof that $\mathscr{A}\mathscr{Q} \subset \mathscr{A}\mathscr{P}$ is quite similar, and hence $\mathscr{A}\mathscr{P} = \mathscr{A}\mathscr{Q}$. Conversely, if $\mathscr{A}\mathscr{P} = \mathscr{A}\mathscr{Q}$, then by 1.1(e), $\mathscr{E}\mathscr{P} = \mathscr{E}\mathscr{A}\mathscr{P} = \mathscr{E}\mathscr{A}\mathscr{Q} = \mathscr{E}\mathscr{Q}$.

(b) Let $X \in \mathscr{P}^*$. By (a), $\mathscr{E}\mathscr{P} = \mathscr{E}\mathscr{Q}$, so $\mathscr{P}(EX) = \mathscr{Q}(EX)$. Since $\mathscr{P} \subset \mathscr{Q}$, $\mathscr{Q}X \subset \mathscr{P}X$ by 1.1(a). These facts and 1.2(b) imply that $\mathscr{P}(EX) = \mathscr{Q}(EX) \subset E(\mathscr{Q}X) \subset E(\mathscr{P}X) = \mathscr{P}(EX)$, so $X \in \mathscr{Q}^*$ and $E(\mathscr{P}X) = E(\mathscr{Q}X)$, and hence it follows from 1.2(b) that $\mathscr{P}X = \mathscr{Q}X$. Conversely, if $X \in \mathscr{Q}^* \cap (\mathscr{P} \otimes \mathscr{Q})$, then $\mathscr{P}(EX) = \mathscr{Q}(EX) = E(\mathscr{Q}X) = E(\mathscr{P}X)$ by our assumption, and hence $X \in \mathscr{P}^*$.

1.4. COROLLARY. *For an extension property* $\mathscr{P}$,

$$\mathscr{P}^* = (\mathscr{A}\mathscr{P})^* \cap (\mathscr{P} \otimes \mathscr{A}\mathscr{P}) \quad and \quad \mathscr{P} = \mathscr{A}\mathscr{P} \cap \mathscr{P}^*.$$

Proof. Since $\mathscr{P} \subset \mathscr{A}\mathscr{P}$ and $\mathscr{A}\mathscr{P} = \mathscr{A}\mathscr{A}\mathscr{P}$, the first equality follows from 1.3(b). Taking intersections of $\mathscr{A}\mathscr{P}$ with both sides of the first equality, we have

$$\mathscr{A}\mathscr{P} \cap \mathscr{P}^* = \mathscr{A}\mathscr{P} \cap (\mathscr{A}\mathscr{P})^* \cap (\mathscr{P} \otimes \mathscr{A}\mathscr{P})$$
$$= \mathscr{A}\mathscr{P} \cap (\mathscr{P} \otimes \mathscr{A}\mathscr{P}) = \mathscr{P}.$$

The inclusion $\mathscr{P} \subset \mathscr{Q}$ does not imply $\mathscr{P}^* \subset \mathscr{Q}^*$ in general. In fact, $\mathscr{C} \subset \mathscr{R}$ and $\mathscr{C}^* = R(\mathscr{C})$ by 1.2(a), but $R(\mathscr{C}) = R(\mathscr{R}) \not\subset \mathscr{R}^*$ (cf. [**38**, p. 344]). The second equality of 1.4 tells us that if f is a perfect map from a $\mathscr{P}$-space X onto a $\mathscr{P}$-regular space Y, then Y has $\mathscr{P}$ if and only if $\mathscr{P}(EY) = E(\mathscr{P}Y)$.

2. Extension properties contained in $\mathscr{A}\mathscr{R}$. Recall from [**9**] that, for a given space E, a space X is *E-compact* if X is homeomorphic to a closed subspace of $E^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$. The class of E-compact spaces is denoted by $\langle E \rangle$. The following theorem was proved by Mrówka in [**25**, 4.10].

2.1. THEOREM. *Let E be a space. An $\langle E \rangle$-regular space X is E-compact if and only if, given an $\langle E \rangle$-regular extension T of X and a point $p \in T - X$, there exists a map $f: X \to E$ that cannot be continuously extended to $X \cup \{p\}$.*

Let I and N denote the closed unit interval of the real line and the space of non-negative integers, respectively.

2.2. *Definition.* A space X is *ultrarealcompact* if it is $(I \times N)$-compact.

Some properties of $(I \times N)$-compact spaces have been studied by Broverman in [**3**] and [**4**]. Let $\mathscr{U}$ denote the class of ultrarealcompact spaces. Then $\mathscr{U}$ is an extension property such that the $\mathscr{U}$-regularity is just complete regularity, and clearly $\mathscr{C} \subset \mathscr{U} \subset \mathscr{R}$. We assume familiarity with the theory of z-filters (cf. [**11**]).

2.3. THEOREM. *Let $\mathscr{P}$ be an extension property and $X \in R(\mathscr{P})$. Then the following conditions are equivalent:*

(a) *X is ultrarealcompact.*

(b) *Every free z-ultrafilter on X contains a countable decreasing sequence of open-and-closed sets with empty intersection.*

(c) *For each $p \in \beta_{\mathscr{P}}X - X$, there is a countable disjoint open cover $\mathfrak{U}$ of X such that $p \notin \mathrm{cl}_{\beta_{\mathscr{P}}X} U$ for each $U \in \mathfrak{U}$.*

(d) *X is homeomorphic to a closed subspace of the product of a $\mathscr{P}$-regular compact space with an N-compact space.*

Proof. (a) → (b). Let $\mathfrak{F}$ be a free z-ultrafilter on X, and let $(I \times N) \cup \{\infty\}$ be the one-point compactification of $I \times N$. There is $p \in \beta X - X$ such that $\{p\} = \bigcap \{\mathrm{cl}_{\beta X} F | F \in \mathfrak{F}\}$. Since $X \in \mathscr{U}$ and βX is $\mathscr{U}$-regular, it follows from 2.1 that there exists a map $f: X \to I \times N$ such that $(\beta f)(p) = \infty$. For each $n \in N$, let

$$G_n = f^{-1}[I \times \{k | k \geqq n\}].$$

Then G_n is open-and-closed in X, $G_n \in \mathfrak{F}$ and $\bigcap G_n = \emptyset$.

(b) → (c). Let $p \in \beta_{\mathscr{P}} X - X$; then there is a z-ultrafilter $\mathfrak{F}$ on X such that $\{p\} = \bigcap \{\mathrm{cl}_{\beta_{\mathscr{P}} X} F | F \in \mathfrak{F}\}$. By (b), $\mathfrak{F}$ contains a decreasing sequence $\{G_n | n \in N\}$ of open-and-closed sets with empty intersection. Setting $U_0 = X - G_0$ and $U_{n+1} = G_n - G_{n+1}$ for each $n \in N$, we have the desired open cover $\{U_n\}$ of X.

(c) → (d). Let $K = \beta_{\mathscr{P}} X$, and note that $R(\langle K \times N \rangle) = R(\mathscr{P})$. To show that X is $(K \times N)$-compact, let T be a $\mathscr{P}$-regular extension of X and $p \in T - X$. The embedding f of X in T extends to a map $\beta_{\mathscr{P}} f: \beta_{\mathscr{P}} X \to \beta_{\mathscr{P}} T$. Pick

$$q \in (\beta_{\mathscr{P}} f)^{-1}(p).$$

Then by (c) there is a countable disjoint open cover $\{U_n | n \in N\}$ of X such that $q \notin \mathrm{cl}_{\beta_{\mathscr{P}} X} U_n$ for each $n \in N$. Define a map g from X into $K \times N$ by setting for each $x \in X$, $g(x) = (x, n)$ if $x \in U_n$. Assume that g extends to a map $G: X \cup \{p\} \to K \times N$; then $G(p) \in K \times \{n\}$ for some $n \in N$. Set

$$V = h^{-1}[K \times \{n\}] - \mathrm{cl}_{\beta_{\mathscr{P}} X} U_n,$$

where $h = G \circ ((\beta_{\mathscr{P}} f) | (X \cup \{q\}))$. Then, since

$$(g \circ f)^{-1}[K \times \{n\}] = U_n,$$

V is a neighborhood of q in $X \cup \{q\}$ with $V \cap X = \emptyset$, which is impossible. Thus g admits no continuous extension over $X \cup \{p\}$, so it follows from 2.1 that X is $(K \times N)$-compact. Since $(K \times N)^{\mathfrak{m}} = K^{\mathfrak{m}} \times N^{\mathfrak{m}}$ and $K^{\mathfrak{m}}$ is $\mathscr{P}$-regular compact, we have (d).

(d) → (a). This follows from Tychonoff's embedding theorem. Hence the proof is complete.

