ANZIAM J. 43(2002), 449–462

FURTHER RESULTS FOR A ONE-DIMENSIONAL LINEAR THERMOELASTIC EQUATION WITH DIRICHLET-DIRICHLET BOUNDARY CONDITIONS

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(Received 23 March, 1999; revised 3 December, 1999)

Abstract

We show that a sequence of generalized eigenfunctions of a one-dimensional linear thermoelastic system with Dirichlet-Dirichlet boundary conditions forms a Riesz basis for the state Hilbert space. This develops a parallel result for the same system with Dirichlet-Neumann or Neumann-Dirichlet boundary conditions.

1. Introduction

In the past two decades, much effort has been concentrated on the heat equation which incorporates the effect of thermomechanical coupling and inertia. In this paper, we study the following one-dimensional linear model for longitudinal vibration within a thermoelastic rod with Dirichlet-Dirichlet boundary conditions (see [2, 3, 5, 6, 8] and the references therein):

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \gamma\theta_x(x,t) = 0, & 0 < x < 1,\ t > 0, \\ \theta_t(x,t) + \gamma u_{xt}(x,t) - k\theta_{xx}(x,t) = 0, & 0 < x < 1,\ t > 0, \\ u(i,t) = \theta(i,t) = 0, & i = 0, 1,\ t \geq 0, \end{cases} \tag{1}$$

where $u = u(x,t)$ represents displacement, $\theta = \theta(x,t)$ represents absolute temperature and $k > 0$ the thermal conductivity. The coupling constant $\gamma > 0$ which is a measure of the mechanical-thermal coupling present in the system is generally much smaller than 1. The following results were collected from [3, 6].

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https://doi.org/10.1017/S1446181100012621 Published online by Cambridge University Press

THEOREM 1. (i) *The system* (1) *associates with a solution of a* C_0*-semigroup of contractions* $T(t) = e^{At}$ *on the state Hilbert space* $\mathscr{H} = H_0^1(0,1) \times (L^2(0,1))^2$, *where* $A : D(A) \to \mathscr{H}$ *is given by*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ D^2 & 0 & -\gamma D \\ 0 & -\gamma D & kD^2 \end{pmatrix},$$

with $D = \partial/\partial x$, $D(A) = (H^2 \times H_0^1 \times (H_0^1 \cap H^2)) \cap \mathscr{H}$.

(ii) A^{-1} *exists and is compact on* $\mathscr{H}$. *Hence* $\sigma(A) = \sigma_p(A)$ *consists of isolated eigenvalues only. We have that* $\lambda \in \sigma_p(A)$ *if and only if* $\lambda \neq 0$ *satisfies the characteristic equation*

$$\begin{aligned} 8\gamma^2\sqrt{k\lambda} + \left[e^{a_1-a_2} + e^{a_2-a_1}\right]\left(k\lambda + \gamma^2 + 1 + 2\sqrt{k\lambda}\right)\left(1 - \sqrt{k\lambda}\right)^2 \\ - \left[e^{a_1+a_2} + e^{-a_1-a_2}\right]\left(k\lambda + \gamma^2 + 1 - 2\sqrt{k\lambda}\right)\left(1 + \sqrt{k\lambda}\right)^2 = 0 \end{aligned} \tag{2}$$

where

$$a_1 = \sqrt{\frac{\lambda}{2k}\left[k\lambda + \gamma^2 + 1 + \sqrt{(k\lambda + \gamma^2 + 1)^2 - 4k\lambda}\right]}, \quad a_2 = \frac{\lambda\sqrt{\lambda/k}}{a_1}. \tag{3}$$

(iii) *The eigenvalues of* A *consist of a real sequence* $\{\sigma_n\}$ *and a sequence of conjugate pairs* $\{\lambda_n, \bar{\lambda}_n\}$ *with the asymptotic properties*:

$$\begin{cases} \sigma_n = -k(n\pi)^2 + \gamma^2/k + \mathscr{O}(n^{-2}), \\ \lambda_n = -\gamma^2/(2k) + in\pi + \mathscr{O}(n^{-1}), \end{cases} \tag{4}$$

where n *is a large positive integer.*

Many other important properties of (1) have been discovered in recent years. It is shown in [6], for example, by a frequency domain criteria for the stability of infinite dimensional linear systems, that the semigroup e^{At} is uniformly exponentially stable. In [5], it is proved that the asymptote of the complex eigenvalues of A, $-\gamma^2/(2k)$, is also the essential spectral bound of A. From the well-known fact that $\omega(A) = \max\{S(A), \omega_{\text{ess}}(A)\}$ and (4), where $\omega(A)$, $S(A)$ and $\omega_{\text{ess}}(A)$ denote the growth order of the semigroup e^{At}, the spectral bound and essential spectral bound of A, respectively, we see that the spectrum-determined growth condition

$$\omega(A) = S(A)$$

is always true of system (1). A significant result on the eigenvalues of A was reported in [2], namely that system (1) has at least one real eigenvalue which is greater than the

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first eigenvalue $-k\pi^2$ of the "pure" heat equation with the same boundary conditions ($\gamma = 0$ in (1)), which, together with an earlier result that $\omega(A) \leq \max\{S(A), -k\pi^2\}$ (see [8]), gives again the spectrum-determined growth condition.

