ON PRODUCTS OF MODULES IN A TOPOS

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Abstract

In an elementary topos if $R$ is a ring and $X$ is a decidable object then there exists a canonical homomorphism from the coproduct of an $X$-family of $R$-modules to the product of the same family. In this paper it is shown that this homomorphisms is a monomorphism.


In the category Set of sets if $X$ is a set and $R$ is a ring then for an $X$-family $\{M_x\}_{x \in X}$ of $R$-modules there is always a canonical monomorphism $\phi: \bigoplus_{x \in X} M_x \rightarrow \prod_{x \in X} M_x$, with $\pi_x \phi i_x = 1_{M_x}$, where $\pi_x$ and $i_x$ are the $x$th projection and injection, respectively. In [5] it is shown, by an example, that in an elementary topos such a homomorphism does not always exist. However, if we choose $X$, the index object, to be decidable, it is proved that such a canonical homomorphism exists.

In this paper we show that the canonical homomorphism given in [5] is a monomorphism. A closely related work can be found for the case of abelian groups in [1].

Throughout the paper, $E$ denotes an elementary topos with natural numbers object, and $R$ is a ring in $E$. All other notation, not explained here, can be found in [2] or [3].

Let $A$ be an $E$-indexed category with $\lim$ and small homs, let $C$ be an internal category of $E$, let $\Gamma: C \rightarrow E$ be an internal functor with $\lim \Gamma = I$ and for each $J$ in $E$ let $\Gamma^J(c) \rightarrow J^* I$ be the canonical injection, where $c \in [J, C]$.  

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1. **Lemma.** If $A \in A^I$ then

$$\sum_{I} A = \lim_{\rightarrow} \sum_{\Gamma'(c)} \lambda'_{c} \pi_2^* A,$$

where $\pi_2: J \times I \to I$ is the projection.

**Proof.** Let $B \in A^I$. Then we have the following natural isomorphisms:

$$\lim_{\rightarrow} \sum_{\Gamma'(c)} \lambda'_{c} \pi_2^* A \to B;$$

 indexed cocone $\langle \sum_{\Gamma'(c)} \lambda'_{c} \pi_2^* A \to J^* B \rangle_{c \in [J, C]}$;

 compatible families $\langle \lambda'_{c} \pi_2^* A \to \Gamma'(c)^* J^* B \rangle_{c \in [J, C]}$;

 compatible families $\langle \lambda'_{c} \pi_2^* A \to \lambda'_{c} \pi_2^* I^* B \rangle_{c \in [J, C]}$;

 indexed cocone $\langle \pi_2 \lambda'_{c} \to \text{Hom}(A, I^* B) \rangle_{c \in [J, C]}$;

$$1 = \lim_{\rightarrow} \pi_2 \lambda'_{c} \to \text{Hom}(A, I^* B);$$

$$A \to I^* B;$$

$$\sum_{I} A \to B.$$

Hence by the Yoneda Lemma,

$$\lim_{\rightarrow} \sum_{\Gamma'(c)} \lambda'_{c} \pi_2^* A = \sum_{I} A.$$

We will use the following theorem, due to D. Schumacher, which is proved in [4].

2. **Theorem.** Let $A$ be a small filtered indexed category and let $F: A \to E$ be an indexed functor.

(a) For every $1 \xrightarrow{x} I^* \lim F$ there exist $J \xrightarrow{\alpha} I$, $A \in A^J$ and $1 \xrightarrow{y} F^J(A)$ such that

$$1 \xrightarrow{\alpha^* x} J^* \lim F \xrightarrow{i_A} F^J(A)$$

commutes, where $i_A$ is indexed.
(b) For \( I \xrightarrow{x_1} F\!^I A_1 \) and \( I \xrightarrow{x_2} F\!^I A_2 \), \( \iota_A(x_1) = \iota_A(x_2) \) if and only if there exist \( J \rightarrow I \) and
\[
\begin{array}{ccc}
\alpha^* A_1 & \xrightarrow{a_1} & A \\
\alpha^* A_2 & \xrightarrow{a_2} & A \\
\end{array}
\]
in \( A' \), such that \( F(I)(\alpha^* x_1) = F(I)(\alpha^* x_2) \).