We denote the class of spaces each of whose countably compact subspaces has compact closure by $\mathscr{S}$. It follows from [**7**, 1.2] and [**8**, 3.11.1] that $\mathscr{A}\mathscr{R} \subset \mathscr{S}$. We are interested in ultrarealcompactness because, roughly speaking, it is the smallest non-compact extension property contained in $\mathscr{S}$:

2.4. THEOREM. *If $\mathscr{P}$ is an extension property contained in $\mathscr{S}$, then either $\mathscr{P} = \mathscr{C}_{\mathscr{P}}$ or $\mathscr{U}_{\mathscr{P}} \subset \mathscr{P}$.*

Proof. Assume that $\mathscr{U}_{\mathscr{P}} \not\subset \mathscr{P}$, and choose $X \in \mathscr{U}_{\mathscr{P}}$ not in $\mathscr{P}$. Then by 2.3, X is homeomorphic to a closed subspace of the product of a $\mathscr{P}$-regular compact space K with an N-compact space. Since $K \in \mathscr{P}$, if $N \in \mathscr{P}$, then X must have $\mathscr{P}$, a contradiction. Thus $N \notin \mathscr{P}$. It follows from 1.1 (d) that $\mathscr{P}$ is contained in the class of countably compact spaces. Since $\mathscr{P} \subset \mathscr{S}$, $\mathscr{P} = \mathscr{C}_{\mathscr{P}}$.

2.5. Corollary. *A space X is N-compact if and only if X is 0-dimensional ultrarealcompact.*

Proof. If we denote the class of N-compact spaces by $\mathscr{N}$, then the class of 0-dimensional ultrarealcompact spaces is $\mathscr{U}_{\mathscr{N}}$, because $\mathscr{N}$-regularity is 0-dimensionality. Since $\mathscr{C}_{\mathscr{N}} \neq \mathscr{N} \subset \mathscr{S}$, it follows from 2.4 that $\mathscr{U}_{\mathscr{N}} \subset \mathscr{N}$. Since $\mathscr{N} \subset \mathscr{U}_{\mathscr{N}}$, $\mathscr{N} = \mathscr{U}_{\mathscr{N}}$.

2.6. *Remarks*. (i) Herrlich and Kim-Peu Chew proved essentially the same results as 2.5 in [**13**, 6.2] and [**5**, Theorem C], respectively.

(ii) In [**23**], Terada defined a space X to be $P_z(\aleph_1)$-*compact* if for each $p \in \beta X - X$ there exists a countable disjoint cover $\mathfrak{Z}$ of X, consisting of zero-sets, such that $p \notin \mathrm{cl}_{\beta X} Z$ for each $Z \in \mathfrak{Z}$, and he showed that $P_z(\aleph_1)$-compactness is an extension property contained in $\mathscr{R}$. By 2.3 (or 2.4), every ultrarealcompact space is $P_z(\aleph_1)$-compact, but the converse is false (see 5.1). The relationship of these extension properties to more familiar ones is summarized as follows:

$$
\begin{array}{ccc}
\text{0-dimensional compact} & \rightarrow & \text{compact} \\
\downarrow & & \downarrow \\
N\text{-compact} & \rightarrow & \text{ultrarealcompact} \\
\downarrow & & \downarrow \\
\text{0-dimensional } P_z(\aleph_1)\text{-compact} & \rightarrow & P_z(\aleph_1)\text{-compact} \\
\downarrow & & \downarrow \\
\text{0-dimensional realcompact} & \rightarrow & \text{realcompact} \\
 & & \downarrow \\
 & & \text{almost realcompact}
\end{array}
$$

The next lemma follows from [**24**, (iv_a), p. 598] and [**7**, 1.2].

2.7. Lemma. *An extremally disconnected, almost realcompact space is N-compact, and hence it is ultrarealcompact.*

Following [**36**], we denote the maximal $\mathscr{AR}$- (resp. $(\mathscr{AR})_{\mathscr{P}}$-) extension of X by aX (resp. $a_{\mathscr{P}}X$).

2.8. Theorem. *Let $\mathscr{P}$ be an extension property for which $\mathscr{C}_{\mathscr{P}} \neq \mathscr{P} \subset \mathscr{AR}$ and $X \in R(\mathscr{P})$. Then:*

(a) *$\mathscr{AP} = (\mathscr{AR})_{\mathscr{P}}$, and hence, if $\mathscr{P} = \mathscr{AP}$, then $\mathscr{P} = (\mathscr{AR})_{\mathscr{P}}$.*

(b) *$\mathscr{P}(EX) = E(\mathscr{P}X)$ if and only if $\mathscr{P}X = a_{\mathscr{P}}X$ and $a_{\mathscr{P}}(EX) = E(a_{\mathscr{P}}X)$.*

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

Proof. Note that $\mathscr{P}$ and $\mathscr{R}_{\mathscr{P}}$ are coregular. By 2.4, $\mathscr{U}_{\mathscr{P}} \subset \mathscr{P}$, so it follows from 2.7 that $\mathscr{E}\mathscr{P} = \mathscr{E}(\mathscr{R}_{\mathscr{P}})$. If $Y \in (\mathscr{A}\mathscr{R})_{\mathscr{P}}$, then by 1.1(b) and 2.7 $EY \in \mathscr{R}_{\mathscr{P}}$, so $Y \in \mathscr{A}(\mathscr{R}_{\mathscr{P}})$. Since $(\mathscr{A}\mathscr{R})_{\mathscr{P}} \supset \mathscr{A}(\mathscr{R}_{\mathscr{P}})$, this shows that $(\mathscr{A}\mathscr{R})_{\mathscr{P}} = \mathscr{A}(\mathscr{R}_{\mathscr{P}})$. Thus both (a) and (b) follow from 1.3.

3. Main theorems. Recall from [**20**], [**30**] and [**31**] that a map $f: X \to Y$ is (*countably*) *bi-quotient* if, whenever $y \in Y$ and $\mathfrak{U}$ is a (countable) cover of $f^{-1}(y)$ by open sets in X, then finitely many $f(U)$, with $U \in \mathfrak{U}$, cover a neighborhood of y in Y. All open and all perfect maps are bi-quotient, and all countably bi-quotient maps are quotient maps. A space X is called *bi-sequential* if it is the bi-quotient image of a metric space (cf. [**21**]), and X is called *strongly* 0-*dimensional* if βX is 0-dimensional (cf. [**8**]). In this section, we consider the following conditions (1) through (10) on a $\mathscr{P}$-regular space Y, where $\mathscr{P}$ is an extension property.

(1) $\mathscr{P}(EY) = E(\mathscr{P}Y)$.

(2) $\mathscr{P}k_Y: \mathscr{P}(EY) \to \mathscr{P}Y$ is perfect onto.

(3) For each perfect irreducible map f from a $\mathscr{P}$-regular space X onto Y, $\mathscr{P}f: \mathscr{P}X \to \mathscr{P}Y$ is perfect onto.

(4) For each perfect map f from a $\mathscr{P}$-regular space X onto Y, there exists a closed subset X_0 of $\mathscr{P}X$ such that $(\mathscr{P}f)|X_0: X_0 \to \mathscr{P}Y$ is perfect onto.

(5) $\mathscr{P}k_Y: \mathscr{P}(EY) \to \mathscr{P}Y$ is bi-quotient onto.

(6) $\mathscr{P}k_Y: \mathscr{P}(EY) \to \mathscr{P}Y$ is countably bi-quotient onto.

(7) Every locally finite family, of nonmeasurable cardinal, of open sets in Y is locally finite in $\mathscr{P}Y$.

(8) Every countable, locally finite family of open sets in Y is locally finite in $\mathscr{P}Y$.

(9) $Y \times T$ is $\mathscr{P}$-embedded in $\mathscr{P}Y \times T$ for each bi-sequential space T.

(10) $Y \times M$ is $\mathscr{P}$-embedded in $\mathscr{P}Y \times M$ for each strongly 0-dimensional metric space M.

Conditions (7) and (8) are formal generalizations of the necessary and sufficient condition, due to Hardy and Woods [**12**], for $\upsilon(EY) = E(\upsilon Y)$ to hold. Let (A_3) denote the following axiom: There exist a $\mathscr{P}$-space E and a fixed pair of distinct points e_0 and e_1 such that for every $\mathscr{P}$-regular space X, every closed subset F of X and every $x \in X - F$, there is a map $f: X \to E$ such that $f(x) = e_0$ and $f(F) = \{e_1\}$. If $\mathscr{P}$-regularity is complete regularity or 0-dimensionality, then $\mathscr{P}$ satisfies (A_3). We begin by dividing conditions (1)–(10) into two groups.

3.1. THEOREM. *Conditions* (1)–(10) *are related as follows:* (1) $\rightleftarrows$ (2) $\rightleftarrows$ (3) $\rightleftarrows$ (4) $\to$ (5) $\to$ (6) *and* (7) $\rightleftarrows$ (8) $\to$ (9) $\rightleftarrows$ (10). *Moreover, if* $\mathscr{P}$ *satisfies* (A_3), *then* (9) $\to$ (8) *is valid.*

To prove 3.1 and subsequent results, we need the following lemmas.