On the other hand, it was shown in [4] that for the same thermoelastic system with Dirichlet-Neumann or Neumann-Dirichlet boundary conditions, there is a sequence of generalized eigenfunctions of the system which forms a Riesz basis for the state Hilbert space. The success in obtaining this result lies in the simplicity of the corresponding characteristic equation as well as the explicit structure of the eigenfunctions. However, for system (1), the characteristic equation (2) is a complicated transcend equation, and the eigenfunctions satisfy a fourth-order ordinary differential equation. The method used in [4] is not practically applicable to this case. In this paper, we shall overcome this difficulty in a different way. By use of an abstract result of the Riesz basis perturbation in Hilbert space which was reported in [1], we are able to develop the Riesz basis property of system (1). Meanwhile, our approach can also be easily used to treat other boundary conditions particularly those such as the *natural boundary conditions* considered in [4].

2. Approximate normalized eigenfunctions

It is known from [3] that $\lambda \in \sigma(A)$ (as $\operatorname{Re}\lambda < 0$) if and only if there exists $(\phi, \psi) \neq 0$ such that

$$\begin{cases} k\phi^{(4)}(x) - \lambda(k\lambda + \gamma^2 + 1)\phi''(x) + \lambda^3\phi(x) = 0, \\ \phi(i) = \lambda(k\lambda + \gamma^2)\phi'(i) - k\phi'''(i) = 0, \qquad i = 0, 1, \end{cases} \tag{5}$$

and

$$\gamma\lambda\psi(x) = k\phi'''(x) - \lambda(k\lambda + \gamma^2)\phi'(x). \tag{6}$$

Moreover, λ is geometrically simple and an associated eigenfunction is $(\phi, \lambda\phi, \psi)$.

The characteristic equation of (5) is

$$ka^4 - \lambda(k\lambda + \gamma^2 + 1)a^2 + \lambda^3 = 0 \tag{7}$$

which has four different roots a_1, a_2, $-a_1$, $-a_2$, where a_1, a_2 depend on λ and are defined as in (3). For any λ,

$$\begin{aligned} \phi(x) &= (g_2 \sinh a_1 - g_1 \sinh a_2)(\cosh a_1 x - \cosh a_2 x) \\ &\quad - (\cosh a_1 - \cosh a_2)(g_2 \sinh a_1 x - g_1 \sinh a_2 x) \\ &= g_2 \sinh a_1(1-x) + g_1 \sinh a_2(1-x) \\ &\quad + g_2 \cosh a_2 \sinh a_1 x + g_1 \cosh a_1 \sinh a_2 x \\ &\quad - g_2 \sinh a_1 \cosh a_2 x - g_1 \sinh a_2 \cosh a_1 x \end{aligned} \tag{8}$$

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satisfies

$$\begin{cases} k\phi^4(x) - \lambda(k\lambda + \gamma^2 + 1)\phi''(x) + \lambda^3\phi(x) = 0, \\ \phi(i) = \lambda(k\lambda + \gamma^2)\phi'(0) - k\phi'''(0) = 0, \qquad i = 0, 1, \end{cases} \tag{9}$$

where

$$g_i = a_i(k\lambda^2 + \gamma^2\lambda - ka_i^2) \neq 0, \quad i = 1, 2. \tag{10}$$

To make $\phi(x)$ (as defined by (8)) be a solution of (5), a necessary and sufficient condition is that ϕ satisfies the last boundary condition $\lambda(k\lambda+\gamma^2)\phi'(1) - k\phi'''(1) = 0$ which leads to the deduction that

$$\begin{aligned} &-2[\lambda(k\lambda + \gamma^2)\phi'(1) - k\phi'''(1)] \\ &\quad = 4g_1g_2 - 4g_1g_2\cosh a_1 \cosh a_2 + 2[g_1^2 + g_2^2]\sinh a_1 \sinh a_2 \\ &\quad = 4g_1g_2 + (g_1 - g_2)^2\cosh(a_1 + a_2) - (g_1 + g_2)^2\cosh(a_1 - a_2). \end{aligned} \tag{11}$$

We obtain once again the characteristic equation derived in [3].

By (6) and (8), we can explicitly write the expression

$$\begin{aligned} \psi(x) &= \frac{k}{\gamma\lambda}\phi'''(x) - \frac{1}{\gamma}(k\lambda + \gamma^2)\phi'(x) \\ &= -\frac{1}{\gamma\lambda}\Big[- g_1g_2\cosh a_1(1-x) - g_1g_2\cosh a_2(1-x) \\ &\qquad - g_1^2\sinh a_2 \sinh a_1x - g_2^2\sinh a_1 \sinh a_2x \\ &\qquad + g_1g_2\cosh a_1 \cosh a_2x + g_1g_2\cosh a_2\cosh a_1x\Big]. \end{aligned} \tag{12}$$

