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# THE JOIN OF SPLIT GRAPHS WHOSE QUASI-STRONG ENDOMORPHISMS FORM A MONOID

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**Abstract**

In this paper, we characterise the quasi-strong endomorphisms of the join of split graphs. We give conditions under which the quasi-strong endomorphisms of the join of split graphs form a monoid.

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## 1. Introduction and preliminaries

Endomorphism monoids of graphs are generalisations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained (see [4, 9–11, 13] and references therein). The aim of this research is to develop further relations between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. Hou, Luo and Cheng [5] explored the endomorphism monoid of $\overline{P_n}$, the complement of a path $P_n$ with $n$ vertices. It was shown that $End(\overline{P_n})$ is an orthodox monoid. The endomorphism spectrum and the endomorphism type of $\overline{P_n}$ were given. The endomorphism monoids and endomorphism regularity of split graphs have been considered by several authors (see [2, 6, 14]).

Let $X$ be a graph. Denote by $End(X)$, $hEnd(X)$, $lEnd(X)$, $qEnd(X)$, $sEnd(X)$ and $Aut(X)$ the sets of all endomorphisms, half-strong endomorphisms, locally strong endomorphisms, quasi-strong endomorphisms, strong endomorphisms and automorphisms of $X$, respectively. It is well known that $End(X)$ and $sEnd(X)$ form monoids with respect to composition of mappings and $Aut(X)$ forms a group. However, $hEnd(X)$, $lEnd(X)$ and $qEnd(X)$ do not form monoids in general (see [1]). So Böttcher and Knauer in [1] posed a question: under what conditions do the sets $hEnd(X)$,

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$lEnd(X)$ and $qEnd(X)$ form monoids for a graph $X$? It seems difficult to obtain a general answer to this question. So the strategy for answering the question is to find various kinds of conditions for various kinds of graphs. In [15], Luo *et al.* give an answer to this question in the range of split graphs. In [7], Hou *et al.* explored the half-strong endomorphisms of a join of split graphs and the conditions under which the half-strong endomorphisms of a join of split graphs form a monoid were given. In this paper, we characterise the quasi-strong endomorphisms of a join of split graphs. We give conditions under which the quasi-strong endomorphisms of a join of split graphs form a monoid.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let $X$ be a graph. The vertex set of $X$ is denoted by $V(X)$ and the edge set of $X$ is denoted by $E(X)$ (or simply $E$). If two vertices $x_1$ and $x_2$ are adjacent in the graph $X$, the edge connecting $x_1$ and $x_2$ is denoted by $\{x_1, x_2\}$ and we write $\{x_1, x_2\} \in E(X)$. For a vertex $v$ of $X$, denote by $N_X(v)$ (or simply $N(v)$) the set $\{x \in V(X) | \{x, v\} \in E(X)\}$. A subgraph $H$ is called an *induced subgraph* of $X$ if for any $a, b \in H$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$.

Let $X$ be a graph. A subset $K \subseteq V(X)$ is said to be *complete* if $\{a, b\} \in E(X)$ for any two vertices $a, b \in K$. A subset $S \subseteq V(X)$ is said to be *independent* if $\{a, b\} \notin E(X)$ for any two vertices $a, b \in S$. A *clique* of a graph $X$ is the maximal complete subgraph of $X$. The *clique number* of $X$, denoted by $\omega(X)$, is the maximal order among the cliques of $X$. Let $X$ and $Y$ be two graphs. The *join* of $X$ and $Y$, denoted by $X + Y$, is a graph with $V(X + Y) = V(X) \cup V(Y)$ and $E(X + Y) = E(X) \cup E(Y) \cup \{\{a, b\} | a \in V(X), b \in V(Y)\}$. A graph $X$ is called a *split graph* if its vertex set $V(X)$ can be partitioned into disjoint (nonempty) sets $S$ and $K$, such that $S$ is an independent set and $K$ is a complete set. In the following, we suppose that $K$ is a maximal complete set of $X$. It is easy to see that for any $y \in S$, $0 \leq d_X(y) \leq n - 1$, where $n = |K|$.

Let $X$ and $Y$ be two graphs. A mapping $f$ from $V(X)$ to $V(Y)$ is called a *homomorphism* (from $X$ to $Y$) if $\{a, b\} \in E(X)$ implies that $\{f(a), f(b)\} \in E(Y)$. A homomorphism $f$ from $X$ to itself is called an *endomorphism* of $X$. Denote by $End(X)$ the set of all endomorphisms of $X$. It is known that $End(X)$ forms a monoid with respect to the composition of mappings and is called the endomorphism monoid (or simply monoid) of $X$. A homomorphism $f$ is called a *half-strong homomorphism* if $\{f(a), f(b)\} \in E(Y)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(x_1) = f(a)$ and $f(x_2) = f(b)$ such that $\{x_1, x_2\} \in E(X)$. A homomorphism $f$ is called a *quasi-strong homomorphism* if $\{f(a), f(b)\} \in E(Y)$ implies that there exists a preimage $x_1 \in V(X)$ of $f(a)$ which is adjacent to every preimage of $f(b)$ and analogously for preimage of $f(b)$.

Let $f$ be an endomorphism of a graph $X$. A subgraph of $X$ is called the *endomorphic image* of $X$ under $f$, denoted by $I_f$, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By $\rho_f$ we denote the equivalence relation on $V(X)$ induced by $f$, that is, for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to $\rho_f$.