3.2 is due to Michael [**20**], and the proof of 3.3 is left to the reader, since it is proved quite similarly to [**28**, 2.2].

3.2. LEMMA. *If $f: X \to Y$ and $g: S \to T$ are bi-quotient onto maps, then the product map $f \times g$ is bi-quotient.*

3.3. LEMMA. *Let $F_i: X_i \to Y_i$ $(i = 1, 2)$ be onto maps such that $F_1 \times F_2$ is a quotient map, and let S_i (resp. $T_i = F_i(S_i)$) be a dense $\mathscr{P}$-embedded subspace of X_i (resp. Y_i). If $S_1 \times S_2$ is $\mathscr{P}$-embedded in $X_1 \times X_2$, then $T_1 \times T_2$ is $\mathscr{P}$-embedded in $Y_1 \times Y_2$.*

Proof of Theorem 3.1. (1) $\to$ (2). If $\mathscr{P}(EY) = E(\mathscr{P}Y)$, then $\mathscr{P}k_Y = k_{\mathscr{P}Y}$ by 1.2(b), so $\mathscr{P}k_Y: \mathscr{P}(EY) \to \mathscr{P}Y$ is perfect onto.

(2) $\to$ (3). Let $f: X \to Y$ be the map hypothesized in (3). Since $f \circ k_X$ is perfect irreducible, $EX = EY$ by the uniqueness of EY. Thus $k_Y = f \circ k_X$, so $\mathscr{P}k_Y = \mathscr{P}f \circ \mathscr{P}k_X$. We show that $\mathscr{P}k_X: \mathscr{P}(EX) \to \mathscr{P}X$ is onto. If there is

$$p \in \mathscr{P}X - (\mathscr{P}k_X)[\mathscr{P}(EX)],$$

then $p = (\beta k_X)(q)$ for some $q \in \beta(EX) - \mathscr{P}(EX)$. Since $\beta k_Y = \beta_{\mathscr{P}} f \circ \beta k_X$ and $(\beta_{\mathscr{P}} f)|\mathscr{P}X = \mathscr{P}f$,

$$(\beta k_Y)(q) = (\mathscr{P}f)(p) \in \mathscr{P}Y,$$

which contradicts (2) because of 1.2 (d). Hence it follows from (2) and 1.2(e) that $\mathscr{P}f: \mathscr{P}X \to \mathscr{P}Y$ is perfect onto.

(3) $\to$ (4). Let f be a perfect map from a $\mathscr{P}$-regular space X onto Y. By 1.2(c), there is a map h from EY onto a closed subset X_1 of X such that $k_Y = f \circ h$. Let $X_0 = \mathrm{cl}_{\mathscr{P}X} X_1$. Since $X_0 \in \mathscr{P}$, h extends continuously to $\mathscr{P}h: \mathscr{P}(EY) \to X_0$. The same argument as used in (2) $\to$ (3) to show that $\mathscr{P}k_X$ is onto shows that $(\mathscr{P}h)[\mathscr{P}(EY)] = X_0$. Since $\mathscr{P}k_Y = \mathscr{P}f \circ \mathscr{P}h$ and $\mathscr{P}k_Y$ is perfect onto by (3), it follows from 1.2(e) that $(\mathscr{P}f)|X_0: X_0 \to \mathscr{P}Y$ is perfect onto.

(4) $\to$ (1). By (4), there is a closed subset X_0 of $\mathscr{P}(EY)$ such that $(\mathscr{P}k_Y)|X_0: X_0 \to \mathscr{P}Y$ is perfect onto. Since k_Y is irreducible, EY is contained in X_0, so $X_0 = \mathscr{P}(EY)$, and hence $\mathscr{P}k_Y$ is perfect onto. Thus $\mathscr{P}(EY) = E(\mathscr{P}Y)$ by 1.2(b).

(4) $\to$ (5) $\to$ (6) and (7) $\to$ (8). These are obvious.

(8) $\to$ (7). Let $\mathfrak{G} = \{G_\alpha | \alpha \in A\}$ be a locally finite family, of non-measurable cardinal, of open sets in Y. Suppose that there is $y_0 \in \mathscr{P}Y - Y$ at which $\mathfrak{G}$ is not locally finite. Then by (8) $y_0 \notin \mathrm{cl}_{\mathscr{P}Y} G_\alpha$ for all but finitely many $\alpha \in A$. Let

$$B = \{\alpha \in A | y_0 \notin \mathrm{cl}_{\mathscr{P}Y} G_\alpha\},$$

and let $\mathfrak{U}$ be a neighborhood system of y_0 in $\mathscr{P}Y$. For each $U \in \mathfrak{U}$, let

$$B_U = \{\beta \in B | U \cap G_\beta \neq \emptyset\};$$

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

then $\{B_U | U \in \mathfrak{U}\}$ is a filter base with $\cap B_U = \emptyset$, so it is contained in some ultrafilter $\mathfrak{F}$ on B. To show that $\mathfrak{F}$ has the countable intersection property, let $\{F_n | n \in N\}$ be a decreasing sequence of members of $\mathfrak{F}$, and set

$$V_n = \cup\{G_\beta | \beta \in F_n\}.$$

Then $\{V_n | n \in N\}$ is a decreasing sequence of open sets in Y, and $y_0 \in \cap \mathrm{cl}_{\mathscr{P}Y} V_n$. For, if $y_0 \notin \mathrm{cl}_{\mathscr{P}Y} V_n$, then there is $U \in \mathfrak{U}$ with $U \cap V_n = \emptyset$; but then $B_U \cap F_n = \emptyset$, a contradiction. Thus $\{V_n\}$ is not locally finite in $\mathscr{P}Y$, and hence it follows from (8) that $\cap \mathrm{cl}_Y V_n \neq \emptyset$. Pick $y \in \cap \mathrm{cl}_Y V_n$. For each $n \in N$, since $\mathfrak{G}$ is locally finite, we can find $\beta_n \in F_n$ with $y \in \mathrm{cl}_Y G_{\beta_n}$. Again using local finiteness of $\mathfrak{G}$, we have that $\{\beta_n | n \in N\}$ is a finite set. As $\{F_n\}$ is decreasing, this shows that $\cap F_n \neq \emptyset$, and hence $\mathfrak{F}$ has the countable intersection property. Since $\mathfrak{F}$ is free (i.e., $\cap \mathfrak{F} = \emptyset$), by [**11**, 12.2], this contradicts the fact that the cardinality of B is nonmeasurable.

(8) $\rightarrow$ (10). Let M be a strongly 0-dimensional metric space, and let $X = \mathscr{P}Y \times M$. Note that M is $\mathscr{P}$-regular by 1.1(c). Since $Y \times M$ is $\mathscr{P}$-embedded in $\mathscr{P}(Y \times M)$, it suffices to prove that $X \subset \mathscr{P}(Y \times M)$. First, to show that $X \subset \beta_{\mathscr{P}}(Y \times M)$, let $f: Y \times M \rightarrow K$ be a map with $K \in \mathscr{C}_{\mathscr{P}}$ and let $E_i (i = 1, 2)$ be disjoint closed sets in K. We show that

$$\mathrm{cl}_X F_1 \cap \mathrm{cl}_X F_2 = \emptyset,$$

where $F_i = f^{-1}[E_i]$. Let $(y_0, t_0) \in X - (Y \times M)$. Then there is a map $g: \mathscr{P}Y \rightarrow K$ such that

$$g(y) = f((y, t_0)) \quad \text{for each } y \in Y.$$

Since E_1 and E_2 are disjoint, we may assume that $g(y_0) \notin E_2$. Choose an open set U in K such that

$$E_1 \cup \{g(y_0)\} \subset U \subset \mathrm{cl}_K U \subset K - E_2,$$

and let $\{V_n | n \in N\}$ be a neighborhood base of t_0 in M with $V_n \supset V_{n+1}$. For each $n \in N$, set

$$H_n = \cup\{H | H \text{ is an open set in } Y \text{ such that } H \times V_n \subset f^{-1}[U]\}.$$

Then $(\mathrm{cl}_Y H_n \times V_n) \cap F_2 = \emptyset$ and $g^{-1}[U] \cap Y = \cup H_n$. Setting

$$G_n = (g^{-1}[U] \cap Y) - \mathrm{cl}_Y H_n$$

for each $n \in N$, we have a decreasing sequence $\{G_n\}$ of open sets in Y with $\cap \mathrm{cl}_Y G_n = \emptyset$. Since $\{G_n\}$ is locally finite in Y, it is locally finite in $\mathscr{P}Y$ by (8), so $y_0 \notin \mathrm{cl}_{\mathscr{P}Y} G_m$ for some m. Let

$$W = g^{-1}[U] - \mathrm{cl}_{\mathscr{P}Y} G_m.$$

Then $W \times V_m$ is a neighborhood of (y_0, t_0) in X such that $(W \times V_m) \cap F_2 = \emptyset$, because $W \cap Y \subset \mathrm{cl}_Y H_m$. Thus $(y_0, t_0) \notin \mathrm{cl}_X F_2$; thus $g(y_0) \notin E_2$

implies $(y_0, t_0) \notin \mathrm{cl}_X F_2$, and hence

$$\mathrm{cl}_X F_1 \cap \mathrm{cl}_X F_2 = \emptyset.$$

If follows from [**8**, 3.2.1] that f admits a continuous extension over X, which implies that

$$X \subset \beta_{\mathscr{P}} X = \beta_{\mathscr{P}}(Y \times M).$$

Next, suppose that $X \not\subset \mathscr{P}(Y \times M)$; then there is $(y_1, t_1) \in X - \mathscr{P}(Y \times M)$. If we set

$$Y' = \{y \in \mathscr{P}Y | (y, t_1) \in X \cap \mathscr{P}(Y \times M)\},$$

then $Y \subset Y' \subsetneqq \mathscr{P}Y$ and $Y' \in \mathscr{P}$, because it is homeomorphic to $\mathscr{P}(Y \times M) \cap (\mathscr{P}Y \times \{t_1\})$. This contradicts 1.1(a), and hence $X \subset \mathscr{P}(Y \times M)$.