Let \( X \) be an object in \( E \) and let \( E_{\text{fin}} \) be the internalization of \( E_{\text{fc}} \), the category of finite cardinals, in the sense that its external category of \( I \)-elements is equivalent to \((E/I)_{\text{fc}}\). For more details and the proof of the following lemma see [5] and [4].

3. **Lemma.** The internal category \( E_{\text{fin}}/X \) is filtered.

Let \( M \) be an object in \( \text{Mod}_R(E)^X \). Define a functor \( P: E_{\text{fin}}/X \rightarrow \text{Mod}_R(E) \) as follows: for any \( I \)-object of \( E_{\text{fin}}/X \), i.e. \( (I \rightarrow N, [p] \xrightarrow{X} I^* X) \), \( P(I)[p] \xrightarrow{X} I^* X = \bigoplus_{[p]} x^* \pi_2^* M \), where \( \pi_2: I \times X \rightarrow X \), and for any \( I \)-morphism
\[
[P] \rightarrow [q] \\
\uparrow \quad \uparrow \quad I^* X
\]
\( P(I)(f) \) is defined by
\[
\begin{bmatrix}
\bigoplus_{[p]} x^* \pi_2^* M, L \\
\bigoplus_{[q]} y^* \pi_2^* M, L
\end{bmatrix} = \begin{bmatrix}
\pi_2^* M, \prod_x [p]^* L \\
\prod_y [q]^* L
\end{bmatrix},
\]
where \( L \) is in \( \text{Mod}_R(E)^I \). It is easy to see that \( P \) is an indexed functor.

Let \( I \) be a decidable object in \( E \), i.e. \( \delta: I \rightarrow I \times I \), the diagonal morphism, has a complement \( J \rightarrow I \times I \) such that \( \delta^* \): \( I + J \rightarrow I \times I \) is an isomorphism. Then it is well known that \( E/I \times I \rightarrow E/I \times E/J \) is an equivalence of categories. This extends to an equivalence \( \alpha: \text{Mod}_R(E)^I \times I \rightarrow \text{Mod}_R(E)^I \times \text{Mod}_R(E)^I \). For \( M \) in \( \text{Mod}_R(E)^I \), let \( \alpha(\pi_1^* M) = (M_1, M'_1) \) and \( \alpha(\pi_2^* M) = (M_2, M'_2) \), where \( I \xrightarrow{\pi_1} I \rightarrow I \rightarrow I \) are the projections and \( M'_1, M'_2 \) are in \( \text{Mod}_R(E)^J \). Since \( \pi_1 \delta = \pi_2 \delta = 1_I \), then \( M_1 \cong M = M_2 \) in \( \text{Mod}_R(E)^I \), i.e. there is an isomorphism \( M_1 \xrightarrow{\delta} M_2 \). Thus
we have a morphism
\[ \alpha(\tau_1^* M) = (M_1, M_1') \rightarrow (M_2, M_2') = \alpha(\tau_2^* M) \]
and so there is a homomorphism \( \psi: \tau_1^* M \rightarrow \tau_2^* M \), because \( \alpha \) is an equivalence.
By the Beck condition the canonical homomorphism \( \gamma: \bigoplus \tau_j^* M \rightarrow \mathcal{I}^* \bigoplus_j M \) is an isomorphism, and hence we have the following natural isomorphisms:
\[
\begin{align*}
\tau_1^* M & \rightarrow \tau_2^* M & \text{in } \text{Mod}_R(E)^{I \times I} ; \\
\bigoplus \tau_1^* M & \rightarrow M & \text{in } \text{Mod}_R(E)^I ; \\
\mathcal{I}^* \bigoplus_j M & \rightarrow M & \text{in } \text{Mod}_R(E)^I ; \\
\bigoplus_j M & \rightarrow \prod_j M & \text{in } \text{Mod}_R(E). 
\end{align*}
\]
If \( \eta' \) is the unit for \( \bigoplus \tau_j \rightarrow \tau_2^* \) and \( \epsilon \) is the counit for \( \mathcal{I}^* \rightarrow \prod_j \), then \( \psi \) and \( \phi \) are related by the equation \((\tau_2^* \phi)(\tau_2^* \gamma)\eta_\tau^* M = \psi\), where \( \phi = \epsilon M(I^* \phi) \).
On the other hand the canonical morphism \( \gamma \) satisfies \((\tau_2^* \gamma)\eta_\tau^* M = \tau_1^* \eta_\tau^* M\), where \( \eta \) is the unit for \( \mathcal{I}^* \rightarrow \prod_j \), so we have \((\tau_2^* \phi)(\tau_1^* \eta_\tau^* M) = \psi\). Apply \( \delta^* \) to this equality to get \( \phi \eta_\tau^* M = \delta^* \psi = I^*_M \) (by definition of \( \psi \)), i.e. \( \epsilon M(I^* \phi) \eta_\tau^* M = I^*_M \).

