ON AN INTEGRAL EQUATION OF ŠUB-SIZONENKO

by P. G. ROONEY

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The integral equation of the title is

$$h(x)=\pi^{-1/2}\int_x^\infty (\log t/x)^{-1/2}f(t)\,dt/t+f(x), \qquad (x>0). \tag{1}$$

It was studied in [**4**], though $h(x)$ was written as $x^{-1}g(x^{-1})$ there, and using a method involving orthogonal Watson transformations, it was shown there that if $h\in L_2(0,\infty)$, then the equation has a solution $f\in L_2(0,\infty)$, and that f is given by

$$f(x)=\frac{d}{dx}\int_x^\infty\left\{\int_{\log t/x}^\infty \operatorname{erfc}(u^{1/2})\,du-\operatorname{erfc}\big((\log t/x)^{1/2}\big)\right\}h(t)\,dt+\tfrac{1}{2}h(x). \tag{2}$$

In this paper, using the techniques of [**3**], we shall show that the equation can be solved for h in the space $\mathscr{L}_{\mu,p}$ of [**3**] for $1\le p<\infty$, $\mu>0$, and that for these spaces, which include $L_2(0,\infty)$, f is given by the simpler formula

$$f(x)=\int_x^\infty\big((t/x)\operatorname{erfc}((\log t/x)^{1/2})-\pi^{-1/2}(\log t/x)^{-1/2}\big)h(t)\,dt/t+h(x) \qquad (x>0). \tag{3}$$

We shall further show that these results can be extended to the spaces $\mathscr{L}_{w,\mu,p}$ of [**3**]. This forms the content of our theorem below.

Our notation in this paper will be that of [**3**]; particular notations from [**3**] that we use frequently are $\mathscr{L}_{\mu,p}$, $\mathscr{L}_{w,\mu,p}$, $\mathscr{A}$, $\mathfrak{A}_p$, $\mathscr{M}$ and $[X]$. We shall also use some results from [**2**], and it must be noted that the spaces $L_{\mu,p}$ of [**2**] are slightly different from the spaces $\mathscr{L}_{\mu,p}$ of [**3**], and the results adjusted accordingly.

We shall write (1) as

$$h=Kf, \tag{4}$$

where

$$K=K_0+I, \tag{5}$$

and

$$(K_0f)(x)=\int_x^\infty(\log t/x)^{-1/2}f(t)\,dt/t \qquad (x>0), \tag{6}$$

and similarly we shall write (3) as

$$f=Lh, \tag{7}$$

where

$$L=L_0+I, \tag{8}$$

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and

$$(L_0h)(x)=\int_x^\infty ((t/x)\operatorname{erfc}((\log t/x)^{1/2})-\pi^{-1/2}(\log t/x)^{-1/2})h(t)\,dt/t, \qquad (x>0). \tag{9}$$

First we need a lemma.

LEMMA. *If* $1\le p<\infty$, $\mu>0$, K_0 *and* $L_0\in[L_{\mu,p}]$ *and if* $f\in L_{\mu,p}$, *where* $1\le p\le 2$, $\mu>0$, *then*

$$(\mathcal{M}K_0f)(s)=s^{-1/2}(\mathcal{M}f)(s), \qquad \operatorname{Re} s=\mu, \tag{10}$$

and

$$(\mathcal{M}L_0f)(s)=-(1+s^{1/2})^{-1}(\mathcal{M}f)(s), \qquad \operatorname{Re} s=\mu. \tag{11}$$

Proof. Clearly

$$(K_0f)(x)=\int_0^\infty k(x/t)f(t)\,dt/t,$$

where

$$k(x)=\begin{cases}\pi^{-1/2}(\log(x^{-1}))^{-1/2} & (0<x<1),\\ 0 & (x>1).\end{cases}$$

Thus, if $\mu>0$,

$$\int_0^\infty x^{\mu-1}\,|k(x)|\,dx=\pi^{-1/2}\int_0^1 x^{\mu-1}(\log(x^{-1}))^{-1/2}\,dx=\pi^{-1/2}\int_0^\infty e^{-\mu t}t^{-1/2}\,dt=\mu^{-1/2}. \tag{12}$$

Hence, by [**2**, Lemma 3.1], $K_0\in[\mathcal{L}_{\mu,p}]$, $(1\le p<\infty)$. Also, by the same calculation as (12) with μ replaced by s, if $\operatorname{Re} s>0$

$$(\mathcal{M}k)(s)=\int_0^\infty x^{s-1}k(x)\,dx=s^{-1/2},$$

and thus, by [**2**, Lemma 4.1], (10) holds.