The reader is referred to [3, 8, 9, 12] for all the notation and terminology not defined here.

## 2. Main results

Let $X$ be a split graph with $V(X) = K_1 \cup S_1$, where $S_1$ is an independent set and $K_1$ is a maximal complete set. Let $Y$ be another split graph with $V(Y) = K_2 \cup S_2$, where $S_2$ is an independent set and $K_2$ is a maximal complete set. Denote $n = |K_1|$ and $m = |K_2|$. Then the vertex set $V(X + Y)$ of $X + Y$ can be partitioned into three parts $K$, $S_1$ and $S_2$, that is, $V(X + Y) = K \cup S_1 \cup S_2$, where $K = K_1 \cup K_2$ is a complete set, and $S_1$ and $S_2$ are independent sets. Obviously the subgraph of $X + Y$ induced by $K$ is a complete graph and the subgraph of $X + Y$ induced by $S = S_1 \cup S_2$ is a complete bipartite graph. Hence in the graph $X + Y$, $N(x_i) = N_X(x_i) \cup V(Y)$ for $x_i \in S_1$ and $N(y_i) = N_Y(y_i) \cup V(X)$ for $y_i \in S_2$. It is easy to see that $X + Y$ is a split graph adding to the edge set $\{\{x_i, y_j\} \mid x_i \in S_1, y_j \in S_2\}$.

In this section, we will explore the quasi-strong endomorphisms of a join of split graphs. We give conditions under which the quasi-strong endomorphisms of a join of split graphs form a monoid. The following theorem is our main result.

Theorem 2.1. *Let $X + Y$ be a join of split graphs. Then $qEnd(X + Y)$ forms a monoid if and only if $X + Y$ is qs-monoidal.*

To prove our main result, we need the following characterisations of the quasi-strong endomorphisms of these graphs.

Lemma 2.2. *Let $f$ be an endomorphism of a graph $G$. If $f$ is quasi-strong, then the subgraph of $G$ induced by $f^{-1}(a) \cup f^{-1}(b)$ has no isolated vertex for any $a, b \in V(G) \cap I_f$ with $\{a, b\} \in E$.*

Proof. Let $f \in qEnd(G)$ and $a, b \in V(G) \cap I_f$ be such that $\{a, b\} \in E$. Then there exists $x_1 \in f^{-1}(a)$ such that $\{x_1, y_1\} \in E$ for any $y_1 \in f^{-1}(b)$. Similarly, there exists $y_2 \in f^{-1}(b)$ such that $\{x_2, y_2\} \in E$ for any $x_2 \in f^{-1}(a)$. Therefore the subgraph of $G$ induced by $f^{-1}(a) \cup f^{-1}(b)$ has no isolated vertex. □

Lemma 2.3. *Let $X + Y$ be a join of split graphs. If $f \in qEnd(X + Y)$ and $f(x) = f(y)$ for some $x \in K$ and $y \in S$, then $N(y) \cap K = K \setminus \{x\}$.*

Proof. Let $x \in K$ and $y \in S$ such that $f(x) = f(y)$. If $\{x, y\} \in E$, then $f(x)$ is a loop in $X + Y$, which is a contradiction. Therefore $\{x, y\} \notin E$. If $N(y) \cap K \neq K \setminus \{x\}$, then $|N(y) \cap K| < n + m - 1$. Thus there exists $x_1 \in K \setminus ((N(y) \cap K) \cup \{x\})$. Since $x \neq x_1$ and $x, x_1 \in K$, $\{x, x_1\} \in E$. Thus $\{f(x), f(x_1)\} \in E$. We will show that $[x]_{\rho_f} \cup [x_1]_{\rho_f}$ contains an isolated vertex, which is a contradiction.

If $[x_1]_{\rho_f} = \{x_1\}$, then $y$ is an isolated vertex of the subgraph of $X + Y$ induced by $[x]_{\rho_f} \cup [x_1]_{\rho_f}$. If $|[x_1]_{\rho_f}| \neq 1$, without loss of generality, we suppose that $[x_1]_{\rho_f} = \{x_1, y_{11}, \dots, y_{1s}\}$ for some $y_{11}, \dots, y_{1s} \in S$. Since $\{x_1, y\} \notin E$ and $\{x_1, y_{1k}\} \notin E$ for any $1 \leq k \leq s$, $y$ and $y_{1k}$ lie in the same $S_i$ $(i = 1, 2)$. Thus $\{y, y_{1k}\} \notin E$. Hence $y$ is an isolated vertex of the subgraph of $X + Y$ induced by $[x]_{\rho_f} \cup [x_1]_{\rho_f}$. By Lemma 2.2, $f \notin qEnd(X + Y)$, which is a contradiction. Therefore $N(y) \cap K = K \setminus \{x\}$. □

Lemma 2.4. *Let $X + Y$ be a join of split graphs. If $f \in qEnd(X + Y)$ and $f(y_1) = f(y_2)$ for some $y_1, y_2 \in S$, then $N(y_1) = N(y_2)$.*

