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CHARACTERISING COMPLETE BOOLEAN ALGEBRAS IN TERMS OF PURE ESSENTIALNESS

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We discuss purity and pure essentialness of abelian groups in a topos $Sh\mathcal{L}$ of sheaves on a locale $\mathcal{L}$ and show that purity is not a local property. We prove that $A \in AbSh\mathcal{L}$ is divisible if and only if it is pure in every extension, and give an example of a category in which absolutely pure does not imply divisible. We discuss uniform abelian groups and show that each AU uniform in Ab does not imply that A is uniform in $AbSh\mathcal{L}$.

Banaschewski showed that the pure subgroups of $Z_{\mathcal{L}}$ are exactly of the type $T_U Z_{\mathcal{L}}$ for the different $U \in \mathcal{L}$. We show that $T_U Z_{\mathcal{L}}$ is essential in $Z_{\mathcal{L}}$ if and only if U is dense in $\mathcal{L}$, Finally, we characterise as complete boolean algebras the locales for which the only pure and essential subgroup of $Z_{\mathcal{L}}$ is $Z_{\mathcal{L}}$.

1. BACKGROUND

DEFINITION 1.1: A *locale* denoted by $\mathcal{L}$ is a complete lattice staisfying the following:

$$U \wedge \bigvee U_i = \bigvee U \wedge U_i,$$

for all U and any family $\{U_i\}_{i \in I}$ in $\mathcal{L}$. We denote the minimal element of $\mathcal{L}$ by 0 and the maximal element by E. Some examples of locales are a topology of a space, a complete chain, complete boolean algebra or a finite distributive lattice.

DEFINITION 1.2: Recall that for any $0 \neq n \in \mathsf{N}$ and $A \in AbSh\mathcal{L}$, one has a map $n_A \colon A \to A = A \to A^n \to A$ where $\mathrm{Im}\, n_A$ is denoted by nA. Further A is said to be divisible if and only if $A = nA$ for all $0 \neq n \in \mathsf{N}$, that is, for any $a \in AU$ and $0 \neq n \in \mathsf{N}$ there exists a cover $u = \bigvee U_i$ such that $a \mid U_i = nb_i$ for some $b_i \in AU_i$ for all i.

For reference on background material required here, the reader may refer to [**2, 4, 6, 7, 9, 10**].

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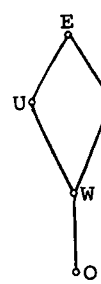
2. PURITY

DEFINITION 2.1: A monomorphism $h\colon A \to B$ in $AbSh\mathcal{L}$ is said to be pure if for all $n \in \mathbb{N}$, the diagram

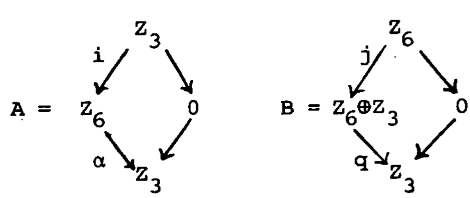
$$\begin{array}{ccc} nA & \xrightarrow{\bar{h}} & nB \\ \downarrow & & \downarrow \\ A & \xrightarrow[h]{} & B \end{array}$$

is a pull back diagram. If A is a subgroup of B and h is a natural embedding then $A \subseteq B$ is pure if $nA = A \cap nB$, that is, $(nA)U = AU \cap (nB)U$ for all $U \in \mathcal{L}$.

Counterexample 2.2. The following counterexample shows that purity is not a local property: that is , there is a local $\mathcal{L}$ and $A, B \in AbSh\mathcal{L}$ such that for some cover $E = \bigvee_{i \in I} U_i$, $A \mid U_i \subseteq B \mid Ui$ is pure in $AbSh \downarrow Ui$ for all $i \in I$, but $A \subseteq B$ is not pure in $AbSh\mathcal{L}$. Consider the locale,



and $A, B \in AbSh\mathcal{L}$ given by,



where α is multiplication by 3 and the other maps are obvious maps. Then $A \mid U \subseteq B \mid U$ and $A \mid V \subseteq B \mid V$ are both pure maps in $AbSh \downarrow U$ and $AbSh \downarrow V$ respectively, but $A \subseteq B$ is not pure in $AbSh\mathcal{L}$, for if this was pure then

$$\begin{array}{ccc} 3A & \longrightarrow & 3B \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

has to be a pull-back which implies that

$$\begin{array}{ccc} 0=(3A)E & \longrightarrow & (3B)E=Z_3 \\ \downarrow & & \downarrow \\ Z_3=AE & \longrightarrow & BE=Z_6 \end{array}$$

is a pull-back, a contradiction, hence the conclusion.