Equations (8) and (12) are our basis for the estimate of the eigenfunctions. We therefore need the following asymptotic expressions of a_i, $i = 1, 2$, which also appeared in [3, 7] (note that there is a typing error in the original equation (20) in [3], we modify it here):

$$\begin{cases} a_1 = \lambda + \dfrac{\gamma^2}{2k} + \dfrac{4\gamma^2 - \gamma^4}{8}\dfrac{1}{\lambda} + \mathcal{O}(|\lambda|^{-2}), \\ a_2 = \dfrac{\sqrt{\lambda}}{\sqrt{k}} - \dfrac{\gamma^2}{2\sqrt{k}}\dfrac{1}{\sqrt{\lambda}} + \mathcal{O}(|\lambda|^{-3/2}), \qquad \text{as } |\lambda| \to \infty, \end{cases} \tag{13}$$

and hence

$$\begin{cases} g_1 = a_1(k\lambda^2 + \gamma^2\lambda - ka_1^2) = -(\gamma^2/k)\lambda\left[1 + \mathcal{O}(|\lambda|^{-1})\right], \\ g_2 = a_2(k\lambda^2 + \gamma^2\lambda - ka_2^2) = \sqrt{k}\lambda^2\sqrt{\lambda}\left[1 + \mathcal{O}(|\lambda|^{-1})\right]. \end{cases} \tag{14}$$

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Therefore

$$\begin{cases} e^{a_1x} = e^{(\lambda+\gamma^2/(2k))x}\left[1+\mathcal{O}(|\lambda|^{-1})\right], \\ e^{a_2x} = e^{\sqrt{\lambda/k}\,x}\left[1-\dfrac{\gamma^2}{2k^{3/2}}\dfrac{x}{\sqrt{\lambda}}+\mathcal{O}(|\lambda|^{-1})\right], \\ e^{-a_1x} = e^{-(\lambda+\gamma^2/(2k))x}\left[1+\mathcal{O}(|\lambda|^{-1})\right], \\ e^{-a_2x} = e^{-\sqrt{\lambda/k}\,x}\left[1+\dfrac{\gamma^2}{2k^{3/2}}\dfrac{x}{\sqrt{\lambda}}+\mathcal{O}(|\lambda|^{-1})\right]. \end{cases} \tag{15}$$

LEMMA 1. *The eigenfunctions* $\{(\phi_n, \sigma_n\phi_n, \psi_n)\}$ *associated with the real eigenvalues* $\{\sigma_n\}$ *of* A *take the following asymptotic expression*:

$$-2\gamma\sigma_n\frac{e^{a_1}}{g_2^2}\begin{pmatrix}\phi_n(x)\\ \sigma_n\phi_n(x)\\ \psi_n(x)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ i\sin n\pi x\end{pmatrix} + F_n(x) \quad \textit{with } \|F_n\|_{\mathscr{H}} = \mathcal{O}(n^{-1})$$

which holds pointwise uniformly for $x \in [0,1]$.

PROOF. It follows from (4) that the real eigenvalue σ_n satisfies

$$\sigma_n = -k(n\pi)^2 + \gamma^2/k + \mathcal{O}(n^{-2})$$

for large positive integer n, hence $\sqrt{\sigma_n}/\sqrt{k} = in\pi + \mathcal{O}(n^{-1})$. By (15),

$$\sinh a_2x = i\sin n\pi x + \mathcal{O}(n^{-1}), \quad \cosh a_2x = \cos n\pi x + \mathcal{O}(n^{-1}). \tag{16}$$

Note that $g_1/g_2 = -\gamma^2/(k^{3/2}\sigma_n\sqrt{\sigma_n}) + \mathcal{O}(|\sigma_n|^{-1})$. We have, by referring to (12) and (16), that

$$\begin{aligned} -2\gamma\sigma_n\frac{e^{a_1}}{g_2^2}\psi_n(x) &= -\frac{g_1}{g_2}e^{a_1x} + \left(\frac{g_1}{g_2}\right)^2\sinh a_2e^{a_1(1-x)} + \sinh a_2x + \frac{g_1}{g_2}\cosh a_2x \\ &\quad + \frac{g_1}{g_2}\cosh a_2e^{a_1(1-x)} + \mathcal{O}(n^{-1}) \\ &= \sinh a_2x + \mathcal{O}(n^{-1}) = i\sin n\pi x + \mathcal{O}(n^{-1}). \end{aligned} \tag{17}$$

Furthermore, by (8)

$$\begin{aligned} \phi_n'(x) = &-a_1g_2\cosh a_1(1-x) - a_2g_1\cosh a_2(1-x) + a_1g_2\cosh a_2\cosh a_1x \\ &+ a_2g_1\cosh a_1\cosh a_2x - a_2g_2\sinh a_1\sinh a_2x - a_1g_1\sinh a_2\sinh a_1x. \end{aligned}$$

We have

$$\begin{aligned} 2\frac{e^{a_1}}{g_2^2}\phi_n'(x) = &-a_1/g_2e^{a_1x} + a_1/g_2\cosh a_2e^{a_1(1-x)} + a_2g_1/g_2^2\cosh a_2x \\ &+ a_2/g_2\sinh a_2x + a_1g_1/g_2^2\sinh a_2e^{a_1(1-x)} + \mathcal{O}(e^{-c|\sigma_n|}) \end{aligned}$$

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where $c > 0$ is a constant independent of σ_n. Hence

$$2\frac{e^{a_1}}{g_2^2}\phi_n'(x) = \mathcal{O}(|\sigma_n|^{-3/2})$$

or

$$-2\gamma\sigma_n\frac{e^{a_1}}{g_2^2}\phi_n'(x) = \mathcal{O}(|\sigma_n|^{-1/2}) = \mathcal{O}(n^{-1}). \tag{18}$$

Similarly,

$$-2\gamma\sigma_n\frac{e^{a_1}}{g_2^2}\sigma_n\phi_n(x) = \mathcal{O}(n^{-1}). \tag{19}$$

Combining (17)–(19) gives

$$-2\gamma\sigma_n\frac{e^{a_1}}{g_2^2}\begin{pmatrix}\phi_n(x)\\ \sigma_n\phi_n(x)\\ \psi_n(x)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ i\sin n\pi x\end{pmatrix} + F_n(x) \quad \text{with } \|F_n\|_{\mathcal{H}} = \mathcal{O}(n^{-1}).$$

This gives the required result.