(10) $\rightarrow$ (9). Let T be a bi-sequential space; then, by the proof of [**21**, 3.D.2], there exist a strongly 0-dimensional, metric space M and a bi-quotient onto map f: $M \rightarrow T$. By 3.2, $\mathrm{id}_{\mathscr{P}Y} \times f$ is a bi-quotient map from $\mathscr{P}Y \times M$ onto $\mathscr{P}Y \times T$, where $\mathrm{id}_{\mathscr{P}Y}$ is the identity of $\mathscr{P}Y$. Since $Y \times M$ is $\mathscr{P}$-embedded in $\mathscr{P}Y \times M$, it follows from 3.3 that $Y \times T$ is $\mathscr{P}$-embedded in $\mathscr{P}Y \times T$.

(9) $\rightarrow$ (10). This is clear.

Finally, assuming (A_3) we prove that (9) $\rightarrow$ (8). Let E, e_0 and e_1 be a $\mathscr{P}$-space and its points as described in (A_3), and let $\{G_n | n \in N\}$ be a countable, locally finite family of open sets in Y. Suppose on the contrary that $\{G_n\}$ is not locally finite at $y_0 \in \mathscr{P}Y - Y$. Set $T = (Y \times N) \cup \{\infty\}$ and define a topology on T as follows: Each point of $Y \times N$ is isolated and $\{W_n | n \in N\}$, where

$$W_n = (Y \times \{i | i > n\}) \cup \{\infty\}$$

is a neighborhood base of ∞. Then T is a metric space. For each $n \in N$ and each $y \in G_n$, there is a map f_{ny}: $Y \rightarrow E$ such that

$$f_{ny}(y) = e_0 \quad \text{and} \quad f_{ny}[Y - G_n] = \{e_1\}.$$

Define a function f: $Y \times T \rightarrow E$ by

$$f((y', t)) = \begin{cases} f_{ny}(y') & \text{if } t = (y, n) \in G_n \times \{n\}, \\ e_1 & \text{otherwise.} \end{cases}$$

To show that f is continuous, let $p_0 = (y, t) \in Y \times T$. If $t \neq \infty$, then there is nothing to prove since $\{t\}$ is open. If $t = \infty$, then $f(p_0) = e_1$. Since $\{G_n\}$ is locally finite, there exist $j \in N$ and a neighborhood U of y in Y such that $U \cap G_n = \emptyset$ for each $n > j$. For each $n \in N$,

$$(U \times W_j) \cap (G_n \times (G_n \times \{n\})) = \emptyset,$$

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

so $f[U \times W_j] = \{e_1\}$, and hence f is continuous. It remains to prove that f admits no continuous extension over $\mathscr{P} Y \times T$. To do this, let $V \times W_k$ be a basic neighborhood of (y_0, ∞) in $\mathscr{P} Y \times T$. Since V meets infinitely many G_n, we can find $m > k$ and $y \in V \cap G_m$. Then both $p_1 = (y, (y, m))$ and $p_2 = (y, \infty)$ belong to $V \times W_k$ and $f(p_1) = f_{my}(y) = e_0$, while $f(p_2) = e_1$. This shows that f does not extend continuously to (y_0, ∞). Hence the proof is complete.

3.4. THEOREM. *Condition* (5) ((6)) *is true if and only if for each perfect map f from a $\mathscr{P}$-regular space X onto Y, $\mathscr{P}f$: $\mathscr{P}X \to \mathscr{P}Y$ is* (*countably*) *bi-quotient onto.*

Proof. The "if" part is obvious. To prove the converse, let f: $X \to Y$ be a perfect onto map with $X \in R(\mathscr{P})$. It is easily checked that if the composition $g \circ h$ of two maps is (countably) bi-quotient onto, then so is g (even if h is not onto). By 1.2 (c), there is a map h from EY to X such that $k_Y = f \circ h$. Since $\mathscr{P}k_Y = \mathscr{P}f \circ \mathscr{P}h$ and $\mathscr{P}k_Y$ is (countably) bi-quotient onto by (5) ((6)), it follows that $\mathscr{P}f$: $\mathscr{P}X \to \mathscr{P}Y$ is (countably) bi-quotient onto.

3.5. *Remarks.* (i) The author does not know if in 3.1 the implications (6) $\to$ (5) $\to$ (4) are true or not, in general, and if (9) $\to$ (8) can be proved without assuming axiom (A_3).

(ii) Let (2′) denote the following condition: For each perfect map f from a $\mathscr{P}$-regular space X onto Y, $\mathscr{P}f$: $\mathscr{P}X \to \mathscr{P}Y$ is perfect onto. In contrast to 3.4, the reader might ask if (2) implies (2′). In 5.2, we give a negative answer to this question.

Next, we connect conditions (4) and (6) with (7).

3.6. THEOREM. *For an extension property $\mathscr{P}$, the following conditions are equivalent:*

(a) *For each $\mathscr{P}$-regular space Y,* (6) *implies* (7).

(b) *$\mathscr{P}$ is not contained in the class of countably compact spaces* (*or equivalently, $N \in \mathscr{P}$*).

(c) $\mathscr{U}_{\mathscr{P}} \subset \mathscr{P}$.

(d) $\mathscr{E}\mathscr{R} \subset \mathscr{E}\mathscr{P}$.

Proof. (a) $\to$ (b). It suffices to show that $N \in \mathscr{P}$. By (a) and (c) of 1.1, N is $\mathscr{P}$-regular and $\beta_{\mathscr{P}}N = \beta N$, so $\mathscr{P}N$ is extremally disconnected. Since $\mathscr{P}(EN) = \mathscr{P}N = E(\mathscr{P}N)$, N satisfies (7) by (a), and thus $\{\{n\}|n \in N\}$ is locally finite at any point of $\mathscr{P}N$. This implies that $N = \mathscr{P}N \in \mathscr{P}$.

(b) $\to$ (c). Let $X \in \mathscr{U}_{\mathscr{P}}$. By 2.3, X is embedded as a closed subspace of the product of a $\mathscr{P}$-regular compact space K with an N-compact space. Since $K \in \mathscr{P}$ and $N \in \mathscr{P}$, $X \in \mathscr{P}$, and thus $\mathscr{U}_{\mathscr{P}} \subset \mathscr{P}$.

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

(c) $\rightarrow$ (d). This follows from 2.7 and 1.1 (c).

(d) $\rightarrow$ (a). Let Y be a $\mathscr{P}$-regular space satisfying (6). To show that Y satisfies (8), let $\{G_n | n \in N\}$ be a countable, locally finite family of open sets in Y and $y \in \mathscr{P}Y - Y$. We may assume that $G_0 = Y$. For each $n \in N$, set

$$H_n = \text{cl}_{EY} k_Y^{-1}[G_n] \quad \text{and}$$
$$D_n = H_n - \bigcup\{H_i | i > n\}.$$

Then $\{D_n\}$ is a countable disjoint open cover of EY. By (d), $N \in \mathscr{P}$, and hence it follows from 1.1(f) that

$$\mathscr{P}(EY) = \oplus \{\mathscr{P}D_n | n \in N\},$$

where $\oplus$ means the topological sum. Since $\mathscr{P}k_Y$ is now countably bi-quotient onto, there exist a neighborhood U of y in $\mathscr{P}Y$ and $m \in N$ such that

$$U \subset \bigcup\{(\mathscr{P}k_Y)[\mathscr{P}D_j] | j \leqq m\}.$$

Since

$$\emptyset = (\bigcup\{\mathscr{P}D_j | j \leqq m\}) \cap (\bigcup\{H_i | i > m\})$$
$$\supset (\bigcup\{\mathscr{P}D_j | j \leqq m\}) \cap (\mathscr{P}k_Y)^{-1}[\bigcup\{G_i | i > m\}],$$

$U \cap (\bigcup\{G_i | i > m\}) = \emptyset$. Thus $\{G_n\}$ is locally finite in $\mathscr{P}Y$. Since (8) always implies (7), the proof is complete.