Similarly

$$(L_0f)(x)=\int_0^\infty l(x/t)f(t)\,dt/t,$$

where

$$l(x)\begin{cases}x^{-1}\operatorname{erfc}((\log(x^{-1}))^{1/2})-\pi^{-1/2}(\log(x^{-1}))^{-1/2} & (0<x<1),\\ 0 & (x>1).\end{cases}$$

Now, integrating by parts, if $u>0$

$$\begin{aligned}\operatorname{erfc} u&=2\pi^{-1/2}\int_u^\infty e^{-t^2}\,dt=2\pi^{-1/2}\left\{-\frac{1}{2}t^{-1}e^{-t^2}\Big|_u^\infty-\frac{1}{2}\int_u^\infty e^{-t^2}\,dt/t^2\right\}\\ &=\pi^{-1/2}\left\{u^{-1}e^{-u^2}-\int_u^\infty e^{-t^2}\,dt/t^2\right\},\end{aligned}$$

so that

$$x^{-1}\operatorname{erfc}((\log(x^{-1}))^{1/2})-\pi^{-1/2}(\log(x^{-1}))^{-1/2}=-x^{-1}\pi^{-1/2}\int_u^\infty e^{-t^2}\,dt/t^2,$$

where $u=(\log(x^{-1}))^{1/2}$, and thus $l(x)\le 0$, $x>0$. Hence if $\mu>0$

$$\begin{aligned}\int_0^\infty x^{\mu-1}|l(x)|\,dx &= -\int_0^1 x^{\mu-1}(x^{-1}\operatorname{erfc}((\log x^{-1})^{1/2})-\pi^{-1/2}(\log(x^{-1}))^{-1/2})\,dx\\ &= -\int_0^\infty e^{-\mu t}e^t\operatorname{erfc} t^{1/2}\,dt+\pi^{-1/2}\int_0^1 x^{\mu-1}(\log(x^{-1}))^{-1/2}\,dx\\ &= -\mu^{-1/2}(\mu^{1/2}+1)^{-1}+\mu^{-1/2}=(\mu^{1/2}+1)^{-1} \qquad (13)\end{aligned}$$

from [**1**, 4.12(10)] and (12). Hence, by [**2**; Lemma 3.1], $L_0\in[\mathscr{L}_{\mu,p}]$, $(1\le p<\infty)$. Also, by a similar calculation as (13), if Re $s>0$, then

$$(\mathscr{M}l)(s)=-(1+s^{1/2})^{-1},$$

and thus, by [**2**, Lemma 4.1], (11) follows.

We can now state our Theorem.

THEOREM. *If* $1\le p<\infty$, $\mu>0$, *then* K *and* $L\in[\mathscr{L}_{\mu,p}]$; K *and* L *map* $\mathscr{L}_{\mu,p}$ *one-to-one onto itself; and*

$$KL=LK=I. \qquad (14)$$

Further, if $1<p<\infty$, $\mu>0$ *and* $w\in\mathfrak{A}_p$, *then* K *and* L *can be extended to* $\mathscr{L}_{w,\mu,p}$ *and if their extensions are still denoted by* K *and* L *respectively, then* K *and* $L\in[\mathscr{L}_{w,\mu,p}]$, K *and* L *map* $\mathscr{L}_{w,\mu,p}$ *one-to-one onto itself; and* (14) *continues to hold.*