LEMMA 2. *The eigenfunctions* $\{(\phi_n, \lambda_n\phi_n, \psi_n), (\bar{\phi}_n, \bar{\lambda}_n\bar{\phi}_n, \bar{\psi}_n)\}$ *associated with the complex conjugate eigenvalue pairs* $\{\lambda_n, \bar{\lambda}_n\}$ *of* A *take the following asymptotic expression*:

$$\frac{2e^{-a_2}}{a_1g_2^2}\begin{pmatrix}\phi_n(x)\\ \lambda_n\phi_n(x)\\ \psi_n(x)\end{pmatrix} = \begin{pmatrix}f_n(x)\\ i\sin n\pi x\\ 0\end{pmatrix} + F_n(x)$$

which holds pointwise uniformly for $x \in [0, 1]$, *where* $f_n'(x) = \cos n\pi x + \mathcal{O}(n^{-1})$, $\|F_n\|_{\mathcal{H}} = \mathcal{O}(n^{-1})$.

PROOF. For the complex conjugate eigenvalue pairs $\{\lambda_n, \bar{\lambda}_n\}$ of A, it holds that

$$\lambda_n = -\gamma^2/(2k) + in\pi + \mathcal{O}(n^{-1}) \tag{20}$$

for large positive integer n and hence $\cosh a_1x = \cos n\pi x + \mathcal{O}(n^{-1})$ and $\sinh a_1x = i\sin n\pi x + \mathcal{O}(n^{-1})$. Moreover, $\sqrt{\lambda_n} = \sqrt{1/2}\,(1+i)\sqrt{n\pi}[1 + \mathcal{O}(n^{-1})]$. By (8), (15) and (20)

$$\begin{aligned}\frac{2e^{-a_2}}{a_1g_2}\phi_n'(x) = &-\frac{a_2g_1}{a_1g_2}e^{-a_2x} + \cosh a_1x + \frac{a_2g_1}{a_1g_2}\cosh a_1e^{-a_2(1-x)}\\ &-\frac{a_2}{a_1}\sinh a_1e^{-a_2(1-x)} - \frac{g_1}{g_2}\sinh a_1x + \mathcal{O}(n^{-1}).\end{aligned}$$

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Notice again that $g_1/g_2 = -\left(\gamma^2/k^{3/2}\lambda_n\sqrt{\lambda_n}\right)[1+\mathcal{O}(|\lambda_n|^{-1})]$, $a_2/a_1 = \left(1/\sqrt{k\lambda_n}\right)[1+\mathcal{O}(|\lambda_n|^{-1})]$. We have thus that $a_2g_1/(a_1g_2) = -\left(\gamma^2/k^2\lambda_n^2\right)[1+\mathcal{O}(|\lambda_n|^{-1})] = \mathcal{O}(n^{-2})$. Therefore

$$\frac{2e^{-a_2}}{a_1g_2}\phi_n'(x) = \cosh a_1x + \mathcal{O}(n^{-1}) = \cos n\pi x + \mathcal{O}(n^{-1}). \tag{21}$$

Similarly,

$$\begin{aligned}\frac{2e^{-a_2}}{a_1g_2}\lambda_n\phi_n(x) &= \lambda_n\frac{g_1}{a_1g_2}e^{-a_2x} + \frac{\lambda_n}{a_1}\sinh a_1x + \lambda_n\frac{g_1}{a_1g_2}\cosh a_1e^{-a_2(1-x)}\\ &\quad - \frac{\lambda_n}{a_1}\sinh a_1e^{-a_2(1-x)} - \frac{g_1}{a_1g_2}\cosh a_1x + \mathcal{O}(n^{-1})\\ &= \sinh a_1x + \mathcal{O}(n^{-1}) = i\sin n\pi x + \mathcal{O}(n^{-1}).\end{aligned} \tag{22}$$

Also

$$\begin{aligned}\frac{2e^{-a_2}}{a_1g_2}\psi_n(x) = -\frac{1}{\gamma\lambda_n}\Bigg[&-\frac{g_1}{a_1}e^{-a_2x} - \frac{g_1^2}{a_1g_2}\sinh a_1x - \frac{g_2}{a_1}\sinh a_1e^{-a_2(1-x)}\\ &+\frac{g_1}{a_1}\cosh a_1e^{-a_2(1-x)} + \frac{g_1}{a_1}\cosh a_1x + \mathcal{O}\left(e^{-c\sqrt{n}}\right)\Bigg] = \mathcal{O}(n^{-1})\end{aligned} \tag{23}$$

where $c > 0$ is a constant independent of λ_n. Combining (21)–(23) gives

$$\frac{2e^{-a_2}}{a_1g_2}\begin{pmatrix}\phi_n(x)\\ \lambda_n\phi_n(x)\\ \psi_n(x)\end{pmatrix} = \begin{pmatrix}f(x)\\ i\sin n\pi x\\ 0\end{pmatrix} + F_n(x)$$

where $f'(x) = \cos n\pi x + \mathcal{O}(n^{-1})$, $\|F_n\|_{\mathcal{H}} = \mathcal{O}(n^{-1})$, proving Lemma 2.

