ON A BOOLEAN ALGEBRA OF PROJECTIONS CONSTRUCTED BY DIEUDONNÉ

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1. Dieudonné (**4**) has constructed an example of a Banach space X and a complete Boolean algebra $\tilde{B}$ of projections on X such that $\tilde{B}$ has uniform multiplicity two, but for no choice of x_1, x_2 in X and non-zero E in $\tilde{B}$ is EX the direct sum of the cyclic subspaces clm $\{Ex_1 : E \in \tilde{B}\}$ and clm $\{Ex_2 : E \in \tilde{B}\}$. Tzafriri observed that it could be deduced from Corollary 4 (**9**, p. 221) that the commutant $\tilde{B}'$ of $\tilde{B}$ is equal to $A(\tilde{B})$, the algebra of operators generated by $\tilde{B}$ in the uniform operator topology. A study of (**3**) suggested the direct proof of the second property given in this note. From this there follows a simple proof that $\tilde{B}$ has the first property.

In this connection, the author has shown in (**5**) that if $\tilde{B}$ is a complete Boolean algebra of projections on X, then when X is a Hilbert space $\tilde{B}' = A(\tilde{B})$ if and only if $\tilde{B}$ has uniform multiplicity one, while if X is a Banach space uniform multiplicity one implies that $\tilde{B}' = A(\tilde{B})$. Therefore in general the reverse implication fails in a Banach space.

2. The reader is referred to (**1**) and (**2**) for terminology used in this paper. Next we recall the definition and properties of a class of Banach spaces studied by Halperin (**6**) and Lorentz (**7**), (**8**).

Let K be a compact interval of the form $[0, y]$ where $y>0$. Two non-negative measurable functions f_1, f_2 on K are said to be *equimeasurable* if and only if for every $k \geqq 0$

$$m\{x \in K: f_1(x) \geqq k\} = m\{x \in K: f_2(x) \geqq k\},$$

where $m(\cdot)$ denotes Lebesgue measure on R. For each non-negative measurable function f on K, which is finite a.e., the *decreasing rearrangement* of f is the function f^* defined by

$$f^*(0) = \operatorname*{ess.\,sup}_{x \in K} |f(x)|$$

$$f^*(x) = \sup\{k \geqq 0: m\{y: f(y) \geqq k\} \geqq x\} \quad x \in K, x \neq 0.$$

f^* is continuous on the left. Also f and f^* are equimeasurable by construction. Now let w be a positive function which is decreasing and integrable over K. The set of equivalence classes of complex measurable functions on K such that $w\,|f|^*$ is integrable forms a Banach space L^1_w under the norm

$$\|f\|_w = \int_0^y w\,|f|^* dx.$$

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If $m>0$, denote by f_m the function equal to f if $|f(x)| \leqq m$ and defined by

$$f_m(x) = \frac{mf(x)}{|f(x)|}, \quad |f(x)|>m.$$

If $f \in L^1_w$, $\| f-f_m \|_w \to 0$ as $m \to \infty$. It follows that the set $B(K)$ of bounded Borel measurable functions on K is norm dense in L^1_w. Now $B(K)$ is a Banach algebra under the supremum norm $\| \ \|$. For f in $B(K)$ and g in L^1_w, we have $fg \in L^1_w$ and $\| fg \|_w \leqq \| f \| \, \| g \|_w$. It follows that the map $T_f : g \to fg$ is a bounded linear operator on L^1_w such that $\| T_f \| \leqq \| f \|$. Moreover the map $f \to T_f$ is a continuous algebra homomorphism of $B(K)$ into an algebra of bounded linear operators on L^1_w.

3. Dieudonné (**4**, p. 10) constructed a compact interval $K = [0, y]$ and functions w_1, w_2, w_3 decreasing and integrable over K such that

$$w_1(y) = w_2(y) = w_3(y) = 1;$$

$$\int_0^y w_1^2 dx = \int_0^y w_2^2 dx = \int_0^y w_3^2 dx = +\infty;$$

$$\int_0^y w_1 w_2 dx < \infty; \quad \int_0^y w_2 w_3 dx < \infty; \quad \int_0^y w_3 w_1 dx < \infty.$$

Let $\tilde{X}$ be the complex Banach space $L^1_{w_1} \oplus L^1_{w_2} \oplus L^1_{w_3}$ under the norm

$$\|(f_1, f_2, f_3)\| = \| f_1 \|_{w_1} + \| f_2 \|_{w_2} + \| f_3 \|_{w_3}$$

where $f_i \in L^1_{w_i}$, $i = 1, 2, 3$. Let X be the closed subspace of $\tilde{X}$ consisting of elements (f_1, f_2, f_3) for which

$$f_1(x) + f_2(x) + f_3(x) = 0 \text{ a.e.}$$

If $\tau \in \Sigma_K$, the Borel subsets of K, then the map

$$E(\tau): (f_1, f_2, f_3) \to (\chi_\tau f_1, \chi_\tau f_2, \chi_\tau f_3)$$

defines a projection on X and the family $\{E(\tau): \tau \in \Sigma_K\}$ forms a σ-complete Boolean algebra $\tilde{B}$ of projections on the separable Banach space X. Hence $\tilde{B}$ is a complete countably decomposable Boolean algebra of projections on X. $\tilde{B}$ has uniform multiplicity two (**4**, pp. 7-8).

