REMARKS ON STABILITY CONDITIONS FOR THE DIFFERENTIAL EQUATION \( x'' + a(t)f(x) = 0 \)

JAMES S. W. WONG

(Received 4 September 1967, revised 5 December 1967)

Consider the following second order nonlinear differential equation:

\[
(1) \quad x'' + a(t)f(x) = 0, \quad t \in [0, \infty),
\]

where \( a(t) \in C[0, \infty) \) and \( f(x) \) is a continuous function of \( x \). We are here concerned with establishing sufficient conditions such that all solutions of (1) satisfy

\[
(2) \quad \lim_{t \to \infty} x(t) = 0.
\]

Since \( a(t) \) is differentiable and \( f(x) \) is continuous, it is easy to see that all solutions of (1) are continuable throughout the entire non-negative real axis. It will be assumed throughout that the following conditions hold:

\[
(A_1) \quad \lim_{t \to \infty} a(t) = \infty,
\]

\[
(A_2) \quad xf(x) > 0, \quad x \neq 0,
\]

\[
(A_3) \quad \lim_{|x| \to \infty} \left| \int_0^x f(u) du \right| = \infty,
\]

\[
(A_4) \quad xf(x) \geq 2\gamma \int_0^x f(u) du, \quad \gamma > 0.
\]

Our main results are the following two theorems:

**Theorem 1.** Let \( 0 < \alpha < 1 \). If \( a(t) \) satisfies

\[
(3) \quad \lim_{T \to \infty} \int_{t_0}^T \frac{a'(t)}{a^\alpha(t)} dt < \infty,
\]

where \( a(t) > 0, \ t \geq t_0 \) and \( a'(t) = \max (-a'(t), 0) \), and

---

1 Research supported in part by the United States Army under Contract No. DA-31-124-ARO-D-462 and in part by Defence Research Board of Canada under Grant DRB-9540-12.

2 On leave from The University of Alberta, Edmonton, Alberta, Canada.
Stability conditions for $x'' + a(t)f(x) = 0$

\[\int_{t_0}^{T} |(a^{-2}(t))'''| dt = 0 \left( a^{1-2}(T) \right), \quad (T \to \infty), \]

then every solution of (1) satisfies (2).

**Theorem 2.** If $a(t)$ satisfies

\[\lim_{T \to \infty} \int_{t_0}^{T} \frac{a'_-(t)}{a(t)} dt < \infty,\]

where $a(t) > 0$ for $t \geq t_0$, and

\[\int_{t_0}^{T} |(a^{-1}(t))'''| dt = o \left( \log a(T) \right), \quad (T \to \infty),\]

then every solution of (1) satisfies (2).

Define for each solution $x(t)$ of (1) the following energy function:

\[V(t, x) = \frac{x'^2}{a(t)} + 2 \int_{0}^{x} f(u) du,\]

which is clearly non-negative, on account of (A2). Under assumptions (A1), (A2), (A3) and (5), we can prove the following two propositions concerning solutions of (1).

**Lemma 1.** $\lim_{t \to \infty} V(t, x)$ exists and is finite.

**Proof.** A simple differentiation shows that

\[V'(t, x) = -\frac{a'(t)}{a^2(t)} x'^2 \leq \frac{a'_-(t)}{a^2(t)} x'^2 \leq \frac{a'_-(t)}{a(t)} V(t, x),\]

from which it follows that

\[V(t, x) \leq V(t_0, x_0) \exp \left( \int_{t_0}^{t} \frac{a'_-(s)}{a(s)} ds \right) \leq M < \infty,\]

where the bound $M$ depends upon $x_0 = x(t_0)$. Integrating the equality in (8), one finds

\[0 \leq V(t, x) = V(t_0, x_0) + \int_{t_0}^{t} \frac{x'^2(s)}{a^2(s)} \left( a'_-(s) - a'_+(s) \right) ds,\]

and hence

\[
\int_{t_0}^{t} \frac{x'^2(s)}{a(s)} \frac{a'_+(s)}{a(s)} ds \leq V(t_0, x_0) + M \int_{t_0}^{t} \frac{a'_-(s)}{a(s)} ds < \infty.
\]

From (9), we may conclude that
exists. Thus,

$$\lim_{t \to \infty} \int_{t_0}^{t} \frac{x''(s)}{a(s)} \frac{a'_+(s)}{a(s)} ds$$

exists, and is finite.

**Lemma 2.** Every solution $x(t)$ of (1) is oscillatory, i.e. there exists a sequence $\{t_k\}$ such that $x(t_k) = 0$, $k = 0, 1, 2, 3, \cdots$ and $t_k \to \infty$ as $k \to \infty$.