Proof. If there exists $x \in K \cap [y_1]_{\rho_f}$, then $N(y_1) \cap K = N(y_2) \cap K = K \setminus \{x\}$. Hence $N(y_1) = N(y_2)$. If $[y_1]_{\rho_f} \subseteq S$, suppose that $N(y_1) \neq N(y_2)$. Without loss of generality, let $x_1 \in (N(y_2) \cap K) \setminus (N(y_1) \cap K)$. Then $\{f(x_1), f(y_1)\} = \{f(x_1), f(y_2)\} \in E$. Let $y \in [x_1]_{\rho_f} \cap S$. Then $\{x_1, y\} \notin E$. Note that $\{x_1, y_1\} \notin E$. Then $y$ and $y_1$ lie in the same $S_i$ $(i = 1, 2)$. Hence $\{y, y_1\} \notin E$ for any $y \in [x_1]_{\rho_f} \cap S$. Thus $y_1$ is an isolated vertex of the subgraph of $X + Y$ induced by $[y_1]_{\rho_f} \cup [x_1]_{\rho_f}$. By Lemma 2.2, $f \notin qEnd(X + Y)$, which is a contradiction. Therefore $N(y_1) = N(y_2)$. □

Lemma 2.5. *Let $X + Y$ be a join of split graphs and $f \in End(X + Y)$. Then $f \in qEnd(X + Y)$ if and only if for any $a, b \in V(X + Y)$ with $\{f(a), f(b)\} \in E$, one of the following conditions holds:*

(1) *$[a]_{\rho_f} = \{x_1, y_{11}, \ldots, y_{1s}\}$ and $[b]_{\rho_f} = \{x_2, y_{21}, \ldots, y_{2t}\}$ for some $x_1, x_2 \in K$ with $x_1 \neq x_2$, $s, t \geq 0$ (where $s = 0$ means $[a]_{\rho_f} = \{x_1\}$ and $t = 0$ means $[b]_{\rho_f} = \{x_2\}$), $y_{1i} \in S$ with $N(y_{1i}) \cap K = K \setminus \{x_1\}$ for $i = 1, \ldots, s$ and $y_{2j} \in S$ with $N(y_{2j}) \cap K = K \setminus \{x_2\}$ for $j = 1, \ldots, t$.*

(2) *$[a]_{\rho_f} = \{x\}$ and $[b]_{\rho_f} = \{y_{31}, \ldots, y_{3r}\}$ for some $r \geq 1$, $x \in K_i$ with $i \in \{1, 2\}$, $y_{3j} \in S_i$ with $x \in N(y_{3j})$ for $j = 1, \ldots, r$ and $N(y_{3u}) = N(y_{3v})$ for $u, v = 1, \ldots, r$.*

(3) *$[a]_{\rho_f} = \{y_{41}, \ldots, y_{4p}\}$ and $[b]_{\rho_f} = \{y_{51}, \ldots, y_{5q}\}$ for some $p, q \geq 1$, $y_{4i} \in S$ with $N(y_{4i}) = N(y_{4j})$ for $i, j = 1, \ldots, p$, $y_{5k} \in S$ with $N(y_{5k}) = N(y_{5l})$ for $k, l = 1, \ldots, q$ and $\{y_{4u}, y_{5v}\} \in E$ for any $u = 1, \ldots, p$ and $v = 1, \ldots, q$.*

(4) *$[a]_{\rho_f} = \{x_3, y_{61}, \ldots, y_{6d}\}$ and $[b]_{\rho_f} = \{y_{71}, \ldots, y_{7e}\}$ for some $d \geq 0$ (where $d = 0$ means $[a]_{\rho_f} = \{x_3\}$), $e \geq 1$, $x_3 \in K_i$ with $i \in \{1, 2\}$, $y_{6t} \in S_i$ with $N(y_{6t}) \cap K = K \setminus \{x_3\}$ for $t = 1, \ldots, d$, $y_{7u} \in S_j$ with $N(y_{7u}) = N(y_{7v})$ for $u, v = 1, \ldots, e$ (where $j \in \{1, 2\}$ and $i \neq j$).*

Proof. Necessity. Let $f \in qEnd(X + Y)$ and $\{f(a), f(b)\} \in E$ for some $a, b \in V(X + Y)$. There are three cases:

*Case 1.* If $[a]_{\rho_f} \cap K \neq \emptyset$ and $[b]_{\rho_f} \cap K \neq \emptyset$, then there exist $x_1 \in [a]_{\rho_f} \cap K$ and $x_2 \in [b]_{\rho_f} \cap K$. Without loss of generality, we may assume that $[a]_{\rho_f} = \{x_1, y_{11}, \ldots, y_{1s}\}$ and $[b]_{\rho_f} = \{x_2, y_{21}, \ldots, y_{2t}\}$ for some $x_1, x_2 \in K$, $s \geq 0$ ($s = 0$ means $[a]_{\rho_f} = \{x_1\}$) and $t \geq 0$ ($t = 0$ means $[a]_{\rho_f} = \{x_2\}$). By Lemma 2.3, $N(y_{1i}) \cap K = K \setminus \{x_1\}$ for $i = 1, \ldots, s$ and $N(y_{2j}) \cap K = K \setminus \{x_2\}$ for $j = 1, \ldots, t$. So (1) holds.

*Case 2.* If $[a]_{\rho_f} \subseteq S$ and $[b]_{\rho_f} \subseteq S$, then we can assume that $[a]_{\rho_f} = \{y_{41}, \ldots, y_{4p}\}$ and $[b]_{\rho_f} = \{y_{51}, \ldots, y_{5q}\}$ for some $p, q \geq 1$. By Lemma 2.4, $N(y_{4i}) = N(y_{4j})$ for $i, j = 1, \ldots, p$ and $N(y_{5k}) = N(y_{5l})$ for $k, l = 1, \ldots, q$. Since $f$ is quasi-strong, $\{y_{4u}, y_{5v}\} \in E$ for any $u = 1, \ldots, p$ and $v = 1, \ldots, q$. So (3) holds.