Summarizing, we have obtained estimates for the approximate normalized eigenfunctions which are referred to in the following theorem.

THEOREM 2. *There are two families of approximate normalized eigenfunctions of operator A: one family $\{\Phi_n\}$, $\Phi_n = (\phi_n, \lambda_n\phi_n, \psi_n)$, associated with the real eigenvalues σ_n takes the following asymptotic expression*:

$$\Phi_n = \begin{pmatrix}\phi_n(x)\\ \sigma_n\phi_n(x)\\ \psi_n(x)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ i\sin n\pi x\end{pmatrix} + F_{1n}(x) \quad \textit{with } \|F_{1n}\|_{\mathcal{H}} = \mathcal{O}(n^{-1});$$

the other family $\{\Psi_n, \overline{\Psi}_n\}$, $\Psi_n = (\phi_n, \lambda_n\phi_n, \psi_n)$, $\overline{\Psi}_n = (\bar{\phi}_n, \bar{\lambda}_n\bar{\phi}_n, \bar{\psi}_n)$, corresponding to the complex conjugate eigenvalue pairs $\{\lambda_n, \bar{\lambda}_n\}$, takes the following asymptotic

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expression:

$$\Psi_n = \begin{pmatrix} \phi_n(x) \\ \lambda_n \phi_n(x) \\ \psi_n(x) \end{pmatrix} = \begin{pmatrix} f_n(x) \\ i \sin n\pi x \\ 0 \end{pmatrix} + F_{2n}(x)$$

where $\|F_{2n}\|_{\mathscr{H}} = \mathscr{O}(n^{-1})$, $f_n'(x) = \cos n\pi x + \mathscr{O}(n^{-1})$.

3. Riesz basis property

Let A_0 denote the operator A with $\gamma = 0$, that is, there is no coupling between the wave and heat equations in (1). It is well-known that A_0 is a self-adjoint operator in $\mathscr{H}$ with compact resolvent. There are two families of eigenvalues of A_0: the real eigenvalues $\lambda_{n0} = -k(n\pi)^2$ for n a positive integer, associated with the normalized eigenfunctions $(0, 0, i \sin n\pi x)$, and the complex conjugate pairs $\{\lambda_{n0}, \bar{\lambda}_{n0}\}$, $\lambda_{n0} = in\pi$ for n a positive integer, associated with the normalized eigenfunctions $(\sin n\pi x/(n\pi), i \sin n\pi x, 0)$. Denote

$$\Phi_{n0} = \begin{pmatrix} 0 \\ 0 \\ i \sin n\pi x \end{pmatrix}, \qquad \Psi_{n0} = \begin{pmatrix} \sin n\pi x/(n\pi) \\ i \sin n\pi x \\ 0 \end{pmatrix}. \tag{24}$$

Then $\{\Phi_{n0}, \Psi_{n0}, \overline{\Psi}_{n0}\}_1^\infty$ is the set of all eigenfunctions (up to a scalar) of A_0, which forms an orthonormal basis for $\mathscr{H}$. By Theorem 2, there is an $N > 0$ such that

$$\sum_{n>N}^{\infty} \left[\|\Phi_n - \Phi_{n0}\|_{\mathscr{H}}^2 + \|\Psi_n - \Psi_{n0}\|_{\mathscr{H}}^2 + \|\overline{\Psi}_n - \overline{\Psi}_{n0}\|_{\mathscr{H}}^2 \right] < \infty. \tag{25}$$

We now introduce a perturbation result for a Riesz basis in Hilbert space which has been recently reported in [1].

THEOREM 3. *Let A be a densely defined discrete operator in a Hilbert space H. Let $\{\phi_n\}_1^\infty$ be a Riesz basis for H. If there exists an $N \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^\infty$ of A such that $\sum_{N+1}^\infty \|\phi_n - \psi_n\|^2 < \infty$ then*

(i) *There exists a constant $M > N$ and generalized eigenvectors $\{\psi_{n0}\}_1^M$ of A such that $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis for H;*

(ii) *Let $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ correspond to eigenvalues $\{\sigma_n\}_1^\infty$ of A. Then $\sigma(A) = \{\sigma_n\}_1^\infty$, where σ_n is counted according to its algebraic multiplicity;*

(iii) *If there is an $M_0 > 0$ such that $\sigma_n \neq \sigma_m$ for all $m, n > M_0$, then there is an $N_0 > M_0$ such that all σ_n are algebraically simple if $n > N_0$.*

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By Theorem 3 and (25), we obtain the main result of this paper.

THEOREM 4. *Let A be the operator associated with thermoelastic system* (1) *by Theorem* 1. *Then*

(a) *There is a sequence of generalized eigenfunctions of A, which forms a Riesz basis for the state space $\mathscr{H}$;*

(b) *All $\lambda \in \sigma(A)$ with sufficiently large modulus are algebraically simple.*

Therefore, for the semigroup $T(t)$ generated by A, the spectrum-determined growth condition holds for any $\gamma \geq 0$.