**Proof.** Let $x(t)$ be a non-oscillatory solution of (1). On account of (A2), we may assume without loss of generality that $x(t) > 0$ for $t \geq t_0$. From (1), it follows that $x'(t)$ is non-increasing, and hence has a limit. If the limit is negative or $-\infty$, then $x(t)$ must eventually be negative, which has been ruled out at the beginning. Thus we may assume that $x'(t)$ is eventually non-negative and so $x(t)$ is non-decreasing and has a limit $c$. If $c$ is finite, then we may choose $T \geq t_0$ such that $c/2 \leq x(t) \leq c$ for $t \geq T$. Denote

$$k = \inf_{\epsilon/2 \leq x \leq \epsilon} f(x), \quad 0 < k < \infty.$$ Integrating (1), we have

$$x'(t) + \int_{T}^{t} a(s)f(x(s))ds = x'(T),$$

from which the desired contradiction follows. On the other hand, if $c = +\infty$, we multiply (1) through by $x'(t)$ and integrate to obtain:

$$\frac{x''(t)}{2} + \int_{T}^{t} a(s)f(x(s))x'(s)ds \leq \frac{x''(T)}{2}.$$ We may assume that $T$ is so chosen such that $a(t) \geq 1$ for $t \geq T$. Thus, (10) becomes

$$\frac{x''(t)}{2} + \int_{x(T)}^{x(t)} f(u)du \leq \frac{x''(T)}{2}.$$ Letting $t$ tend to infinity in (11), one easily obtains a contradiction to (A2).

**Proof of Theorem 1.** Let $x(t)$ be any non-trivial solution of (1) and $V(t, x)$ be defined by (7). Clearly (3) implies (5), so by Lemma 1, $\lim_{t \to \infty} V(t, x) = L$ exists. If $L = 0$, then (2) clearly follows on account of (A2). Now assume that $L > 0$ for some solution $x(t)$ of (1). By Lemma 2, there exists an increasing sequence $\{k\}$ such that $t_k \to \infty$ as $k \to \infty$ and $x'(t_k) = 0$, $k = 0, 1, 2, 3, \cdots$. Let $\varepsilon > 0$, we choose $t_0 \geq 0$ such that
Stability conditions for $x'' + a(t)f(x) = 0$

(12) \[ a(t) > 0 \text{ and } (1-\epsilon)L \leq V(t, x) \leq (1+\epsilon)L, \]
for $t \geq t_0$. Write $V(t) = V(t, x)$ for short and denote $\varphi = a^{-\alpha}$. A simple computation using (1) yields the following identity:

(13) \[ \frac{d}{dt} \left\{ \varphi aV + \frac{1}{2} \varphi'' x^2 - \varphi' xx' \right\} = \frac{1}{2} \varphi''' x^2 + 2(1-\alpha) a^{-\alpha} a' F(x) - \alpha a^{-\alpha} a' xf(x), \]
where

\[ F(x) = \int_{0}^{x} f(u)du. \]

Integrating (13) from $t_0$ to $t_k$, we obtain

(14) \[ a^{1-\alpha}(t_k)V(t_k) = c_0 - \frac{1}{2} \varphi''(t_k)x^2(t_k) + \frac{1}{2} \int_{t_0}^{t_k} \varphi''' x^2 dt + 2(1-\alpha) a^{-\alpha} a' \int_{t_0}^{t_k} F(x) dt - \alpha \int_{t_0}^{t_k} a^{-\alpha} a' xf(x) dt, \]
where $c_0 = a^{1-\alpha}(t_0)v(t_0) + \frac{1}{2} \varphi''(t_0)x^2(t_0)$. By Lemma 1 and (A_4), we conclude that every solution $x(t)$ is bounded, say $|x(t)| \leq B$. Note that

\[ |\varphi''(t_k)| \leq |\varphi''(t_0)| + \int_{t_0}^{t_k} |\varphi'''| dt. \]

Denoting $\beta = \sup_{|x| \leq B} xf(x)$ and using (A_4) and (12) in (14), we get

(15) \[ a^{1-\alpha}(t_k)(1-\epsilon)L \leq |c_1| + B^2 \int_{t_0}^{t_k} |\varphi'''| dt + a^{1-\alpha}(t_k)(1+\epsilon)L \max \left( 1 - \frac{\alpha \gamma}{1-\alpha}, 0 \right) + (2(1-\alpha)(1+\epsilon)L - \alpha \beta) \int_{t_0}^{t_k} \frac{a'}{a^\alpha} dt, \]
where $c_1$ is some appropriate constant. Using (3) and (4), we obtain from (15)

\[ (1-\epsilon) \leq (1+\epsilon) \max \left( 1 - \frac{\alpha \gamma}{1-\alpha}, 0 \right) + o(1), \]
which produces the desired contradiction with any $\epsilon > 0$ if $\alpha \geq (\gamma+1)^{-1}$ and with $\epsilon < \gamma \alpha (2(1-\alpha) - \alpha \gamma)^{-1}$ if $\alpha < (1+\gamma)^{-1}$.