3.7. THEOREM. *For an extension property $\mathscr{P}$, the following conditions are equivalent:*

(a) *For each $\mathscr{P}$-regular space Y,* (7) *implies* (4).

(b) *For each $\mathscr{P}$-regular space Y,* (7) *implies* (6).

(c) $\mathscr{P} = R(\mathscr{P})$ *or* $\mathscr{P} \subset \mathscr{A}\mathscr{R}$.

(d) $\mathscr{P} = R(\mathscr{P})$ *or* $\mathscr{E}\mathscr{P} \subset \mathscr{E}\mathscr{R}$.

Proof. (a) $\rightarrow$ (b). This follows from 3.1.

(b) $\rightarrow$ (c). Suppose on the contrary that $\mathscr{P} \neq R(\mathscr{P})$ and $\mathscr{P} \not\subset \mathscr{A}\mathscr{R}$. Then by axiom ($A_2$) there exist a $\mathscr{P}$-regular space S of nonmeasurable cardinal not in $\mathscr{P}$ and a $\mathscr{P}$-space Z' not in $\mathscr{A}\mathscr{R}$. Since S is homeomorphic to the diagonal of

$$\prod\{\beta_{\mathscr{P}}S - \{s\} | s \in \beta_{\mathscr{P}}S - S\},$$

$\beta_{\mathscr{P}}S - \{s^*\} \notin \mathscr{P}$ for some $s^* \in \beta_{\mathscr{P}}S - S$. For $i = 1, 2$, let K_i be the copy of $\beta_{\mathscr{P}}S$ and s_i the point of K_i corresponding to s^*. Let $K = K_1 \oplus K_2$, and let L be the quotient space obtained from K by identifying s_1 and s_2. Then $L \in \mathscr{C}_{\mathscr{P}}$, because it is homeomorphic to the closed subspace

$$(\{s_1\} \times K_2) \cup (K_1 \times \{s_2\})$$

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

of $K_1 \times K_2$. Let $\phi: K \to L$ be the quotient map, and set $L_i = \phi(K_i)$ and $Z = EZ'$. Then by 1.1(b) $Z \in \mathscr{E}\mathscr{P}$, but $Z \notin \mathscr{R}$. Let us set

$$X = (K \times \upsilon Z) - ((\{s_1\} \cup K_2) \times (\upsilon Z - Z)) \quad \text{and}$$
$$Y = (L \times \upsilon Z) - (L_2 \times (\upsilon Z - Z)).$$

Since υZ is extremally disconnected, it follows from 1.1(c) that both X and Y are $\mathscr{P}$-regular. Note that $\upsilon Z \in \mathscr{U}_{\mathscr{P}}$ by 2.7. Pick $z_0 \in \upsilon Z - Z$, and set $p_0 = (s_0, z_0)$, where $s_0 = \phi(s_1)$ $(= \phi(s_2))$.

Claim 1.

$$\mathscr{P}X = (K_1 \times \upsilon Z) \oplus (K_2 \times Z) \quad \text{and} \quad p_0 \in \mathscr{P}Y \subset L \times \upsilon Z.$$

To prove the first equality, let

$$X_1 = (K_1 \times \upsilon Z) - (\{s_1\} \times (\upsilon Z - Z)) \quad \text{and} \quad X_2 = K_2 \times Z;$$

then by axiom (A_1) and 1.1(f), $\mathscr{P}X = \mathscr{P}X_1 \oplus \mathscr{P}X_2$. Clearly $\mathscr{P}X_2 = X_2$. Since K_1 is a compact space of nonmeasurable cardinal, it follows from [**6**, 5.3] that $\upsilon(K_1 \times Z) = K_1 \times \upsilon Z$, so $\upsilon X_1 = K_1 \times \upsilon Z$. Since $K_1 \times \upsilon Z \in \mathscr{U}_{\mathscr{P}}$, $K_1 \times \upsilon Z = \mathscr{U}_{\mathscr{P}}X_1$. We distinguish two cases. If $N \in \mathscr{P}$, then $\mathscr{U}_{\mathscr{P}} \subset \mathscr{P}$ by 3.6, so $\mathscr{P}X_1 \subset \mathscr{U}_{\mathscr{P}}X_1$. If $N \notin \mathscr{P}$, then it follows from 1.1(d) that Z is countably compact. Since υZ is then compact by [**8**, 3.11.1], $K_1 \times \upsilon Z = \beta_{\mathscr{P}}X_1$. Thus, in any case, $\mathscr{P}X_1 \subset K_1 \times \upsilon Z$. For each $z \in \upsilon Z - Z$, since $(K_1 - \{s_1\}) \times \{z\}$ is homeomorphic to $\beta_{\mathscr{P}}S - \{s^*\}$, it is not closed in $\mathscr{P}X_1$. This shows that (s_1, z) must be contained in $\mathscr{P}X_1$, so $\mathscr{P}X_1 = K_1 \times \upsilon Z$, and hence

$$\mathscr{P}X = (K_1 \times \upsilon Z) \oplus (K_2 \times Z).$$

The second inequality can be proved similarly.

Claim 2. Y satisfies (7).

Since (8) implies (7), we prove that Y satisfies (8). Let $\{G_n | n \in N\}$ be a countable, locally finite family of open sets in Y. If we set

$$U_n = \bigcup\{G_i \cap (L \times Z) | i \geqq n\},$$

then $U_n \supset U_{n+1}$, $\mathrm{cl}_Y U_n \supset G_n$ and $\bigcap \mathrm{cl}_Y U_n = \emptyset$. Let $H_n = \mathrm{cl}_Z \pi[U_n]$, where π is the projection from $L \times Z$ to Z; then H_n is open-and-closed in Z. Since π is perfect, $\bigcap H_n = \emptyset$, and so $\bigcap \mathrm{cl}_{\upsilon Z} H_n = \emptyset$ by [**11**, 8.7]. Note that $\mathscr{P}Y \subset L \times \upsilon Z$ by claim 1. Since

$$\mathrm{cl}_{\mathscr{P}Y} U_n \subset \mathrm{cl}_{L \times \upsilon Z} U_n \subset L \times \mathrm{cl}_{\upsilon Z} H_n,$$

we have $\bigcap \mathrm{cl}_{\mathscr{P}Y} U_n = \emptyset$. Consequently $\{G_n\}$ is locally finite in $\mathscr{P}Y$.

Claim 3. Y does not satisfy (6).

To prove this, let

$$f = (\phi \times \mathrm{id}_{\upsilon Z}) | X.$$

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

Then f is a perfect map from X onto Y, and

$$\mathscr{P}f = (\phi \times \mathrm{id}_{vZ})|\mathscr{P}X.$$

Since $(\mathscr{P}f)^{-1}(p_0) = \{(s_1, z_0)\}$, $K_1 \times vZ$ is an open neighborhood of $(\mathscr{P}f)^{-1}(p_0)$ in $\mathscr{P}X$, but $(\mathscr{P}f)[K_1 \times vZ]$ $(= L_1 \times vZ)$ contains no neighborhood of p_0 in $\mathscr{P}Y$. This shows that $\mathscr{P}f$ is not countably bi-quotient, and thus, by 3.4, Y does not satisfy (6). Hence we have (c).

(c) $\rightarrow$ (d). If $\mathscr{P} \subset \mathscr{A}\mathscr{R}$, then by 1.1 (e) $\mathscr{E}\mathscr{P} \subset \mathscr{E}\mathscr{A}\mathscr{R} = \mathscr{E}\mathscr{R}$.

(d) $\rightarrow$ (a). Let Y be a $\mathscr{P}$-regular space satisfying (7). It suffices to prove that Y satisfies (2). If $\mathscr{P} = R(\mathscr{P})$, then Y clearly satisfies (2), so suppose that $\mathscr{E}\mathscr{P} \subset \mathscr{E}\mathscr{R}$ and

$$\mathscr{P}k_Y: \mathscr{P}(EY) \rightarrow \mathscr{P}Y$$

is not perfect onto. Then by 1.2(d) there is $p \in \beta(EY) - \mathscr{P}(EY)$ such that $(\beta k_Y)(p) \in \mathscr{P}Y$. Since $\mathscr{E}\mathscr{P} \subset \mathscr{E}\mathscr{R}$, $v(EY) \subset \mathscr{P}(EY)$, and hence by [**8**, 3.11.10], there is a map h: $\beta(EY) \rightarrow I$ such that $h(p) = 0$ and $h(y) > 0$ for each $y \in EY$. For each $n \in N$, let

$$G_n = Y - k_Y[EY - H_n],$$

where

$$H_n = \{y \in EY | h(y) < 1/n\}.$$

Then, k_Y being perfect irreducible, $\{G_n\}$ is a locally finite family of open sets in Y and

$$\mathrm{cl}_Y G_n = k_Y[\mathrm{cl}_{EY} H_n].$$

Since $p \in \mathrm{cl}_{\beta(EY)} H_n$ for each $n \in N$,

$$(\beta k_Y)(p) \in \bigcap \mathrm{cl}_{\beta_{\mathscr{P}} Y} G_n,$$

and so $\bigcap \mathrm{cl}_{\mathscr{P}Y} G_n \neq \emptyset$. This contradicts (7). Hence the proof is complete.