4. Application to other boundary conditions

To further demonstrate our approach, we consider, in this section, the following thermoelastic equation with *natural boundary conditions* (see [4]):

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \gamma\theta_x(x,t) = 0, & 0 < x < 1, t > 0, \\ \theta_t(x,t) + \gamma u_{xt}(x,t) - k\theta_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ u_x(0,t) = u(1,t) = \theta(0,t) = \theta_x(1,t) = 0, & t \geq 0. \end{cases} \tag{26}$$

Let $H = H_e^1(0,1) \times L^2(0,1) \times L^2(0,1)$, $H_e^1(0,1) = \{u \in H^1 \mid u(1) = 0\}$. As with (1), we can write (26) as an evolutionary equation in H:

$$\frac{dw(t)}{dt} = Bw(t) \tag{27}$$

with $w(t) = (u(\cdot,t), u_t(\cdot,t), \theta(\cdot,t))^T$, where the operator $B : D(B) \to H$ is defined in the following way:

$$\begin{cases} B(u,v,\theta) = (v, u_{xx} - \gamma\theta_x, k\theta_{xx} - \gamma v_x), \\ D(B) = \{(u,v,\theta) \in H^2 \times H^1 \times H^2, u'(0) = u(1) = \theta(0) = \theta'(1) = v(1) = 0\}. \end{cases} \tag{28}$$

The following lemma is trivially verified.

LEMMA 3. *Let B be defined by* (28). *Then B^{-1} is compact on H and hence $\sigma(B)$ consists of isolated eigenvalues only.*

As for (5)–(6), we find that for any $\lambda \in \sigma(B)$, there is a unique eigenfunction $(f, \lambda f, g)$ corresponding to λ, where f satisfies

$$\begin{cases} kf''''(x) - \lambda(k\lambda + \gamma^2 + 1)f''(x) + \lambda^3 f(x) = 0, \\ f'(0) = f(1) = 0, \quad f'''(0) = 0, \quad f''(1) = 0 \end{cases} \tag{29}$$

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and $\gamma\lambda g(x) = kf'''(x) - \lambda(k\lambda + \gamma^2)f'(x)$. Therefore, the eigenvalue problem for operator B is equivalent to finding a pair $(\lambda, f) \in \mathbb{C} \times H^4(0, 1)$ such that $f \neq 0$ and (29) is fulfilled.

The general solution of

$$\begin{cases} kf''''(x) - \lambda(k\lambda + \gamma^2 + 1)f''(x) + \lambda^3 f(x) = 0, \\ f'(0) = f'''(0) = 0 \end{cases}$$

is

$$f(x) = c_1(e^{a_1 x} + e^{-a_1 x}) + c_2(e^{a_2 x} + e^{-a_2 x}) \tag{30}$$

where a_i, $i = 1, 2$, are defined by (3) and c_i, $i = 1, 2$, are arbitrary constants. In order that f defined by (30) be a solution of (29), the other boundary conditions $f(1) = f''(1)$ at $x = 1$ should be satisfied. This gives

$$(e^{2a_1} + 1)(e^{2a_2} + 1) = 0. \tag{31}$$

PROPOSITION 1. *It holds that $\lambda \in \sigma(B)$ if and only if λ is a root of* (31).

THEOREM 5. *Asymptotically, the solutions of* (31) *consist of a real sequence $\{\sigma_n\}$ and a sequence of conjugate pairs $\{\lambda_n, \bar{\lambda}_n\}$ with*

$$\begin{aligned} \sigma_n &= -k((n - 1/2)\pi)^2 + \gamma^2/k + \mathcal{O}(n^{-2}), \\ \lambda_n &= -\gamma^2/(2k) + (n - 1/2)\pi i + \mathcal{O}(n^{-1}) \end{aligned} \tag{32}$$

where n is a large positive integer.

PROOF. Equation (31) can be decomposed into

$$e^{2a_1} = -1 \quad \text{or} \quad e^{2a_2} = -1. \tag{33}$$

Choosing $e^{2a_1} = -1$ implies that there is an integer n such that $a_1 = (n - 1/2)\pi i$. By (13),

$$\begin{aligned} \lambda &= (n - 1/2)\pi i - \gamma^2/(2k) - (4\gamma^2 - \gamma^4)/(8\lambda) + \mathcal{O}(|\lambda|^{-2}) \\ &= (n - 1/2)\pi i - \gamma^2/(2k) + \mathcal{O}(|n|^{-1}). \end{aligned}$$

This is the first part. Secondly, $e^{2a_2} = -1$ implies that $a_2 = (n - 1/2)\pi i$ for some integer n. It follows from (13) that $\lambda = -k((n-1/2)\pi)^2 + \gamma^2/k + \mathcal{O}(n^{-2})$. Since for large n, both positive and negative n give the same asymptotic expression, Lemma 3 is proved.