PROPOSITION 2.3. *If $A \in AbSh\mathcal{L}$ is divisible, then for all extensions B of A, the monomorphism $h\colon A \to B$ is pure.*

PROOF: If A is divisible then $nA = A$. Hence, it is clear that the diagram

$$\begin{array}{ccc} nA & \xrightarrow{\bar{h}} & nB \\ \downarrow & & \downarrow \\ A & \xrightarrow[h]{} & B \end{array}$$

is a pull-back, that is, h is a pure map. ∎

PROPOSITION 2.4. *If $h\colon A \to B$ is a pure map with B a divisible group, then A is also a divisible group.*

PROOF: By the given hypothesis, we have a pull back diagram

$$\begin{array}{ccc} nA & \xrightarrow{\bar{h}} & nB \\ \downarrow & & \downarrow \\ A & \xrightarrow[h]{} & B \end{array}$$

for all $n \in \mathbb{N}$. So, there exists a unique $\alpha\colon A \to nA$ such that in the diagram,

$$\begin{array}{ccccc} A & = & A & = & A \\ \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle h} \\ A & & nA & \xrightarrow{\bar{h}} & nB(=B) \\ \| & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow{1_A} & A & \xrightarrow[h]{} & B \end{array}$$

$\bar{h}\alpha = h$ and $i\alpha = 1_A$. So i is an epimorphism, hence an isomorphism and therefore $A = nA$, that is A is divisible. Thus we obtain the following: ∎

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COROLLARY 2.5. *$A \in AbSh\mathcal{L}$ is divisible if and only if it is pure in every extension.*

PROOF: ($\Rightarrow$) This is clear from Proposition 2.3. For the converse, since $AbSh\mathcal{L}$ has enough injectives [**10**], there exists an injective $B \in AbSh\mathcal{L}$ and a monomorphism $h\colon A \to B$, and since injective implies divisible, the result now follows from Proposition 2.4. ∎

Remark 2.6. It is clear that $A \in AbSh\mathcal{L}$ is injective if and only if A is an absolute retract. Since the category $AbSh\mathcal{L}$ has the special property that injective always implies divisible, by the above Corollary we have that injective implies absolute purity. Of course in $AbSh\mathcal{L}$, A an absolutely pure group does not necessarily imply that A is injective. This is so because for a non-boolean locale $\mathcal{L}$, there are divisible (=absolutely pure) groups which are not injective [**1**]. Here is an example of an abelian category, where we show that injective does not imply divisible, which also shows that absolutely pure does not imply divisible.

Example 2.7. Consider the category P of elementary abelian p-groups. Then P is an abelian category and is the same as the category of vector spaces over the field $\mathbb{Z}/p\mathbb{Z}$. Therefore each $A \in P$ is an injective group, hence absolutely pure but no nonzero A is divisible, since $0 = pA \neq A$.

PROPOSITION 2.8. *For $A \to B$ in $AbSh\mathcal{L}$, if each $AU \to BU$ is pure in Ab, then $A \to B$ is pure in $AbSh\mathcal{L}$.*

PROOF: This is clear, since the sheaf reflection preserves finite limits and co-limits it preserves pull backs and satisfies the condition $(nA)^{\sim} = n\tilde{A}$. ∎

PROPOSITION 2.9. *The torsion subgroup of a group is a pure subgroup.*

PROOF: Let T denote the torsion subgroup of a given group $A \in AbSh\mathcal{L}$. Then $T = \cup_{0 \neq n \in \mathbb{N}} \operatorname{Ker} n_A$, which is the same as saying that $TU \doteq T(AU)$, where $T(AU)$ is the torsion subgroup of AU. Now at each $U \in L$, the diagram

$$\begin{array}{ccc} nT(AU) & \longrightarrow & nAU \\ \downarrow & & \downarrow \\ T(AU) & \longrightarrow & AU \end{array}$$

is a pull-back in Ab, therefore

$$\begin{array}{ccc} nT_o & \longrightarrow & nA \\ \downarrow & & \downarrow \\ T_o & \longrightarrow & A \end{array}$$

is a pull-back in $AbPSh\mathcal{L}$, where T_o is the presheaf $U \rightsquigarrow T(AU)$. By 2.8 it follows that the square

$$\begin{array}{ccc} nT & \longrightarrow & nA \\ \downarrow & & \downarrow \\ T & \longrightarrow & A \end{array}$$

