GENERATING GROUPS FOR NILPOTENT VARIETIES

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(Received 2 May 1968; revised 14 October 1968)

To Bernhard Hermann Neumann on his 60th birthday

Communicated by G. E. Wall

Let $\mathfrak{N}_c$ denote the variety of all nilpotent groups of class $\leqq c$, that is, $\mathfrak{N}_c$ is the class of all groups satisfying the law

$$[x_1, \cdots, x_{c+1}] = 1,$$

where we define, as usual, $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and, inductively, $[x_1, \cdots, x_n] = [[x_1, \cdots, x_{n-1}], x_n]$. Further, let $F_k(\mathfrak{N}_c)$ denote a free group of $\mathfrak{N}_c$ of rank k. In her book Hanna Neumann ([4], Problem 14) poses the following problem: Determine $d(c)$, the least k such that $F_k(\mathfrak{N}_c)$ generates $\mathfrak{N}_c$. Further, she suggests, incorrectly, that $d(c) = [c/2]+1$. However, as we shall prove here, the correct answer is $d(c) = c-1$, for $c \geqq 3$. [2] More generally, we shall prove the following result.

THEOREM. *Let* var $F_i(\mathfrak{N}_c)$ *denote the variety generated by* $F_i(\mathfrak{N}_c)$. *Then*

(1) $\quad$ var $F_1(\mathfrak{N}_c) <$ var $F_2(\mathfrak{N}_c) < \cdots <$ var $F_{c-1}(\mathfrak{N}_c) = \mathfrak{N}_c$ *for all* $c \geqq 3$.

For convenience we will divide the proof into two parts. In part I the inequalities in (1) are established by constructing, for each $k \leqq c$, a law in $F_{k-2}(\mathfrak{N}_c)$ which is not a law in $F_{k-1}(\mathfrak{N}_c)$. In part II the final equality in (1) is established by showing that $F_k(\mathfrak{N}_c)$ is residually a $(c-1)$ generator group, for any $k \geqq c$. [3]

Part I:

(2) $$\text{var } F_{k-1}(\mathfrak{N}_c) < \text{var } F_k(\mathfrak{N}_c), \qquad 2 \leqq k \leqq c-1.$$

PROOF OF PART I. To show (2) for $3 \leqq k \leqq c-1$ (the case $k = 2$ is trivial) it is sufficient to find a law $Q_{k,c} = 1$ which holds in $F_{k-1}(\mathfrak{N}_c)$ but

[1] The author gratefully acknowledges the support of the National Science Foundation.

[2] Since preparing this manuscript the author has received a letter reporting two independent proofs of this result from M. F. Newman in Canberra [1], both based on somewhat less elementary arguments, however.

[3] All notation and terminology not specified follows that of [3] or [4].

not in $F_k(\mathfrak{N}_c)$. The particular law we have chosen is constructed as follows: Let

$$(3) \qquad Q_k = \prod_\sigma [x_k, x_{\sigma(1)}, \cdots, x_{\sigma(k-1)}]^{|\sigma|}$$

where σ runs through all permutations of $\{1, \cdots, k-1\}$ and $|\sigma| = 1$ if σ is even, $|\sigma| = -1$ if σ is odd. Then we define $Q_{c-1,c}$ to be Q_c and, for $3 \leqq k \leqq c-2$, $Q_{k,c} = [Q_{k+1}, x_{k+2}, \cdots, x_c]$. We first prove the following.

(A) $Q_c = 1$ holds in $F_{c-2}(\mathfrak{N}_c)$ but not in $F_{c-1}(\mathfrak{N}_c)$, $c \geqq 3$.

Proof of (A). Let $R_k = Z[y_1, \cdots, y_k]$ be a free associative ring over Z in the free non-commuting indeterminates $y_1, \cdots, y_k$, and let $I_{k,c+1}$ be the (two-sided) ideal in R_k generated by all monomials of degree $c+1$. In $R_{k,c} = R_k/I_{k,c+1}$ any element $1+y_i$ has an inverse $1-y_i+y_i^2- \cdots \pm y_i^c$, and, hence, we may consider the multiplicative group $G_{k,c}$ in $R_{k,c}$ generated by the elements $1+y_i$, $i = 1, \cdots, k$. We define $(z_1, z_2) = z_1z_2-z_2z_1$ and, inductively, $(z_1, \cdots, z_n) = ((z_1, \cdots, z_{n-1}), z_n)$, $z_i \in R_k$. A direct computation shows that

$$(4) \qquad [1+z_1, \cdots, 1+z_n] = 1+(z_1, \cdots, z_n)+\text{terms of higher degree},$$

for any $1+z_i \in G_{k,c}$. Since the n-length commutator $(z_1, \cdots, z_n)$ is a homogeneous polynomial of degree n in the z_i, it follows that $G_{k,c} \in \mathfrak{N}_c$. In fact (cf., [3], Chapter 5), $G_{k,c} \cong F_k(\mathfrak{N}_c)$. In particular,