**Proof of Theorem 2.** The general argument is similar to that of Theorem 1. Here instead of (13), we have the following identity:

\[ \frac{a'}{a} xf(x) + \frac{d}{dt} \left\{ V + \frac{1}{2} \varphi'' x^2 - \varphi' xx' \right\} = \frac{1}{2} \varphi''' x^2 + \frac{a'}{a} xf(x), \]
from which we have the following inequality:

$$\gamma \frac{a'}{a} V + \frac{d}{dt} \left\{ (1+\gamma) V + \frac{1}{2} q'' x^2 - q' x x' \right\}$$

$$= \frac{1}{2} q'''' x^2 + \frac{a'}{a} (x f(x) - 2 \gamma F(x)).$$

Integrating (16) from \( t_0 \) to \( t_k \), we obtain

$$\gamma (1-\varepsilon) L \log a(t_k) \leq |c_0| + B^2 \int_{t_0}^{t_k} |q'''| dt,$$

where \( c_0 \) is some appropriate integration constant. Using (5) and (6), one easily derives a contradiction from (17).

**Remark 1.** Theorem 1 is a nonlinear extension of some stability conditions recently obtained for the linear equation:

$$x'' + a(t)x = 0.$$

However, even in the special case of equation (18), Theorem 1 is an improvement over its predecessors where it is assumed that \( a'(t) \geq 0 \) instead of (3), (cf. Meir, Willett and Wong [3] and an independent result for the case \( \frac{1}{2} \leq \alpha < 1 \) by Chang [1].) The assumption (3), or its stronger substitute that \( a'(t) \geq 0 \), is essential here and in [3] as compared to the result of Lazer [2] where no such assumption is made.

**Remark 2.** Assumptions (A3) and (A4) are easily realized if \( f(x) \) is non-decreasing in \( x \). As typical examples, one may take \( f(x) = x^\lambda \), where \( \lambda \) is the quotient of two odd integers and \( \lambda > 0 \), or take

$$f(x) = \begin{cases} x & |x| \leq 1, \\ \frac{x}{|x|^\mu} & |x| > 1, \end{cases}$$

with \( 1 \leq \mu < 2 \).

**Remark 3.** It is easily verified that the elementary functions \( a(t) = t^\sigma, \sigma > 0, e^t, \) and \( \log t \) satisfy both (4) and (6). An example is given in [3] which satisfies (6) but not (4).

**Remark 4.** Results on asymptotic properties of solutions of (1) may be transferred to the following slightly more general equation:

$$\left( \rho(t)x' \right)' + q(t)f(x) = 0, \quad \rho(t) > 0,$$
by standard Louiville transformations. The transformation necessary depends on the convergence and divergence of the integral

$$\int_0^\infty \frac{dt}{\dot{p}(t)}.$$ 

In case

$$\int_0^\infty \frac{dt}{\dot{p}(t)} = \infty,$$

we let

$$s = \int_t^\infty \frac{d\tau}{\dot{p}(\tau)}$$

and $y(s) = x(t)$ and transform (19) into:

$$\frac{d^2y}{ds^2} + \dot{p}(t)q(t)f(y) = 0,$$

which is of the form of equation (1). On the other hand, if

$$\int_0^\infty \frac{dt}{\dot{p}(t)} < \infty,$$

we let

$$s = \left(\int_t^\infty \frac{d\tau}{\dot{p}(\tau)}\right)^{-1} \quad \text{and} \quad y(s) = x(t) \left(\int_t^\infty \frac{d\tau}{\dot{p}(\tau)}\right)$$

and transform (19) into

$$\frac{d^2y}{ds^2} + \dot{p}(t)q(t)\frac{f(y)}{s^4} = 0,$$

which is again of the form of equation (1). To preserve asymptotic properties under Louiville transformations, it is essential here that $s$ tends to infinity as $t$ does.

**Remark 5.** Finally, we note that the present hypothesis does not imply that equation (1) is globally asymptotically stable, i.e. all solutions and their derivatives tend to zero as $t$ tends to infinity. In fact, the interesting fact is that every non-trivial solution $x(t)$ of (1) satisfies

$$\lim_{t \to \infty} \sup |x'(t)| > 0.$$ 

To see this, define an energy-like function $W(t, x)$ as follows

$$W(t, x) = x'^2 + 2a(t) \int_x^\infty f(u)du.$$
Using (1), we obtain

$$W'(t, x) = 2a'(t) \int_0^x f(u) \, du$$

$$\geq -a'(t)2 \int_0^x f(u) \, du$$

$$\geq -\frac{a'(t)}{a(t)} W(t, x),$$

from which it follows that

$$W(t, x) \geq W(t_0, x_0) \exp \left( -\int_{t_0}^t \frac{a'(\tau)}{a(\tau)} \, d\tau \right).$$

Since for every non-trivial solution we must have $W(t_0, x_0) > 0$, (22) yields $W(t, x) \geq \zeta^2 > 0$ for all $t$. Let $\{t_k\}$ be the sequence of zeros of $x(t)$ such that $t_k \to \infty$. We have from (21) that $|x'(t_k)| \geq \zeta > 0$ for all $k$, and in particular (20) holds.

References


Mathematics Research Centre
University of Wisconsin
Madison, Wisconsin, U.S.A.

Department of Mathematics
Carnegie Mellon University
Pittsburgh, Pa., 15213, U.S.A.