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Starting from (30), we find a solution f of (29) being

$$f(x) = \cosh a_2 \cosh a_1 x - \cosh a_1 \cosh a_2 x. \tag{34}$$

Hence

$$\begin{aligned}
\gamma\lambda g(x) &= kf'''(x) - \lambda(k\lambda + \gamma^2)f'(x) \\
&= k(a_1^3 \cosh a_2 \sinh a_1 x - a_2^3 \cosh a_1 \sinh a_2 x) \\
&\quad - k\lambda^2(a_1 \cosh a_2 \sinh a_1 x - a_2 \cosh a_1 \sinh a_2 x) \\
&\quad - \gamma^2\lambda(a_1 \sinh a_2 \cosh a_1 x - a_2 \cosh a_1 \sinh a_2 x) \\
&= ka_2(\lambda^2 - a_2^2) \cosh a_1 \sinh a_2 x + ka_1(a_1^2 - \lambda^2) \cosh a_2 \sinh a_1 x \\
&\quad - \gamma^2\lambda(a_1 \sinh a_2 \cosh a_1 x - a_2 \cosh a_1 \sinh a_2 x).
\end{aligned}$$

When $\lambda = -k((n - 1/2)\pi)^2 + \gamma^2/k + \mathcal{O}(n^{-2})$, similar to (16), we have

$$\sinh a_2 x = i \sin(n - 1/2)\pi x + \mathcal{O}(n^{-1}). \tag{35}$$

It then follows from (13) that

$$\begin{aligned}
2\frac{\gamma e^{a_1}}{k\lambda a_2} g(x) &= (1 - \lambda^{-2}a_2^2)2e^{a_1} \cosh a_1 \sinh a_2 x \\
&\quad + a_1/a_2\lambda^{-2}(a_1^2 - \lambda^2)2e^{a_1} \sinh a_1 x \cosh a_2 \\
&\quad - \gamma^2\lambda^{-1}(a_1/a_2 2e^{a_1} \cosh a_1 x \sinh a_2 - 2e^{a_1} \cosh a_1 \sinh a_2 x) \\
&= \sinh a_2 x + \mathcal{O}(|\lambda|^{-1/2}) = i \sin(n - 1/2)\pi x + \mathcal{O}(n^{-1}).
\end{aligned} \tag{36}$$

Similarly

$$2\frac{\gamma e^{a_1}}{k\lambda a_2} f'(x) = \mathcal{O}(n^{-1}), \quad 2\lambda\frac{\gamma e^{a_1}}{k\lambda a_2} f(x) = \mathcal{O}(n^{-1}).$$

Therefore

$$2\frac{\gamma e^{a_1}}{k\lambda a_2}\begin{pmatrix} f'(x) \\ \lambda f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ i \sin(n - 1/2)\pi x \end{pmatrix} + \mathcal{O}(n^{-1}). \tag{37}$$

When $\lambda = -\gamma^2/(2k) + i(n - 1/2)\pi + \mathcal{O}(n^{-1})$, $\sqrt{\lambda/k} = \sqrt{1/2}\,(1 + i)\sqrt{(n - 1/2)\pi}$. It follows from (13) that

$$\begin{aligned}
(2/\lambda)\, e^{-a_2}\lambda f(x) &= 2e^{-a_2} \cosh a_2 \cosh a_1 x - e^{-a_2} \cosh a_2 \cosh a_1 \\
&= \cosh a_1 x + \mathcal{O}(n^{-1}) = \cos(n - 1/2)\pi x + \mathcal{O}(n^{-1}), \\
(2/\lambda)\, e^{-a_2} f'(x) &= (1/\lambda)\, a_1 2e^{-a_2} \cosh a_2 \sinh a_1 x - (2/\lambda)\, e^{-a_2} a_2 \sinh a_2 x \cosh a_1 \\
&= \sinh a_1 x + \mathcal{O}(n^{-1}) = i \sin(n - 1/2)\pi x + \mathcal{O}(n^{-1}), \\
(2/\lambda)\, e^{-a_2} g(x) &= \mathcal{O}(n^{-1}).
\end{aligned}$$

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Therefore

$$2\frac{2}{\lambda}e^{-a_2}\begin{pmatrix} f'(x) \\ \lambda f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} \cos(n-1/2)\pi x \\ i\sin(n-1/2)\pi x \\ 0 \end{pmatrix} + \mathcal{O}(n^{-1}). \tag{38}$$

As for (24) and (25), we have, by virtue of (37) and (38), that

$$\sum_{n>N}^{\infty}\left[\|F_n - F_{n0}\|_{\mathcal{H}}^2 + \|G_n - G_{n0}\|_{\mathcal{H}}^2 + \|\overline{G}_n - \overline{G}_{n0}\|_{\mathcal{H}}^2\right] < \infty \tag{39}$$

where $N > 0$ is some integer, F_n and $\{G_n, \overline{G}_n\}$ are eigenfunctions corresponding to σ_n and $\{\lambda_n, \bar{\lambda}_n\}$ which are determined by Theorem 5, respectively. Thus

$$F_{n0} = \begin{pmatrix} 0 \\ 0 \\ i\sin(n-1/2)\pi x \end{pmatrix}, \quad G_{n0} = \begin{pmatrix} \sin(n-1/2)\pi x/((n-1/2)\pi) \\ i\sin(n-1/2)\pi x \\ 0 \end{pmatrix} \tag{40}$$

and $\{F_{n0}, G_{n0}, \overline{G}_{n0}\}_1^\infty$ is the set of all eigenfunctions (up to a scalar) of the operator B with $\gamma = 0$ which is self-adjoint with compact resolvent in H. The set $\{F_{n0}, G_{n0}, \overline{G}_{n0}\}_1^\infty$ forms an orthonormal basis for H. By Theorem 2, we obtain again the Riesz basis property for system (26) obtained in [4].