The main result of this paper is as follows.

**4. Theorem.** Let \( X \) be a decidable object in \( E \) and \( M \) be an object in \( \text{Mod}_R(E)^X \).
Then the homomorphism \( \bigoplus \chi M \rightarrow \prod_X M \), defined above, is a monomorphism.

**Proof.** Let \( T \rightarrow \bigoplus_X M \) be a \( T \)-element of \( \bigoplus_X M \) such that \( \phi(t) = 0 \). We have to show that \( t = 0 \). By Lemma 1, \( \bigoplus_X M = \lim_c \bigoplus_{[p]} x^* \pi_2^* M \), where \( [p] \rightarrow X^* X \) is in \( E/I \) and \( C = E_{\text{fin}}/X \). Apply the first part of Theorem 2 to the filtered indexed category \( E_{\text{fin}}/X \) and the indexed functor \( P \), defined above (we can do that because filtered colimits in \( \text{Mod}_R(E) \) are the same as in \( E \)) to get \( L \rightarrow T, p: L \rightarrow N, x: [p] \rightarrow L^* X \), and \( 1 \rightarrow \bigoplus_{[p]} x^* \pi_2^* M \) such that \( \alpha^* m = i_x \gamma \), where \( i_x: P^L(x) \rightarrow L^* \lim_c P \) is the indexed injection. But by the properties of colimit the following diagram commutes:

\[
\begin{array}{ccc}
[p]^* L^* \rightarrow \lim_c & \bigoplus_{[p]} x^* \pi_2^* M = x^* \pi_2^* X^* \bigoplus_X M & \rightarrow x^* \pi_2^* X^* \prod_X M \\
& \phi \downarrow & \phi \\
[p]^* \bigoplus_{[p]} x^* \pi_2^* M & \rightarrow x^* \pi_2^* M. 
\end{array}
\]
Since \( x^*\pi_2^*X^* = [p]^*L^* \), then by transpose of the above diagram along \([p]^*\) we get

\[
\begin{array}{ccc}
L^* & \lim_{\to C} & \bigoplus_{[p]} x^*\pi_2^*M \\
& & \downarrow i_x \\
& \bigoplus_{[p]} x^*\pi_2^*M & \longrightarrow \prod_{[p]} x^*\pi_2^*M,
\end{array}
\]

where \( j_x \) is the transpose of \( x^*\pi_2^*\epsilon_M \). Now, by Theorem 2.3 [5],

\[
\bigoplus_{[p]} x^*\pi_2^*M \xrightarrow{\phi} \prod_{[p]} x^*\pi_2^*M
\]

is an isomorphism. Therefore \( j_xL^*\phi(\alpha^*m) = j_xL^*\phi(i_xy) = \phi(y) = 0 \) implies \( y = 0 \), so \( \alpha^*m = 0 \). But \( \alpha \) is an epimorphism and hence \( m = 0 \), as required.

References


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