3.8 THEOREM. *For an extension property* $\mathscr{P}$, *the following conditions are equivalent:*

(a) *For each* $\mathscr{P}$*-regular space* Y, *conditions* (1) *through* (8) *are equivalent.*

(b) $\mathscr{P} = R(\mathscr{P})$ *or* $\mathscr{C}_{\mathscr{P}} \neq \mathscr{P} \subset \mathscr{A}\mathscr{R}$.

(c) $\mathscr{P} = R(\mathscr{P})$ *or* $\mathscr{E}\mathscr{P} = \mathscr{E}\mathscr{R}$.

Furthermore, if $\mathscr{P}$ *satisfies* (A_3), *then we can replace* "(1) *through* (8)" *by* "(1) *through* (10)" *in* (a).

Proof. This is a consequence of 3.1, 3.6 and 3.7.

3.9. THEOREM. *Let* $\mathscr{P}$ *be an extension property. Then* $\mathscr{P}^* = R(\mathscr{P})$ *if and only if either* $\mathscr{P} = R(\mathscr{P})$ *or* $\mathscr{P} = \mathscr{C}_{\mathscr{P}}$.

Proof. If $\mathscr{P} = R(\mathscr{P})$, then by 1.1(c) $\mathscr{P}(EY) = EY = E(\mathscr{P}Y)$ for each $Y \in R(\mathscr{P})$, and thus $\mathscr{P}^* = R(\mathscr{P})$. If $\mathscr{P} = \mathscr{C}_{\mathscr{P}}$, then it follows from

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

1.2 (a) that $\mathscr{P}^* = R(\mathscr{P})$. To prove the converse, assume that $\mathscr{P}^* = R(\mathscr{P})$. Then, since each $\mathscr{P}$-regular space Y satisfies (6), it follows from 3.7 that either $\mathscr{P} = R(\mathscr{P})$ or $\mathscr{P} \subset \mathscr{A}\mathscr{R}$. Let X be the space constructed in [**19**, Example, p. 240]. In [**36**, p. 206], Woods essentially proved that $a(EX) \neq E(aX)$ and aX is 0-dimensional. By 1.1(c), $X \in R(\mathscr{P})$ and

$$a_{\mathscr{P}}(EX) = a(EX) \neq E(aX) = E(a_{\mathscr{P}}X).$$

If $\mathscr{C}_{\mathscr{P}} \neq \mathscr{P} \subset \mathscr{A}\mathscr{R}$, then it follows from 2.8(b) that $\mathscr{P}(EX) \neq E(\mathscr{P}X)$, so $X \notin \mathscr{P}^*$. This contradicts $\mathscr{P}^* = R(\mathscr{P})$, and hence, if $\mathscr{P} \subset \mathscr{A}\mathscr{R}$, then $\mathscr{P} = \mathscr{C}_{\mathscr{P}}$.

3.10. *Remark.* Axiom (A_2) is useful only for the implication (b) $\rightarrow$ (c) in 3.7 (and hence, also for 3.8 and 3.9). The author does not know if 3.7 can be proved without assuming (A_2). We note that, by 5.4 below, the following are equivalent:

(a) Every cardinal is nonmeasurable.

(b) Every extension property satisfies (A_2).

3.11. *Remarks.* (i) A space is called *Dieudonné complete* if it is homeomorphic to a closed subspace in a product of metric spaces (cf. [**8**, 8.5.13]). If we denote the class of Dieudonné complete spaces by $\mathscr{T}$, then $\mathscr{T}$ is an extension property such that the $\mathscr{T}$-regularity is just complete regularity. Let $\mathscr{V}$ denote the class of spaces which are homeomorphic to a closed subspace in a product of a compact space with discrete spaces, and let (7′) denote the following condition on a $\mathscr{P}$-regular space Y: Every locally finite family of open sets in Y is locally finite in $\mathscr{P}Y$. By 2.3 $\mathscr{U} \subset \mathscr{V}$, and (7′) implies (7). If we use [**8**, 8.5.13(b)], then the following results, concerning an extension property $\mathscr{P}$, will be proved analogously to 3.6 and 3.7:

(3.11.1) *For each $\mathscr{P}$-regular space Y* (5) *implies* (7′) *if and only if $\mathscr{V}_{\mathscr{P}} \subset \mathscr{P}$ (or equivalently, every discrete space has $\mathscr{P}$).*

(3.11.2) *For each $\mathscr{P}$-regular space Y* (7′) *implies* (4) *if and only if either $\mathscr{P} = R(\mathscr{P})$ or $\mathscr{P} \subset \mathscr{A}\mathscr{T}$.*

For internal characterizations of members of $\mathscr{A}\mathscr{T}$, see [**27**].

(ii) For a space X, $\mathscr{T}X$ is usually denoted by μX, and pX denotes the largest subspace S of βX containing X such that $X \times T$ is C^*-embedded in $S \times T$ for each paracompact p-space T, where a *paracompact p-space* (= a paracompact M-space in the sense of Morita [**22**]) is a perfect preimage of a metric space (cf. [**1**]). Recently, in [**29**], Przymusiński proved that for a space X of nonmeasurable cardinal $\mu X = pX$ is equivalent to $\mu(EX) = E(\mu X)$, and he asked whether this equivalence holds for every space X. In 5.6 below, we show that if there exists a measurable cardinal, then there exists a space X such that $\mu X = pX$ but $\mu(EX) \neq E(\mu X)$. Hence his question is equivalent to the set theoretic question: Is every cardinal nonmeasurable? His result quoted above follows also from 3.8 and [**26**, 19.1] since $\mu X = \upsilon X$ if the cardinality of X is nonmeasurable (cf. [**11**.20]).

4. Applications. Let us call a property of spaces a *strongly fitting property* if it is preserved by closed subspaces and perfect images. There are several classes of spaces which are determined by a strongly fitting property of the maximal $\mathscr{P}$-extensions; for example, an M'-space in the sense of Isiwata [**18**] is characterized as a space X whose Dieudonńe completion μX is a paracompact M-space (cf. [**18**] and [**23**]). Condition (4) considered in the preceding section concerns the preservation of such classes under perfect maps. The following theorem follows immediately from 3.1.

4.1. THEOREM. *Let $\mathscr{P}$ be an extension property, and let f be a perfect map from a $\mathscr{P}$-regular space X onto Y with $Y \in \mathscr{P}^*$. If $\mathscr{P}X$ has a strongly fitting property, then $\mathscr{P}Y$ has the same property.*

4.2. COROLLARY. *Suppose that $f: X \to Y$ is a perfect onto map and υX is locally compact. Then υY is locally compact if and only if $\upsilon(EY) = E(\upsilon Y)$.*

Proof. Since local compactness is a strongly fitting property, the necessity follows from 4.1. The sufficiency is due to Woods [**35**, 2.10].

Conditions (5), (6), (9) and (10) concern the problem of when the relation $\mathscr{P}(X \times Y) = \mathscr{P}X \times \mathscr{P}Y$ is valid.

4.3. THEOREM. *Let $\mathscr{P}$ be an extension property, satisfying* (A_3), *such that $\mathscr{C}_{\mathscr{P}} \neq \mathscr{P} \subset \mathscr{A}\mathscr{R}$, and let Y be a $\mathscr{P}$-regular space of nonmeasurable cardinal. Then the following conditions on Y are equivalent:*

(a) $\mathscr{P}(EY) = E(\mathscr{P}Y)$.

(b) *For each perfect onto map $f: X \to Y$ with $X \in R(\mathscr{P})$ and each $Z \in R(\mathscr{P})$, $\mathscr{P}(Y \times Z) = \mathscr{P}Y \times \mathscr{P}Z$ whenever $\mathscr{P}(X \times Z) = \mathscr{P}X \times \mathscr{P}Z$.*

(c) *For each perfect onto map $f: X \to Y$ with $X \in R(\mathscr{P})$ and each perfect onto map $g: S \to T$ with $S \in R(\mathscr{P})$ and with $T \in \mathscr{P}^*$, $\mathscr{P}(Y \times T) = \mathscr{P}Y \times \mathscr{P}T$ whenever $\mathscr{P}(X \times S) = \mathscr{P}X \times \mathscr{P}S$.*

(d) *$\mathscr{P}(Y \times T) = \mathscr{P}Y \times \mathscr{P}T$ for each bi-sequential $\mathscr{P}$-space T.*

(e) *$\mathscr{P}(Y \times M) = \mathscr{P}Y \times \mathscr{P}M$ for each strongly* 0*-dimensional, metric space M of nonmeasurable cardinal.*

Proof. (a) $\to$ (b). By 3.1 and 3.4, $\mathscr{P}f$ is bi-quotient onto, so it follows from 3.2 that $\mathscr{P}f \times \mathrm{id}_{\mathscr{P}Z}$ is bi-quotient onto. Hence (b) follows from 3.3.