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*Case 3.* Assume that $[a]_{\rho_f} \cap K \neq \emptyset$ and $[b]_{\rho_f} \subseteq S$, or $[b]_{\rho_f} \cap K \neq \emptyset$ and $[a]_{\rho_f} \subseteq S$. Without loss of generality, we may suppose that $[a]_{\rho_f} \cap K \neq \emptyset$ and $[b]_{\rho_f} \subseteq S$. Then there exists $x_3 \in [a]_{\rho_f} \cap K_i$ for some $i \in \{1, 2\}$. Let $[a]_{\rho_f} = \{x_3, y_{61}, \ldots, y_{6d}\}$ and $[b]_{\rho_f} = \{y_{71}, \ldots, y_{7e}\}$ for some $d \geq 0$ (where $d = 0$ means $[a]_{\rho_f} = \{x_3\}$) and $e \geq 1$. By Lemma 2.3 $N(y_{6t}) \cap K = K \setminus \{x_3\}$ for $t = 1, \ldots, d$ and by Lemma 2.4 $N(y_{7u}) = N(y_{7v})$ for $u, v = 1, \ldots, e$.

(i) If $y_{7u} \in S_j$ for $u = 1, \ldots, e$ (where $j \in \{1, 2\}$ and $j \neq i$), then (4) holds.

(ii) If $y_{7u} \in S_i$ for $u = 1, \ldots, e$, then $d = 0$. Otherwise, there exists $y_{6d} \in [a]_{\rho_f}$ such that $\{y_{6d}, y_{7u}\} \notin E$ for any $u = 1, \ldots, e$. Thus $y_{6d}$ is an isolated vertex of the subgraph of $X + Y$ induced by $[a]_{\rho_f} \cup [b]_{\rho_f}$. This is a contradiction. Hence (2) holds.

Sufficiency. Let $f \in End(X + Y)$. For any $a, b \in V(X + Y)$ with $\{f(a), f(b)\} \in E$, if $f$ satisfies condition (1), then $x_1 \neq x_2$, $x_1$ is adjacent to every vertex of $[b]_{\rho_f}$ and $x_2$ is adjacent to every vertex of $[a]_{\rho_f}$. If $f$ satisfies condition (2) or (3), then every vertex of $[b]_{\rho_f}$ is adjacent to every vertex of $[a]_{\rho_f}$ and every vertex of $[a]_{\rho_f}$ is adjacent to every vertex of $[b]_{\rho_f}$. If $f$ satisfies condition (4), then $x$ is adjacent to every vertex of $[b]_{\rho_f}$ and every vertex of $[b]_{\rho_f}$ is adjacent to $x$. Hence $f$ is quasi-strong. □

Let $f \in qEnd(X + Y)$ and $\{f(a), f(b)\} \in E$ for some $a, b \in V(X + Y)$. In view of Lemma 2.5, (1) if there exists $x \in K \cap [a]_{\rho_f}$, then $x$ is adjacent to every vertex of $[b]_{\rho_f}$; (2) if $[a]_{\rho_f} \subseteq S$, then every vertex of $[a]_{\rho_f}$ is adjacent to every vertex of $[b]_{\rho_f}$.

Now we find conditions for a join of split graphs under which $qEnd(X + Y)$ forms a monoid.

Lemma 2.6. *Let $X + Y$ be a join of split graphs and $i, j \in \{1, 2\}$ with $i \neq j$. Suppose that $X + Y$ satisfies the following conditions:*

(1) *there exists $x_0 \in K_i$ such that $|N(y) \cap K| = n + m - 1$ for any $y \in N(x_0) \cap S_i$;*

(2) *there exists $y_1, y_2 \in S_i$ such that $N(y_1) \cap K = K \setminus \{x_1\}$, $N(y_2) \cap K = K \setminus \{x_2\}$ for some $x_1, x_2 \in K_i$ and $x_1 \neq x_2$;*

(3) *there exists $y_3 \in S_i$ such that $N(y_3) \cap K = K \setminus \{x_0, x_3\}$ for some $x_3 \in K_i$ and $x_0 \neq x_3$;*

(4) *there exists a mapping $h$ from $K \setminus \{x_0\}$ to a clique of $X + Y$ not containing $x_2$ such that $h(x_3) = x_1$ and there exists a mapping $k$ from $S_0 = \{y \in S_i \mid |N(y) \cap K| \leq n + m - 2\}$ to $S_i$ with $k(y_3) = y_1$ such that either $h(N(y) \cap K) = N(k(y)) \cap K$ or $h(N(y) \cap K) = (N(k(y)) \cap K) \setminus \{x_2\}$ for any $y \in S_0$.*

*Then $qEnd(X + Y)$ does not form a monoid.*

Proof. Let $X + Y$ be a join of split graphs satisfying conditions (1)–(4) and let

$$g(z) = \begin{cases} h(z) & z \in K \setminus \{x_0\}, \\ y_2 & z = x_0, \\ y_1 & z = y_3, \\ h(x) & z \in S \text{ with } N(z) \cap K = K \setminus \{x\} \text{ for some } x \in K, \\ k(z) & z \in S_0, \\ z & \text{otherwise.} \end{cases}$$

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Then it is easy to check that $g \in qEnd(X + Y)$. Let $f$ be an endomorphism of $X + Y$ such that $f(y_1) = x_1$, $f(y_2) = x_2$ and $f(z) = z$ if $z \neq y_1, y_2$. Then $f \in qEnd(X + Y)$. Now $fg(y_3) = fg(x_3) = x_1$ and $|N(y_3) \cap K| = n + m - 2$. By Lemma 2.3, $fg$ is not quasi-strong. Therefore, $qEnd(X + Y)$ does not form a monoid. □

For $i, j \in \{1, 2\}$ and $i \neq j$, denote $S_{01} = \{y \in S_i \mid |N(y) \cap K| \leq n + m - 2\}$, $S_{02} = \{y \in S_j \mid |N(y) \cap K| \leq n + m - 2\}$ and $S'_0 = S_{01} \cup S_{02}$.