4. Now let $T \in \tilde{B}'$ and let $T(1, 0, -1) = (u, v, -u-v)$. We shall show that v is 0 a.e. If not, then since v is integrable over K, there would exist a compact subset δ of K with $m(\delta)>0$ in which v is continuous and non-zero. The hypothesis $T \in \tilde{B}'$ implies that for each function f in $B(K)$

$$T(f, 0, -f) = (fu, fv, -fu-fv).$$

Since T is continuous, the map A of the subspace $\{(f, 0, -f): f \in B(K)\}$ under the $\tilde{X}$-norm into $L^1_{w_2}$ defined by

$$A: (f, 0, -f) \to fv$$

is continuous. We shall show that this gives a contradiction.

For every x in δ, define

$$h(x) = \int_0^x \chi_\delta(t)dt$$

$$g(x) = w_2(h(x)).$$

Put $g(x) = 0$ for x in $K \backslash \delta$. It follows that g is equimeasurable to the restriction of w_2 to an interval $[0, y']$ where $y' = m(\delta)$. Hence $g \in L^1_{w_1}$ and $g \in L^1_{w_3}$ but $g \notin L^1_{w_2}$. Since $|v|$ is bounded below in δ by a positive number c, we have $|gv|^* \geqq cg^*$ and so $gv \notin L^1_{w_2}$. Now the sequence $\{(g_m, 0, -g_m)\}, m = 1, 2, 3, \ldots$ converges to $(g, 0, -g)$ in X and so by the continuity of the map A the sequence $\{g_m v\colon\ m = 1, 2, 3, \ldots\}$ ought to converge to a limit in $L^1_{w_2}$. However

$$|g_m v|^* \geqq cg_m^*.$$

This gives a contradiction since the norm of g_m^* in $L^1_{w_2}$ becomes arbitrarily large with m.

Hence $v = 0$. It is now easy to see that u is essentially bounded. If not then for every $m>0$ there would be a measurable set $\tau \subseteq K$ with $m(\tau)>0$ where $|u(x)| \geqq m$. Now

$$\| T(\chi_\tau, 0, -\chi_\tau)\| \geqq m \|(\chi_\tau, 0, -\chi_\tau)\|$$

and this contradicts the continuity of T. It follows that there is a bounded Borel measurable function u_1 such that

$$T(1, 0, -1) = (u_1, 0, -u_1).$$

Similarly there are bounded Borel measurable functions u_2 and u_3 such that

$$T(0, 1, -1) = (0, u_2, -u_2),$$

$$T(1, -1, 0) = (u_3, -u_3, 0).$$

By the linearity of T, $u_1 = u_2 = u_2$ a.e. It follows that $\tilde{B}' = A(\tilde{B})$.

Finally, suppose there is $F \neq 0$ in $\tilde{B}$ and x_1, x_2 in X such that FX is the direct sum of the cyclic subspaces

$$M(x_1) = \text{clm}\,\{Ex_1\colon E \in \tilde{B}\} \text{ and } M(x_2) = \text{clm}\,\{Ex_2\colon E \in \tilde{B}\}.$$

Then $x_1, x_2 \in FX$, $x_1 \neq 0$, and $x_2 \neq 0$, since $\tilde{B}$ has uniform multiplicity two. It follows readily that there is a projection P in $\tilde{B}'$ with range $M(x_1)$. However, from above every projection in $\tilde{B}'$ lies in $\tilde{B}$. This contradicts the fact that $\tilde{B}$ has uniform multiplicity two.

5. It is possible to construct a similar counterexample in which X is uniformly convex. It suffices to take in Section **3**

$$\tilde{X} = L^2_{w_1} \oplus L^2_{w_2} \oplus L^2_{w_3}$$

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under the norm defined by

$$\|(f_1, f_2, f_3)\| = \left[\int_0^y \{w_1(|f_1|^*)^2 + w_2(|f_2|^*)^2 + w_3(|f_3|^*)^2\} dx\right]^{\frac{1}{2}}$$

and in Section **4** to define $g(x) = (w_2(h(x)))^{\frac{1}{2}}$ for x in δ.

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