$$\prod_\sigma [1+z_c, 1+z_{\sigma(1)}, \cdots, 1+z_{\sigma(c-1)}]^{|\sigma|} = 1+ \sum_\sigma |\sigma| (z_c, z_{\sigma(1)}, \cdots, z_{\sigma(c-1)}),$$

for elements $1+z_i \in G_{k,c}$. Hence, to prove (A) it is sufficient to prove

$$(5) \qquad Q'_c = \sum_\sigma |\sigma| (x_c, x_{\sigma(1)}, \cdots, x_{\sigma(c-1)}) = 0$$

in $R_{c-2,c}$ but not in $R_{c-1,c}$. (In this context, Q'_c may be considered as element in the free associative ring $R_c(x) = Z[x_1, \cdots, x_c]$ just as Q_c may be considered as an element of the free group on $x_1, \cdots, x_c$ (cf., [3]).)

The proof of (5) is based on the following lemma.

Lemma. *Let $P \neq 0$ be a homogeneous polynomial in R_c of total degree c ($\geqq 3$) and of degree 1 in each indeterminate $y_1, \cdots, y_c$. If P is a linear combination of c-fold commutators, i.e., elements of the form $(a_1, \cdots, a_c)$, then for some $i \neq j$, $P \neq 0$ modulo $y_i = y_j$. (The latter statement will be abbreviated by $P\{y_i = y_j\} \neq 0$.)*

Proof. First we note that the polynomial

$$(6) \qquad P_c = P_c(y_1, \cdots, y_c) = \sum_\sigma |\sigma| y_{\sigma(1)} \cdots y_{\sigma(c)},$$

where σ runs through all permutations of $\{1, \cdots, c\}$ satisfies $P_c\{y_i = y_j\} = 0$, for any $i \neq j$. P_c, $c \geqq 3$, is not a linear combination of c-fold commutators. For, if it were, then by the Dynkin-Specht-Wever Theorem (cf., [3]), we would have

$$\{P_c\} = \sum_\sigma |\sigma|(y_{\sigma(1)}, \cdots, y_{\sigma(c)}) = cP_c.$$

However, a straightforward induction starting with

$$P_3 = 2(x_1, x_2, x_3)+2(x_2, x_3, x_1)+2(x_3, x_1, x_2),$$

which is 0 by the Jacobi identity, and noting that, for $c > 3$,

$$\{P_c\} = \sum_{k=1}^{c} \sum_{\sigma(c)=k} |\sigma|(y_{\sigma(1)}, \cdots, y_{\sigma(c)}),$$

shows that $\{P_c\} = 0$ for all $c \geqq 3$.

To complete the proof of the Lemma it suffices to show that any P described in the Lemma which satisfies $P\{y_i = y_j\} = 0$ for all $i \neq j$, is a multiple of P_c. The proof is by induction on c (starting with $c = 2$, however). For $c = 2$, $P = ny_1y_2+my_2y_1$, and $P\{y_1 = y_2\} = 0$ implies that $m = -n$, i.e., $P = nP_2$.

Next, let $c > 2$ and write P in the form

$$P = \sum_{k=1}^{c} A_k y_k,$$

where the A_i are homogeneous of total degree $c-1$ in $y_1, \cdots, \hat{y}_i, \cdots, y_c$ (y_i omitted). Since $P\{y_p = y_q\} = 0$ for any $p, q \neq k$, it follows by induction that

$$A_k = n_k P_{c-1,k}, \qquad n_k \in Z,$$

where $P_{c-1,k} = P_{c-1}(y_1, \cdots, \hat{y}_k, \cdots, y_c)$, as defined by (6). Thus,

$$P = \sum_{k=1}^{c} n_k P_{c-1,k} y_k, \tag{7}$$

However, since $P\{y_1 = y_k\} = 0$ for any $k \neq 1$, it follows from a comparison of the first and k-th summands in (7) that this is possible only if

$$n_1 P_{c-1,1} y_1 + n_k P_{c-1,k} y_k = n_1 \sum_\sigma |\sigma| y_{\sigma(1)} \cdots y_{\sigma(c)},$$

where the summation is restricted to all those σ for which either $\sigma(c) = 1$ or $\sigma(c) = k$. Since this is to be true for all k, it follows that $P = n_1 P_c$. This proves the Lemma.

We may now apply the Lemma to (5). Since the component of Q'_c (considered as a polynomial in R_c) of terms with left factor x_c is precisely

$$\sum_{\sigma} x_c x_{\sigma(1)} \cdots x_{\sigma(c-1)},$$

it is clear that $Q'_c \neq 0$. Further, Q'_c is antisymmetric in the $x_1, \cdots, x_{c-1}$, so that $Q'_c\{x_i = x_j\} = 0$ for any $i, j \neq c$, $i \neq j$. Since $Q'_c \neq 0$, it follows from the Lemma that $Q'_c\{x_c = x_j\} \neq 0$ for some $j \neq c$. Thus, $Q'_c = 0$ is not a law in $R_{c-1,c}$, which means that $Q_c = 1$ is not a law in $F_{c-1}(\mathfrak{N}_c)$ as well.