Lemma 2.7. *Let $X + Y$ be a join of split graphs and $i, j \in \{1, 2\}$ with $i \neq j$. Suppose that $X + Y$ satisfies the following conditions:*

(1) *there exists $x_0 \in K_i$ such that $|N(y) \cap K| = n + m - 1$ for any $y \in N(x_0) \cap S_i$;*
(2) *there exists $y_1, y_2 \in S_j$ such that $N(y_1) \cap K = K \setminus \{x_1\}$, $N(y_2) \cap K = K \setminus \{x_2\}$ for some $x_1, x_2 \in K_j$ and $x_1 \neq x_2$;*
(3) *there exists $y_3 \in S_i$ such that $N(y_3) \cap K = K \setminus \{x_0, x_3\}$ for some $x_3 \in K_i$ and $x_0 \neq x_3$;*
(4) *there exists a bijection $h$ from $K \setminus \{x_0\}$ to $K \setminus \{x_2\}$ such that $h(x_3) = x_1$ and there exists a mapping $k$ from $S'_0$ to $S$ with $k(y_3) = y_1$ such that $k(y) \in S_j$ with either $h(N(y)) = N(k(y))$ or $h(N(y)) = N(k(y)) \setminus \{x_2\}$ for any $y \in S_{01}$ and $k(y) \in S_i$ with $h(N(y)) = N(k(y)) \setminus \{x_2\}$ for any $y \in S_{02}$.*

*Then $qEnd(X + Y)$ does not form a monoid.*

Proof. Let $X + Y$ be a join of split graphs satisfying conditions (1)–(4) and let

$$g(z) = \begin{cases} h(z) & z \in K \setminus \{x_0\}, \\ y_2 & z = x_0, \\ y_1 & z = y_3, \\ h(x) & z \in S \text{ with } N(z) \cap K = K \setminus \{x\} \text{ for some } x \in K, \\ k(z) & \text{otherwise.} \end{cases}$$

Then it is easy to check that $g \in qEnd(X + Y)$. Let $f$ be an endomorphism of $X + Y$ such that $f(y_1) = x_1$, $f(y_2) = x_2$ and $f(z) = z$ if $z \neq y_1, y_2$. Then $f \in qEnd(X + Y)$ . Now $fg(y_3) = x_1$ and $(fg)^{-1}(x_1) \not\subseteq S$. Since $|N(y_3) \cap K| \neq n + m - 1$, by Lemma 2.3, $fg$ is not quasi-strong. Therefore, $qEnd(X + Y)$ does not form a monoid. □

A join of split graphs $X + Y$ is said to be *qs-dismonoidal* if $X + Y$ satisfies the conditions stated in Lemma 2.6 or 2.7. Otherwise, $X + Y$ is called *qs-monoidal*. The following example shows that there exists a join of split graphs which is qs-monoidal.

Example 2.8. Let $X$ be a split graph with $K_1 = \{x_1, x_2, x_3, x_4\}$ and $S_1 = \{y_1, y_2, y_3, y_4\}$ such that $N(y_1) = K_1 \setminus \{x_1\}$, $N(y_2) = K_1 \setminus \{x_2\}$, $N(y_3) = \{x_1, x_2\}$ and $N(y_4) = \{x_4\}$. Let $Y$ be another split graph with $K_2 = \{r_1, r_2, r_3\}$ and $S_2 = \{z_1, z_2, z_3\}$ such that $N(z_1) = \{r_1\}$, $N(z_2) = \{r_2\}$ and $N(z_3) = \{r_3\}$. Then $X + Y$ satisfies conditions (1)–(3) stated in Lemma 2.6, $x_3$ is the only vertex in $K$ such that $|N(y) \cap K| = 6$ for any $y \in N(x_3) \cap S$, and $y_3$ is the only vertex in $S$ of degree 5. Let $h$ be a bijection from $K \setminus \{x_3\}$ to $K \setminus \{x_2\}$ such that $h(x_4) = x_1$. Note that $h(N(y_4)) = \{x_1\}$ and there is no vertex $y \in S_1$ such that either $N(y) = \{x_1\}$ or $N(y) = \{x_1, x_2\}$. Then there is no mapping $k$ from $\{y_3, y_4\}$ to $S$

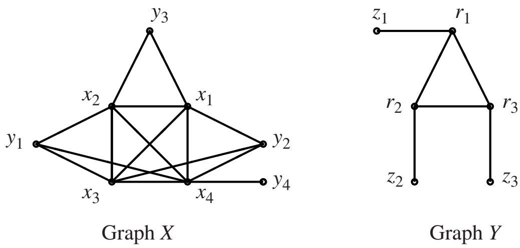


FIGURE 1. Graphs in Example 1.

with $k(y_4) = y_1$ such that either $h(N(y)) = N(k(y))$ or $h(N(y)) = N(k(y)) \setminus \{x_2\}$ for any $y \in \{y_3, y_4\}$. Hence $X + Y$ does not satisfy condition (4) stated in Lemma 2.6. Since there is no $x \in K_2$ such that $|N(y) \cap K| = n + m - 1$ for any $y \in N(x) \cap S_2$, $X + Y$ does not satisfy the conditions stated in Lemma 2.7. Therefore $X + Y$ is qs-monoidal.