(b) $\to$ (c). By (b), $\mathscr{P}(Y \times S) = \mathscr{P}Y \times \mathscr{P}S$. Since $\mathscr{P}(ET) = E(\mathscr{P}T)$ and (a) implies (b) as proved above, $\mathscr{P}(Y \times T) = \mathscr{P}Y \times \mathscr{P}T$.

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

(c) $\rightarrow$ (d). Let T be a bi-sequential $\mathscr{P}$-space. Then $T \in \mathscr{P}^*$ since $\mathscr{P} \subset \mathscr{P}^*$. Let $X = EY, f = k_Y, S = T$ and $g = \mathrm{id}_S$. Then $\mathscr{P}(EX) = \mathscr{P}X = E(\mathscr{P}X)$, because X and $\mathscr{P}X$ are extremally disconnected. Since $\mathscr{C}_{\mathscr{P}} \neq \mathscr{P} \subset \mathscr{A}\mathscr{R}$, it follows from 3.8 that $X \times S$ is $\mathscr{P}$-embedded in $\mathscr{P}X \times S$, so $\mathscr{P}(X \times S) = \mathscr{P}X \times S$. Since $f: X \rightarrow Y$ and $g: S \rightarrow T$ are perfect onto, $\mathscr{P}(Y \times T) = \mathscr{P}Y \times T$ $(= \mathscr{P}Y \times \mathscr{P}T)$ by (c).

(d) $\rightarrow$ (e). Note that a strongly 0-dimensional, metric space of nonmeasurable cardinal is N-compact (cf. [**24**, (iv$_a$), p. 598] and [**11**, 15.20]). Since $\mathscr{C}_{\mathscr{P}} \neq \mathscr{P} \subset \mathscr{A}\mathscr{R}$, it follows from 2.4 that M is a $\mathscr{P}$-space. Thus (d) implies (e).

(e) $\rightarrow$ (a). Observe that the space T constructed in the proof that (9) $\rightarrow$ (8) in 3.1 is a strongly 0-dimensional, metric space whose cardinality is equal to that of Y. From 3.8 and this fact we have (a). Hence the proof is complete.

The next theorem improves [**28**, 3.4], and shows that "$T \in \mathscr{P}^*$" in 4.3(c) cannot be replaced by "$T \in R(\mathscr{P})$" even when $\mathscr{P} = \mathscr{R}$ (see 5.3).

4.4. THEOREM. *The following conditions on a space Y of nonmeasurable cardinal are equivalent:*

(a) *υY is locally compact.*

(b) *For each perfect onto map $f: X \rightarrow Y$ and each quotient onto map $g: S \rightarrow T$, $\upsilon(Y \times T) = \upsilon Y \times \upsilon T$ whenever $\upsilon(X \times S) = \upsilon X \times \upsilon S$.*

(c) *As in* (b) *with "perfect" instead of "quotient".*

Proof. (a) $\rightarrow$ (b). By [**35**, 2.10], $\upsilon(EY) = E(\upsilon Y)$, so it follows from 4.3 that $\upsilon(Y \times S) = \upsilon Y \times \upsilon S$. Thus $\upsilon(Y \times T) = \upsilon Y \times \upsilon T$ by [**28**, 3.4]. The implication (b) $\rightarrow$ (c) is obvious, and (c) $\rightarrow$ (a) is a special case of [**28**, 3.4].

If $\mathscr{P} \neq \mathscr{R}$, then a theorem analogous to 4.4 is not necessary true. In fact, if $\mathscr{P}$ is ultrarealcompactness, T is the real line and $S = ET$, then T is the image of S under a perfect map and by 2.7 $\mathscr{P}(I \times S) = I \times S$ $(= \mathscr{P}I \times \mathscr{P}S)$, but it follows from 2.3 and Glicksberg's theorem (cf. [**8**, p. 298]) that

$$\mathscr{P}(I \times T) = \beta(I \times T) \neq \beta I \times \beta T = \mathscr{P}I \times \mathscr{P}T.$$

For an extension property $\mathscr{P}$, the class of $\mathscr{P}$-regular spaces X for which $\mathscr{P}X = \beta_{\mathscr{P}}X$ is denoted by $\mathscr{P}'$. In [**37**, 2.10], Woods proved that if $\mathscr{P}$-regularity is 0-dimensionality, then either $\mathscr{P} = \mathscr{C}_{\mathscr{P}}$ or $\mathscr{P}'$ does not properly contain the class of pseudocompact $\mathscr{P}$-regular spaces, and Broverman remarked in [**3**] that this result is not valid for arbitrary extension properties. If we denote the class of $\mathscr{P}$-regular spaces X for which $\mathscr{P}(EX) = \beta(EX)$ by $\mathscr{P}''$, then we have the following theorem.

4.5 THEOREM. *Let $\mathscr{P}$ be an extension property. Then:*

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

(a) $\mathscr{P}'' = \mathscr{P}' \cap \mathscr{P}^*$.

(b) *Either* $\mathscr{P} = \mathscr{C}_{\mathscr{P}}$ *or* $\mathscr{P}''$ *does not properly contain the class of pseudocompact* $\mathscr{P}$*-regular spaces.*

Proof. (a) Let $X \in \mathscr{P}''$. Then, since

$$\mathscr{P}X \supset (\mathscr{P}k_X)[\mathscr{P}(EX)] = (\mathscr{P}k_X)[\beta(EX)],$$

$\mathscr{P}X$ is compact, so $X \in \mathscr{P}'$. From this fact and 1.2 (a),

$$\mathscr{P}(EX) = \beta(EX) = E(\beta_{\mathscr{P}}X) = E(\mathscr{P}X),$$

and hence $X \in \mathscr{P}^*$. Conversely, if $X \in \mathscr{P}' \cap \mathscr{P}^*$, then it follows from 1.2(a) that

$$\mathscr{P}(EX) = E(\mathscr{P}X) = E(\beta_{\mathscr{P}}X) = \beta(EX),$$

i.e., $X \in \mathscr{P}''$.

(b) Assume that $\mathscr{P} \neq \mathscr{C}_{\mathscr{P}}$, and choose a $\mathscr{P}$-space X not in $\mathscr{C}_{\mathscr{P}}$. If $N \notin \mathscr{P}$, then X is pseudocompact by 1.1(d), but $X \notin \mathscr{P}''$ since $\mathscr{P}(EX) = EX \neq \beta(EX)$. If $N \in \mathscr{P}$, then $\mathscr{E}\mathscr{R} \subset \mathscr{E}\mathscr{P}$ by 3.6, and hence it follows from [**11**, 8A4] that each space in $\mathscr{P}''$ is pseudocompact. In any case, $\mathscr{P}''$ does not properly contain the class of pseudocompact $\mathscr{P}$-regular spaces.

5. Examples and questions.

5.1. *Example.* There exists a $P_z(\aleph_1)$-compact space but not ultrarealcompact.

Proof. Let $X = \cup\{I_n \cup S_n | n \in N\}$, where I_n and S_n are subspaces of the Euclidean plane as follows:

$$I_n = \{(x, y) | x = 1/n, 0 \leqq y \leqq 1\},$$
$$S_n = \{(x, y) | x^2 + y^2 = 1/n^2, \text{ and } x \leqq 0 \text{ or } y \leqq 0\}.$$

Then, each $I_n \cup S_n$ being a compact zero-set, X is $P_z(\aleph_1)$-compact. Since X is connected but not compact, it is not ultrarealcompact.

5.2. *Example.* Condition (2) does not imply (2′) even when $\mathscr{P} = \mathscr{R}$.

Proof. Let Y be the Tychonoff Plank (cf. [**11**, 8.20]), and let $E = \{\omega_1\} \times N$ be the right edge of Y. Since Y is pseudocompact, it follows from 3.7 that Y satisfies (2) for $\mathscr{R}$. Let $X = Y \oplus E$, and let $f: X \to Y$ be the natural map. Then f is perfect onto, but $\mathscr{R}f: \upsilon X \to \upsilon Y$ is not even a closed map, because $\upsilon X = \upsilon Y \oplus E$.

5.3. *Example.* There exists a space Y such that $\upsilon(EY) = E(\upsilon Y)$ but υY is not locally compact.