is a pull-back square in $AbSh\mathcal{L}$, hence T is a pure subgroup of A. ∎

PROPOSITION 2.10. *If $h\colon A \to B$ is retractable, then h is pure.*

PROOF: Let $g\colon B \to A$ be such that $gh = 1_A$. Consider the diagram

$$\begin{array}{ccccc} C & = & C & = & C \\ \| & & & & \downarrow t \\ C & & nA & \longrightarrow & nB \\ \| & & i_A\downarrow & & \downarrow i_B \\ C & \xrightarrow[s]{} & A & \xrightarrow[h]{} & B \end{array}$$

where $hs = i_B t$. Now it can be seen as follows that $gh = 1_A$ implies $\overline{g}\overline{h} = 1_{nA}$ (where $\overline{g}\colon nB \to nA$ is the unique map such that $i_A\overline{g} = gi_B$). Since $i_B\overline{h} = hi_A$, $gi_B = i_A\overline{g}$, $gi_B\overline{h} = ghi_A = i_A$. Now $i_A\overline{g}\overline{h} = gi_B\overline{h} = i_A = i_A 1_{nA}$. Since i_A is a monomorphism, $\overline{g}\overline{h} = 1_{nA}$. Consider the map $\alpha = \overline{g}t\colon C \to nA$. We claim this is the desired map, that is α is unique such that $i_A\alpha = s$ and $\overline{h}\alpha = t$. That α is unique is clear, since 1_A is a monomorphism. Also $i_A\alpha = i_A\overline{g}t = gi_Bt = ghs = s$, and $i_B\overline{h}\overline{g}t = hi_A\overline{g}t = hs = i_Bt$. Since i_B is a monomorphism, it follows $\overline{h}\overline{g}t = \overline{h}\alpha = t$, hence $h\colon A \rightarrowtail B$ is pure. ∎

Motivated by module theory we say,

DEFINITION 2.11: A in $AbSh\mathcal{L}$ is uniform if every subgroup of A is essential in A.

PROPOSITION 2.12. *A is uniform if and only if $E(A)$, the injective hull of A, is uniform.*

PROOF: Let $0 \neq B, C$ be non-zero subgroups of $E(A)$. Since A is an essential subgroup, therefore $A \cap B$, $A \cap C$ are non-zero subgroups of A. But A uniform implies $(A \cap B) \cap (A \cap C) \neq 0$, hence $B \cap C \neq 0$.
Conversely, if $E(A)$ is uniform, then it is clear that every subgroup is uniform. ∎

PROPOSITION 2.13. *A uniform implies that A is indecomposable and A an indecomposable injective implies that A is uniform.*

PROOF: ($\Rightarrow$) Let $A = B \oplus C$, $B \neq 0$, then A uniform and $B \cap C = 0$ implies $C = 0$.
($\Leftarrow$): Let $0 \neq B \subseteq A$, then $E(B) \subseteq A$ and so $A = E(B) \oplus C$. But A indecomposable implies $E(B) = 0$ or $C = 0$. Since $0 \neq B$ it follows $C = 0$ and so $E(B) = A$. Hence A is an essential extension of each of its subgroups, that is A is uniform. ∎

Counterexample 2.14. If all $AU (U \in \mathcal{L})$ are uniform in Ab then A need not be uniform in $AbSh\mathcal{L}$. Consider the locale $\mathcal{L} = 3$, and an A in $AbSh3$ given by

$$A = \begin{array}{c} Z \\ \big\downarrow o \\ Z \end{array}$$

We claim A is not uniform although each $AU = Z$ is uniform in Ab.

$$B = \begin{array}{c} Z \\ \big\downarrow \\ 0 \end{array} \qquad C = \begin{array}{c} 0 \\ \big\downarrow \\ Z \end{array}$$

are non zero subgroups of A but certainly

$$B \cap C = \begin{array}{c} 0 \\ \big\downarrow \\ 0. \end{array}$$

PROPOSITION 2.15. *For any $U \in \mathcal{L}$ the restriction functors $R_U: AbSh\mathcal{L} \to AbSh \downarrow U$ preserve uniform groups.*