LEMMA 2.9. *If $X + Y$ is qs-monoidal, then $qEnd(X + Y)$ forms a monoid.*

PROOF. Let $X + Y$ be a qs-monoidal join of split graphs and $f, g \in qEnd(X + Y)$. We only need to show that $fg \in qEnd(X + Y)$. Let $\{(fg)(a), (fg)(b)\} \in E(X + Y)$ for some $a, b \in V(X + Y)$. We first show that there exists $g(c) \in [g(a)]_{\rho_f} \cap I_g$ such that $g(c)$ is adjacent to every vertex of $[g(b)]_{\rho_f} \cap I_g$ and there exists $g(d) \in [g(b)]_{\rho_f} \cap I_g$ such that $g(d)$ is adjacent to every vertex of $[g(a)]_{\rho_f} \cap I_g$. Since $f$ is quasi-strong, by Lemma 2.5 there are four cases.

*Case 1.* $[g(a)]_{\rho_f} = \{x_1, y_{11}, \dots, y_{1s}\}$ and $[g(b)]_{\rho_f} = \{x_2, y_{21}, \dots, y_{2t}\}$ for some $x_1, x_2 \in K$ and $s, t \geq 0$, $y_{1i}, y_{2j} \in S$, $i = 1, \dots, s$, $j = 1, \dots, t$ such that $N(y_{1i}) \cap K = K \setminus \{x_1\}$ and $N(y_{2j}) \cap K = K \setminus \{x_2\}$. If $x_1$ and $x_2$ lie in the different $K_i$, then the subgraph of $X + Y$ induced by $([g(a)]_{\rho_f} \cap I_g) \cup ([g(b)]_{\rho_f} \cap I_g)$ is isomorphic to a complete bipartite graph. The result holds. In the following, we suppose that $x_1$ and $x_2$ lie in the same $K_i$. Without loss of generality, we can suppose that $K_i = K_1$. Since any endomorphism $f$ maps a clique to a clique of the same size, $I_g$ contains a clique of size $n + m$. Note that any clique of $X + Y$ can miss at most one vertex of $K_1$. Then $\{x_1, x_2\} \cap I_g \neq \emptyset$.

(i) If $x_1 \in I_g$ and $x_2 \in I_g$, then $x_1$ is adjacent to every vertex of $[g(b)]_{\rho_f}$ and $x_2$ is adjacent to every vertex of $[g(a)]_{\rho_f}$.

(ii) If $x_1 \in I_g$ and $x_2 \notin I_g$, then $g(K) \neq K$ and there exists $x_0 \in K$ such that $g(x_0) = y_2 \in S_1$ with $N(y_2) \cap K = K \setminus \{x_2\}$. Since $K$ is a clique of size $n + m$ in $X + Y$, $g(K)$ is also a clique of size $n + m$ in $X + Y$. Note that any clique of size $n + m$ in $X + Y$ can miss at most one vertex of $K_1$. It follows from $x_2 \notin g(K)$ that $x_1 \in g(K)$. Hence there exists $x_3 \in K$ such that $x_1 = g(x_3)$.

If $[g(a)]_{\rho_f} \cap I_g \neq \{x_1\}$, then $\{y_{11}, \dots, y_{1s}\} \cap I_g \neq \emptyset$. Since $\{x_1, y_2\} = \{g(x_3), g(x_0)\} \in E(X + Y)$, we have $y_2 \notin \{y_{11}, \dots, y_{1s}\}$ and so $x_2 \in N(y_{1i})$ for any $i = 1, \dots, s$. Since $x_2 \notin I_g$, $g^{-1}(z) \subseteq S$ for any $z \in \{y_{11}, \dots, y_{1s}\} \cap I_g$. Let $y_1 \in \{y_{11}, \dots, y_{1s}\} \cap I_g$ and $y_1 = g(y_3)$ for some $y_3 \in S$. Since $\{g(x_3), g(y_3)\} = \{x_1, y_1\} \notin E(X + Y)$ and $\{g(y_3), g(x_0)\} = \{y_1, g(x_0)\} \notin E(X + Y)$, we have $x_0, x_3 \notin N(y_3)$.