Proof. Let W be the space of all countable ordinals with the order topology and Q the space of rational numbers. Set $Y = W \times Q$. Then a

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

similar argument to that of [**11**, 8.20] shows that $\upsilon Y = \beta W \times Q$, so υY is not locally compact. Since the projection from Y to Q is a closed map with countably compact fibers, it is easily checked that Y satisfies (8) for $\mathscr{R}$, and hence it follows from 3.8 (or [**12**, 2.4]) that $\upsilon(EY) = E(\upsilon Y)$.

5.4. *Example.* If there exists a measurable cardinal, then there exists an extension property which fails to satisfy (A_2).

Proof. Let $\mathscr{M}$ be the class of spaces which are embedded as a closed subspace in a product of spaces of nonmeasurable cardinal. Then $\mathscr{M}$ is an extension property and, by Tychonoff's theorem, $\mathscr{M}$-regularity is just complete regularity. Clearly, every space of nonmeasurable cardinal has $\mathscr{M}$. Let D be the discrete space of measurable cardinal; then $D \notin \mathscr{R}$ by [**11**, 12.2]. If D is a closed subspace in a product of spaces of nonmeasurable cardinal, then D remains homeomorphic and closed if one changes the topology of each factor to the discrete topology, so D must be realcompact by [**11**, 12.2]. This contradiction shows that D is an $\mathscr{M}$-regular space not in $\mathscr{M}$, and hence $\mathscr{M}$ does not satisfy (A_2).

5.5. *Question.* Does every almost realcompact space have $\mathscr{M}$? This question is closely related to the questions asked by Hušek in [**16**, p. 43].

5.6. *Example.* If there exists a measurable cardinal, then there exists a space X such that $\mu X = pX$ but $\mu(EX) \neq E(\mu X)$.

Proof. Let W^* be the space of all ordinals less than or equal to the first uncountable ordinal ω_1 with the order topology and let $\omega N = N \cup \{\infty\}$ be the one-point compactification of N. Let K be the quotient space obtained from $W^* \oplus \omega N$ by identifying ω_1 and ∞ and let $\psi: W^* \oplus \omega N \rightarrow K$ be the quotient map. Let D be the discrete space of measurable cardinal; then $D = \mu D \neq \upsilon D$. Let

$$X = (K \times \upsilon D) - (\psi[\omega N] \times (\upsilon D - D)).$$

Then it follows from [**6**, 5.3] that

$$\upsilon X = \upsilon(K \times D) = K \times \upsilon D, \quad \text{and}$$

$$\mu X = \upsilon X - (\psi[N] \times (\upsilon D - D))$$

since it is the smallest Dieudonné complete subspace of υX containing X. Following [**29**], let mX denote the largest subspace S of βX containing X such that $X \times M$ is C^*-embedded in $S \times M$ for each metric space M. Since $K \times D$ is paracompact, it follows from [**28**, 3.5 (1)] that $m(K \times D) = \upsilon(K \times D)$, so $mX = \upsilon X$, and hence $pX = mX \cap \mu X = \mu X$ by [**29**, Corollary 1]. If we set

$$\mathfrak{U} = \{\psi[N] \times \{d\} | d \in D\},$$

then $\mathfrak{U}$ is a locally finite family of open sets in X, but it is not locally finite

https://doi.org/10.4153/CJM-1982-088-4 Published online by Cambridge University Press

at any point of $\mu X - X$. Hence it follows from 3.11.1 that $\mu(EX) \neq E(\mu X)$.

A continuous image of I containing two distinct points is called a *non-trivial arc*. Let $\mathscr{P}$ be the class of compact spaces containing no non-trivial arcs. Then $R(\mathscr{P})$ is known to be the largest extension property whose regularity is not complete regularity (cf. [**15**, p. 329]). We conclude this paper by asking a question about this property.

5.7. *Question*. Does every space containing no non-trivial arcs belong to $R(\mathscr{P})$? In other words, does every space containing no non-trivial arcs have a compactification possessing the same property?

The referee kindly informed me that $\beta\mathbf{R}^+ - \mathbf{R}^+$, where $\mathbf{R}^+$ is the space of non-negative real numbers, is an example of a compact connected space containing no non-trivial arcs. The author wishes to thank the referee for his helpful suggestions.

References

1. A. V. Arhangel'skiĭ, *On a class of spaces containing all metric spaces and all locally bi-compact spaces*, Amer. Math. Soc. Transl. (2) *92* (1970), 1–39.

2. B. Banaschewski, *Über nulldimensionale Raüme*, Math. Nachr. *13* (1955), 129–140.

3. S. Broverman, *Pseudocompactness properties*, Proc. Amer. Math. Soc. *59* (1976), 175–178.

4. ——— *N-compactness and weak homogeneity*, Proc. Amer. Math. Soc. *62* (1977), 173–176.

5. Kim-Peu Chew, *A characterization of N-compact spaces*, Proc. Amer. Math. Soc. *26* (1970), 679–682.

6. W. W. Comfort and S. Negrepontis, *Extending continuous functions on $X \times Y$ to subsets of $\beta X \times \beta Y$*, Fund. Math. *59* (1966), 1–12.

7. N. Dykes, *Mappings and realcompact spaces*, Pacific J. Math. *31* (1969), 347–358.

8. R. Engelking, *General topology* (Polish Scientific Publishers, Warszawa, 1977).

9. R. Engelking and S. Mrówka, *On E-compact spaces*, Bull. Acad. Polon. Sci. *6* (1958), 429–436.

10. Z. Frolík, *A generalization of realcompact spaces*, Czech. Math. J. *13* (1963), 123–137.

11. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, N.J., 1960).

12. K. Hardy and R. G. Woods, *On c-realcompact spaces and locally bounded normal functions*, Pacific J. Math. *43* (1972), 647–656.

13. H. Herrlich, *$\mathfrak{E}$-kompakte Raüme*, Math. Z. *96* (1967), 228–255.

14. H. Herrlich and J. van der Slot, *Properties which are closely related to compactness*, Indag. Math. *29* (1967), 524–529.

15. S. S. Hong, *On $\mathfrak{k}$-compactlike spaces and reflective subcategories*, Gen. Top. Appl. *3* (1973), 319–330.

16. M. Hušek, *Perfect images of E-compact spaces*, Bull. Acad. Polon. Sci. *20* (1972), 41–45.

17. S. Iliadis and S. Fomin, *The methods of centered systems in the theory of topological spaces*, Russian Math. Surveys *21* (1966), 37–62.

18. T. Isiwata, *Generalizations of M-spaces I, II*, Proc. Japan Acad. Ser. A Math. Sci. *45* (1969), 359–363, 364–367.

19. J. E. Mack and D. G. Johnson, *The Dedekind completion of* $C(X)$, Pacific J. Math. *20* (1967), 231–243.
20. E. A. Michael, *Bi-quotient maps and cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble), *18* (1968), 287–302.
21. ——— *A quituple quotient quest*, Gen. Top. Appl. *2* (1972), 91–138.
22. K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. *154* (1964), 365–382.
23. ——— *Topological completions and M-spaces*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A *10* (1970), 271–288.
24. S. Mrówka, *On E-compact spaces II*, Bull. Acad. Polon. Sci. *14* (1966), 597–605.
25. ——— *Further results on E-compact spaces. I*, Acta Math. *120* (1968), 161–185.
26. H. Ohta, *On Hewitt realcompactifications of products*, Ph.D. Thesis at the University of Tsukuba, (1979).
27. ——— *Topologically complete spaces and perfect maps*, Tsukuba J. Math. *1* (1977), 77–89.
28. ——— *The Hewitt realcompactification of products*, Trans. Amer. Math. Soc. *263* (1981), 363–375.
29. T. C. Przymusiński, *Extending functions from products with a metric factor and absolutes*, Pacific J. Math. *101* (1982), 463–475.
30. F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, Gen. Top. Appl. *1* (1971), 143–154.
31. F. Siwiec and V. J. Mancuso, *Relations among certain mappings and conditions for their equivalence*, Gen. Top. Appl. *1* (1971), 34–41.
32. D. Strauss, *Extremally disconnected spaces*, Proc. Amer. Math. Soc. *18* (1967), 305–309.
33. T. Terada, *New topological extension properties*, Proc. Amer. Math. Soc. *67* (1977), 162–166.
34. R. G. Woods, *Co-absolutes of remainders of Stone-Čech compactifications*, Pacific J. Math. *37* (1971), 545–560.
35. ——— *Ideals of pseudocompact regular closed sets and absolutes of Hewitt realcompactifications*, Gen. Top. Appl. *2* (1972), 315–331.
36. ——— *A Tychonoff almost realcompactification*, Proc. Amer. Math. Soc. *43* (1974), 200–208.
37. ——— *Topological extension properties*, Trans. Amer. Math. Soc. *210* (1975), 365–385.
38. ——— *A survey of absolutes of topological spaces*, Topological Structures II, Math. Centre Tracts *116* (1979), 323–362.

Shizuoka University,
Ohya, Shizuoka, Japan