PROOF: Let $A \in AbSh\mathcal{L}$, be uniform. We claim $A \mid U = R_U A$ is uniform in $AbSh \downarrow U$. Let B, C be non zero subgroups of $A \mid U$. Then $BW \neq 0$ for $W \subseteq U$ and $CV \neq 0$ for some $V \subseteq U$. This means $E_U B$ and $E_U C$ are non-zero subgroups of $E_U(A \mid U) = T_U(A)$. Since $T_U(A) \subseteq A$ [2] and A is uniform, $E_U B \cap E_U C \neq 0$, that is $(E_U B \cap E_U C)W$ for some $W \in \mathcal{L}$. By definition of the functor E_U, [2], there exists some $W_i \subseteq W$, $W_i \subseteq U$ such that $(B \cap C)W_i \neq 0$, hence $B \cap C \neq 0$, that is $A \mid U$ is uniform in $AbSh \downarrow U$. ∎

PROPOSITION 2.16. *Let C be pure in B and B pure in A such that A is a pure exxential extension of C. Then B is a pure essential extension of C and A is a pure essential extension of B.*

PROOF: Let $\alpha: B \to E$ be such that αi is pure where $i: C \to B$ is the natural embedding. Embed E into its pure injective hull denoted by $\overline{E}$. Since $j: B \to A$ is pure and $\overline{E}$ is pure injective, there exists $\overline{\alpha}\colon A \to \overline{E}$ such that $\overline{\alpha}j = k\alpha$.

$$\begin{array}{ccccc} C & \xrightarrow{\ i\ } & B & \xrightarrow{\ j\ } & A \\ & & {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\alpha}} \\ & & E & \xrightarrow[\ k\ ]{} & \overline{E} \end{array}$$

Now $\overline{\alpha}ji = k\alpha i$ and therefore $\overline{\alpha}ji$ is pure. But ji pure essential implies that $\overline{\alpha}$ is a monomorphism and so $\overline{\alpha}j$ is a monomorphism. Since $k\alpha = \overline{\alpha}j$, α is a mono and hence i is pure essential. Now to see that A is pure essential extension of B, we consider $\beta\colon A \to D$ such that βj is pure. Then βji is pure, but ji pure essential implies β is monomorphism, hence j is pure essential. ∎

Remark. In [**2**], Banaschewski has shown that the pure subgroups of $Z_{\mathcal{L}}$ are exactly of the form $T_U Z_{\mathcal{L}}$ for $U \in \mathcal{L}$. The aim is now to characterise those $u \in \mathcal{L}$ for which the pure subgroup $T_U Z_{\mathcal{L}}$ is essential.

PROPOSITION 2.17. *For any $U \in \mathcal{L}$, $T_U Z_{\mathcal{L}} \subseteq Z_{\mathcal{L}}$ is an essential subgroup if and only if U is dense in $\mathcal{L}$.*

PROOF: ($\Rightarrow$) Consider $0 \neq V \in \mathcal{L}$ and $0 \neq a \in Z_{\mathcal{L}}V$. By hypothesis there exist $W \leqslant V$ and $m \in Z$ such that $0 \neq ma \mid W \in (T_U Z_{\mathcal{L}})W$. By definition of $T_U Z_{\mathcal{L}}$, this means there is a cover $W = \bigvee_{i\in I} W_i$ such that for some $W_i \subseteq U$, $0 \neq (ma \mid W) \mid W_i = ma \mid W_i \in Z_{\mathcal{L}}W_i$. That shows $0 \neq W_i \leqslant U \wedge V$, that is $U \wedge V \neq 0$ and therefore U is dense in $\mathcal{L}$.
($\Leftarrow$) Consider any $0 \neq \phi \in Z_{\mathcal{L}}V$ for some $V \in \mathcal{L}$. Then $V = \bigvee_{n\in Z} \phi(n)$, and since U is dense in $\mathcal{L}$, $0 \neq U \wedge V = \bigvee_{n\in Z} U \wedge \phi(n)$, therefore $U \wedge \phi(n) \neq 0$ for some n. Thus $0 \neq \phi \mid (U \wedge \phi(n)) \in Z_{\mathcal{L}}(U \wedge \phi(n)) = (T_U Z_{\mathcal{L}})(U \wedge \phi(n))$ which shows $T_U Z_{\mathcal{L}} \subseteq Z_{\mathcal{L}}$ is essential. ∎

PROPOSITION 2.18. *A locale $\mathcal{L}$ is Boolean if and only if the only pure and essential subgroup of $Z_{\mathcal{L}}$ is $Z_{\mathcal{L}}$.*

PROOF: ($\Rightarrow$) If $\mathcal{L}$ is Boolean, then $\mathcal{L}$ has no dense elements and so the result follows by 2.17.
($\Leftarrow$) The given conditions imply that $\mathcal{L}$ has no dense elements, hence $\mathcal{L}$ is Boolean. ∎

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