If $x_0 \in K_1$, then $y_3 \in S_1$ and so $x_3 \in K_1$. Now $\{g(x_0), g(y)\} \in E(X+Y)$ implies that $g(y) \in K$ or $g(y) \in S_2$ for any $y \in N(x_0) \cap S_1$. If $g(y) \in S_2$, then $\{g(y), g(y_3)\} = \{g(y), y_1\} \in E$. Note that $\{y, y_3\} \notin E$, in contradiction to $g$ being quasi-strong. Hence $g(y) \in K$ for any $y \in N(x_0) \cap S_1$. By Lemma 2.3 we have $|N(y) \cap K| = n+m-1$ for any $y \in N(x_0) \cap S_1$. Note that $N(g(y_3)) \cap K = N(y_1) \cap K = K \setminus \{x_1\}$. Hence $y_3$ is adjacent to all vertices in $K \setminus \{x_0, x_3\}$ since $g$ is quasi-strong and so $N(y_3) \cap K = K \setminus \{x_0, x_3\}$. Since $g$ is quasi-strong, for any $y \in S$ with $|N(y) \cap K| \le n+m-2$, by Lemma 2.3 we have $[y]_{\rho_g} \subseteq S$ and $g(y) \in S$. Hence $A^g_{g(y)} = N(y)$ by Lemma 2.4. Note that $g$ is half-strong. Then $g(N(y)) = g(A^g_{g(y)}) = N(g(y)) \cap I_g$. It follows that $g(N(y)) = N(g(y))$ or $g(N(y)) = N(g(y)) \setminus \{x_2\}$. Consequently, $X+Y$ satisfies the conditions of Lemma 2.6 and so $X+Y$ is qs-dismonoidal. This is a contradiction. Thus we must have $[g(a)]_{\rho_f} \cap I_g = \{x_1\}$. Therefore the subgraph of $X$ induced by $([g(a)]_{\rho_f} \cap I_g) \cup ([g(b)]_{\rho_f} \cap I_g) = \{x_1\} \cup (\{y_{21}, \ldots, y_{2t}\} \cap I_g)$ is a complete bipartite graph.

If $x_0 \in K_2$, then $y_3 \in S_2$ and so $x_3 \in K_2$. Now $\{g(x_0), g(y)\} = \{y_2, g(y)\} \in E(X+Y)$ implies that $g(y) \in K$ or $g(y) \in S_2$ for any $y \in N(x_0) \cap S_2$. If $g(y) \in S_2$, then $\{g(y), g(y_3)\} = \{g(y), y_1\} \in E$. Note that $\{y, y_3\} \notin E$, in contradiction to $g$ being quasi-strong. Hence $g(y) \in K$ for any $y \in N(x_0) \cap S_2$. By Lemma 2.3 we have $|N(y) \cap K| = n+m-1$ for any $y \in N(x_0) \cap S_2$. Note that $N(g(y_3)) = N(y_1) = K \setminus \{x_1\}$. Hence $y_3$ is adjacent to all vertices in $K \setminus \{x_0, x_3\}$ since $g$ is quasi-strong and so $N(y_3) \cap K = K \setminus \{x_0, x_3\}$. Since $g$ is quasi-strong, for any $y \in S$ with $|N(y) \cap K| \le n+m-2$, by Lemma 2.3 we have $[y]_{\rho_g} \subseteq S$ and $g(y) \in S$. Hence $A^g_{g(y)} = N(y)$ by Lemma 2.4. Note that $g$ is half-strong. Then $g(N(y)) = g(A^g_{g(y)}) = N(g(y)) \cap I_g$. Denote $S_{01} = \{y \in S_2 \mid |N(y) \cap K| \le n+m-2\}$, $S_{02} = \{y \in S_1 \mid |N(y) \cap K| \le n+m-2\}$ and $S'_0 = S_{01} \cup S_{02}$. Since $g$ is quasi-strong, $g(y) \in S_1$ with either $g(N(y)) = N(g(y))$ or $g(N(y)) = N(g(y)) \setminus \{x_2\}$ for any $y \in S_{01}$ and $g(y) \in S_2$ with $g(N(y)) = N(g(y)) \setminus \{x_2\}$ for any $y \in S_{02}$. Consequently, $X+Y$ satisfies the conditions of Lemma 2.7 and so $X+Y$ is qs-dismonoidal. This is a contradiction. Thus we must have $[g(a)]_{\rho_f} \cap I_g = \{x_1\}$. Therefore the subgraph of $X$ induced by $([g(a)]_{\rho_f} \cap I_g) \cup ([g(b)]_{\rho_f} \cap I_g) = \{x_1\} \cup (\{y_{21}, \ldots, y_{2t}\} \cap I_g)$ is a complete bipartite graph.

*Case 2.* $[g(a)]_{\rho_f} = \{x\}$ and $[g(b)]_{\rho_f} = \{y_{31}, \ldots, y_{3r}\}$ for some $x \in K_i$ with $i \in \{1, 2\}$, $y_{3j} \in S_i$ with $j = 1, \ldots, r$ and $x \in N(y_{3j})$, $N(y_{3u}) = N(y_{3v})$ for $u, v = 1, \ldots, r$. Then $x = g(a) \in I_g$. Hence $x$ is adjacent to every vertex of $[g(b)]_{\rho_f} \cap I_g$.

*Case 3.* $[g(a)]_{\rho_f} = \{y_{41}, \ldots, y_{4p}\}$ and $[g(b)]_{\rho_f} = \{y_{51}, \ldots, y_{5q}\}$ for some $p, q \ge 1$, $y_{4i} \in S$ with $N(y_{4i}) = N(y_{4j})$ for $i, j = 1, \ldots, p$, $y_{5k} \in S$ with $N(y_{5k}) = N(y_{5l})$ for $k, l = 1, \ldots, q$ and $\{y_{4u}, y_{5v}\} \in E$ for any $u = 1, \ldots, p$ and $v = 1, \ldots, q$. Clearly, $g(a)$ is adjacent to every vertex of $[g(b)]_{\rho_f}$ and $g(b)$ is adjacent to every vertex of $[g(a)]_{\rho_f}$.