THEOREM 6. *Let B be the operator defined in* (27). *Then*

(i) *There is a sequence of generalized eigenfunctions of B, which forms a Riesz basis for the state space $\mathcal{H}$*;

(ii) *All $\lambda \in \sigma(B)$ with sufficiently large modulus are algebraically simple.*

Therefore, for the semigroup e^{Bt} generated by B, the spectrum-determined growth condition holds for any $\gamma \geq 0$.

We have seen earlier in this section that the case of *natural boundary conditions* is much simpler than the case of Dirichlet-Dirichlet boundary conditions, discussed in previous sections. The reason for this is that the characteristic equation (31) and expression of eigenfunctions (34) for natural boundary conditions have much simpler forms than the corresponding forms (2) and (8) for Dirichlet-Dirichlet boundary conditions. On the other hand, our approach is based on Theorem 2 which allows us to treat "high frequency" cases only. This is the main difference between our approach and that available in the literature. Recall an earlier result due to Bari for Riesz basis generation in Hilbert space that if $\{\phi_n\}_1^\infty$ is a Riesz basis in a Hilbert space $\mathbf{H}$ and another ω-linearly independent sequence $\{\psi_n\}_1^\infty$ in $\mathbf{H}$ is quadratically close to $\{\phi_n\}_1^\infty$ in the sense that $\sum_{n=1}^\infty \|\phi_n - \psi_n\|^2 < \infty$, then $\{\psi_n\}_1^\infty$ is a Riesz basis itself for $\mathbf{H}$. In order to use Bari's theorem, we have to arrange the eigenfunctions

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corresponding to "low eigenfrequencies". From the characteristic equation (2), this is relatively difficult to carry out for (1). However, for the case of *natural boundary conditions*, this had already been done in [4]. For example, our success in estimating the "low eigenfrequencies" for (26) is due to the following result which shows that the characteristic equation (31) actually consists of a sequence of cubic polynomials.

PROPOSITION 2. *For any solution* λ *of* (33), *there is an integer* $n \geq 1$ *such that*

$$p_n(\lambda) = \lambda^3 + k\mu_n\lambda^2 + (\gamma^2+1)\mu_n\lambda + k\mu_n^2 = 0 \tag{41}$$

where $\mu_n = (n-1/2)^2\pi^2$. *Moreover, for each* $n > 0$, (41) *admits a real solution* σ_n *and one conjugate pair solution* $\{\lambda_n, \bar{\lambda}_n\}$.

PROOF. Obviously, any solution λ of $e^{2a_1} = -1$ or $e^{2a_2} = -1$ must satisfy $a_1 = (n-1/2)\pi i$ or $a_2 = (n-1/2)\pi i$ for some integer n, that is,

$$\frac{\lambda}{2k}\left[k\lambda + \gamma^2 + 1 + \sqrt{(k\lambda+\gamma^2+1)^2 - 4k\lambda}\right] = -(n-1/2)^2\pi^2$$

or

$$\frac{\lambda}{2k}\left[k\lambda + \gamma^2 + 1 - \sqrt{(k\lambda+\gamma^2+1)^2 - 4k\lambda}\right] = -(n-1/2)^2\pi^2.$$

Rearranging terms yields

$$\pm\frac{\lambda}{2k}\sqrt{(k\lambda+\gamma^2+1)^2 - 4k\lambda} = -(n-1/2)^2\pi^2 - \frac{\lambda}{2k}(k\lambda+\gamma^2+1)$$

or

$$\lambda^3 + k((n-1/2)^2\pi^2)^2 + \lambda(k\lambda+\gamma^2+1)(n-1/2)^2\pi^2 = 0.$$

This is (41).

When $k^2\mu_n - 3(\gamma^2+1) < 0$, $p_n'(\lambda) > 0$ for all real λ. Therefore there is a unique real solution λ_n to $p_n(\lambda) = 0$ since $p(\pm\infty) = \pm\infty$. When $k^2\mu_n - 3(\gamma^2+1) \geq 0$, $p_n'(\lambda) = 3\lambda^2 + 2k\mu_n\lambda + (\gamma^2+1)\mu_n = 0$ has real roots

$$\eta_{1,2} = -\frac{1}{3}k\mu_n \pm \frac{1}{3}\sqrt{k^2\mu_n^2 - 3(\gamma^2+1)\mu_n} > -k\mu_n. \tag{42}$$

Since

$$9p_n(\eta_{1,2}) = -2[k^2\mu_n^2 - 3(\gamma^2+1)\mu_n]\eta_{1,2} + k(8-\gamma^2)\mu_n^2 > 0$$

and $p_n'(\lambda) = 3(\lambda-\eta_2)(\lambda-\eta_1)$, we see that $p_n'(\lambda) > 0$ for all $\lambda > \eta_1$ or $\lambda < \eta_2$. Therefore there is a unique real solution λ_n to $p_n(\lambda) = 0$ with $\lambda_n < \eta_2$.

Proposition 2 is similar to [7, Lemma 6.42]. By this result, we can arrange the eigenfunctions corresponding to "low eigenfrequencies" to use Bari's theorem. Because the result is already known in [4], we omit the details here.

https://doi.org/10.1017/S1446181100012621 Published online by Cambridge University Press

Acknowledgments

The support of the National Key Project of China and the National Natural Science Foundation of China are gratefully acknowledged.

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