As just observed, $Q'_c = 0$ if any two of the $x_1, \cdots, x_{c-1}$, are identified. Thus, in $R_{c-2,c}$ if the x_i are replaced by the y_i, then since there are just $c-2$ distinct y_i it follows that Q'_c will vanish. To decide whether $Q'_c = Q'_c(x_1, \cdots, x_c)$ (i.e., considered as a function of the x_i) vanishes over all of $R_{c-2,c}$ or not it is enough modulo $I_{c-2,c+1}$ to consider linear substitutions of the y_i for the x_i. However, since $Q'_c(x_1, \cdots, x_c)$ is multilinear in the x_i, such a substitution yields a linear combination of terms of the form $Q'_c(v_1, \cdots, v_c)$, $v_i \in \{y_1, \cdots, y_{c-2}\}$. By the previous remark, each of these terms vanishes. Hence, $Q'_c = 0$ is a law in $R_{c-2,c}$ and $Q_c = 1$ is a law in $F_{c-2}(\mathfrak{N}_c)$. This completes the proof of (5) and, hence, of (A).

The above argument shows that $\text{var}\, F_{c-2}(\mathfrak{N}_c) < \text{var}\, F_{c-1}(\mathfrak{N}_c)$. To complete the proof of Part I we must show that $Q_{k,c}$ is trivial over $F_{k-1}(\mathfrak{N}_c)$ but not over $F_k(\mathfrak{N}_c)$, $3 \leqq k \leqq c-2$.[4] That $Q_{k,c} = 1$ is a law for $F_{k-1}(\mathfrak{N}_c)$ follows immediately from the above arguments regarding Q_c. Further, as we have seen above, $Q'_c\{x_k = x_i\} \neq 0$ for some $i \neq k$. This implies, however, that

(8) $$[Q_k\{x_k = x_c\}, z_{k+1}, \cdots, z_c] \neq 1$$

where $z_i \in \{x_1, \cdots, x_k\}$, so that $Q_{k,c}$ will not be a law for $F_k(\mathfrak{N}_c)$. This is a direct consequence of the more general remark that if $P \in R_n$ is a polynomial with at least two distinct y_i appearing in each term, then for any y_i, $(P, y_i) \neq 0$. To see this, express P in the form $P = \sum y_i^m A_m$, where the A_m are polynomials without terms with left y_i factors. We may assume that no A_m has a term of degree 0. Then,

$$y_i P = \sum y_i^{m+1} A_m \neq P y_i = \sum y_i^m A_m y_i.$$

This completes the proof of Part I.

Part II:

For any $k \geqq c \geqq 3$, $F_k(\mathfrak{N}_c)$ is residually a $(c-1)$ generator group.

PROOF OF PART II. The proof is by induction on $c \geqq 3$. First, it is known, for any $c \geqq 3$, that the variety of metabelian groups nilpotent of

[4] This portion of the proof of Part 1 based on the law $Q_{k,c}$ for $k \leqq c-2$, has been suggested to the author by M. F. Newman and is included here with his permission. It replaces the author's original proof which was based on a slightly more complicated law with a resulting lengthier argument.

class c is generated by its 2-generator groups (Baumslag, Neumann, Neumann, Neumann [2], cf., [4], 36. 34). This proves the case $c = 3$. Next, let $c > 3$ and set $F = F_k(\mathfrak{N}_c)$, for an arbitrary $k \geqq c$. By induction,

$$F/\gamma_c F = F_k(\mathfrak{N}_{c-1}),$$

where $\gamma_c F$ denotes the c-th term of the lower central series of F, is residually $(c-2)$ generator. Hence, consider $g \in \gamma_c F$. Note that g is a product of commutators of the form $[a_1, \cdots, a_c]$. If g involves more than c generators, say $g_1, \cdots, g_m$, set $g' = g\{g_{c+1} = \cdots = g_m = 1\}$. We may assume, after a possible reordering of the indices of the g_i, that $g' \neq 1$. From the Lemma it follows that $g'\{g_i = g_j\} \neq 1$ for some $i \neq j$, $i, j \leqq c$. Thus, if N is the normal closure of the elements $g_i g_j^{-1}, g_{c+1}, \cdots, g_m$ in F, it follows that $g' \notin N$, and since F/N is generated by $(c-1)$ elements, it follows that $F_k(\mathfrak{N}_c)$ is residually a $(c-1)$ generator group, as desired. This completes the proof of Part II and, hence, of the theorem.

References

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