*Case 4.* $[g(a)]_{\rho_f} = \{x_3, y_{61}, \ldots, y_{6d}\}$ and $[g(b)]_{\rho_f} = \{y_{71}, \ldots, y_{7e}\}$ for some $d, e \ge 1$, $x_3 \in K_i$ with $i \in \{1, 2\}$, $y_{6t} \in S_i$ with $N(y_{6t}) \cap K = K \setminus \{x_3\}$ for $t = 1, \ldots, d$, $y_{7u} \in S_j$ with $N(y_{7u}) = N(y_{7v})$ for $u, v = 1, \ldots, e$ (where $i \neq j$). Then $g(a)$ is adjacent to every vertex of $[g(b)]_{\rho_f}$ and $g(b)$ is adjacent to every vertex of $[g(a)]_{\rho_f}$.

So far we have proved that if $\{(fg)(a), (fg)(b)\} \in E(X+Y)$ for some $a, b \in V(X+Y)$, then there exist $g(c) \in [g(a)]_{\rho_f} \cap I_g$ such that $g(c)$ is adjacent to every vertex of $[g(b)]_{\rho_f} \cap I_g$, and $g(d) \in [g(b)]_{\rho_f} \cap I_g$ such that $g(d)$ is adjacent to every vertex of $[g(a)]_{\rho_f} \cap I_g$. Let $v \in [g(b)]_{\rho_f} \cap I_g$. Then $\{g(c), v\} \in E(X+Y)$. In view of Lemma 2.5, if there exists $x \in K \cap g^{-1}(g(c))$, then $x$ is adjacent to every vertex of $g^{-1}(v)$. Hence $x \in [a]_{\rho_{fg}}$ is adjacent to every vertex of $[b]_{\rho_{fg}}$. If $g^{-1}(g(c)) \subseteq S$, then every vertex of $g^{-1}(g(c))$ is adjacent to every vertex of $g^{-1}(v)$. Take any vertex $s \in g^{-1}(g(c))$. Then $s \in [a]_{\rho_{fg}}$ and $s$ is adjacent to every vertex of $[b]_{\rho_{fg}}$. Dually, there exists a vertex $t \in [b]_{\rho_{fg}}$ such that $t$ is adjacent to every vertex of $[a]_{\rho_{fg}}$. Consequently, $fg \in qEnd(X+Y)$. □

With these preparations, the proof of our main result is straightforward.

Proof of Theorem 2.1. Necessity follows directly from Lemmas 2.6 and 2.7. Sufficiency follows directly from Lemma 2.9. □

In [7], Hou *et al.* investigated the half-strong endomorphisms of the join of split graphs and gave the conditions under which the half-strong endomorphisms of the join of split graphs form a monoid.

Lemma 2.10 [7]. *Let $X+Y$ be a join of split graphs. Then $hEnd(X+Y)$ forms a monoid if and only if*

(A) $N(y_i) \not\subset N(y_j)$ *for any* $y_i, y_j \in S$,

(B) *there are no* $y, y_1, \ldots, y_t \in S\,(t \geq 2)$ *such that* $|N(y_i) \cap K| < |N(y) \cap K|$ *and* $|N(y) \cap K| = |\bigcup_{i=1}^{t} N(y_i) \cap K|$ *and*

(C) *for any* $r_1 \in K$ *with* $N(y_1) \cap K = K \setminus \{r_1\}$ *for some* $y_1 \in S$, *there are no* $y, y_2, \ldots, y_t \in S\,(t \geq 2)$ *such that* $r_1 \in N(y)$ *and* $|N(y) \cap K| - 1 = |\bigcup_{i=2}^{t} N(y_i) \cap K|$.

For a join of split graphs $X+Y$, we have the following corollary.

Corollary 2.11. *Let $X+Y$ be a join of split graph. If $hEnd(X+Y)$ forms a monoid, then $X+Y$ is qs-monoidal and so $qEnd(X+Y)$ forms a monoid.*

Proof. If $hEnd(X+Y)$ forms a monoid, then by Lemma 2.10, $X+Y$ satisfies conditions (A), (B) and (C). Suppose that $X+Y$ is qs-dismonoidal. Then $X+Y$ satisfies the conditions of Lemma 2.6 or Lemma 2.7. If $x_1$ or $x_2 \in \{x_0, x_3\}$, then $N(y_3)$ is strictly contained in $N(y_1)$ or $N(y_2)$. This contradicts (A). If $x_1, x_2 \notin \{x_0, x_3\}$, then $|N(y_2) \cap K| - 1 = |N(y_3) \cap K|$. This contradicts (C). We have proved that if $hEnd(X+Y)$ forms a monoid, then $X+Y$ is qs-monoidal. Now the fact that $qEnd(X+Y)$ forms a monoid follows from Theorem 2.1. □

Let $S$ be a semigroup. An element $a$ of $S$ is called *regular* if there exists $x \in S$ such that $axa = a$. A semigroup $S$ is called regular if all its elements are regular. A graph $X$ is said to be End-regular if its endomorphism monoid $End(X)$ is a regular semigroup. Recall that for any graph $X$, every regular endomorphism of $X$ must be half-strong. Hence if $X$ is End-regular, then $hEnd(X)$ forms a monoid. Thus we have the following corollary.

Corollary 2.12. *Let $X + Y$ be a join of split graphs. If $X + Y$ is End-regular, then $X + Y$ is qs-monoidal and so $qEnd(X + Y)$ forms a monoid.*

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