Proof. Since K_0 and $L_0\in[\mathscr{L}_{\mu,p}]$ for $1\le p<\infty$, $\mu>0$, so are K and L. If $\mu>0$ and $f\in\mathscr{L}_{\mu,2}$, then from (10) and (11), if Re $s=\mu$

$$\begin{aligned}(\mathscr{M}KLf)(s)&=(s^{-1/2}+1)(\mathscr{M}Lf)(s)=(s^{-1/2}+1)(1-(s^{1/2}+1)^{-1})(\mathscr{M}f)(s)\\ &=(\mathscr{M}f)(s),\end{aligned}$$

so that $KLf=f$, and similarly $LKf=f$. Hence, on $\mathscr{L}_{\mu,2}$ (14) holds. But from [**2**, Lemma 2.2], $\mathscr{L}_{\mu,2}\cap\mathscr{L}_{\mu,2}\cap\mathscr{L}_{\mu,p}$ is dense in $\mathscr{L}_{\mu,p}$, $(1\le p<\infty)$, and thus since both sides of (14) are operators in $[\mathscr{L}_{\mu,p}]$, (14) holds on $\mathscr{L}_{\mu,p}$. It follows from this that K and L are one-to-one onto on $\mathscr{L}_{\mu,p}$. For if $g\in\mathscr{L}_{\mu,p}$, $(1\le p<\infty$, $\mu>0)$, and we let $f=Lg$, then $Kf=KLg=g$, so that K is onto, and if $Kf_1=Kf_2$, $f_i\in L_{\mu,p}$, $(i=1,2)$, then $f_1=LKf_1=LKf_2=f_2$; similarly for L.

From (10), if $f\in\mathscr{L}_{\mu,p}$, $(1\le p\le 2$, $\mu>0)$, then

$$(\mathscr{M}Kf)(s)=m(s)(\mathscr{M}f)(s) \quad\text{and}\quad (\mathscr{M}Lf)(s)=(1/m(s))(\mathscr{M}f)(s), \qquad (15)$$

where $m(s)=s^{-1/2}+1$. Clearly m is holomorphic in $0=\alpha(m)<\text{Re } s<\beta(m)=\infty$. Also if

$0<\sigma_1\le\sigma_2$, then in $\sigma_1\le\operatorname{Re} s\le\sigma_2$, $m(s)$ is bounded. Further $|m'(\sigma+it)|=\frac{1}{2}|\sigma+it|^{-3/2}=O(|t|^{-1})$ as $|t|\to\infty$. Thus $m\in\mathcal{A}$, with $\alpha(m)=0$, $\beta(m)=\infty$. In an exactly similar way $1/m\in\mathcal{A}$ with $\alpha(m)=0$, $\beta(m)=\infty$. Hence by [**3**, Theorem 1], there are operators H_m and $H_{1/m}\in[\mathcal{L}_{w,\mu,p}]$ for $1<p<\infty$, $\mu>0$, $w\in\mathfrak{A}_p$ and such that for $f\in\mathcal{L}_{\mu,p}$, with $\mu>0$, $1<p\le 2$

$$(\mathcal{M}H_mf)(s)=m(s)(\mathcal{M}f)(s) \quad \text{and} \quad (\mathcal{M}H_{1/m}f)(s)=(1/m(s))(\mathcal{M}f)(s). \tag{16}$$

Comparing (15) and (16), it is clear that on $\mathcal{L}_{\mu,p}$, $(1<p\le 2,\ \mu>0)$, $H_m=K$ and $H_{1/m}=L$, and this must hold on all $\mathcal{L}_{\mu,p}$, $(\mu>0,\ 1<p<\infty)$, since $\mathcal{L}_{\mu,2}\cap\mathcal{L}_{\mu,p}$ is dense in $\mathcal{L}_{\mu,p}$ and all operators in question are in $[\mathcal{L}_{\mu,p}]$. Thus we can extend K and L to $\mathcal{L}_{w,\mu,p}$ for $1<p<\infty$, $\mu>0$, $w\in\mathfrak{A}_p$ as members of $[\mathcal{L}_{w,\mu,p}]$ by defining them to be H_m and $H_{1/m}$ respectively, and then by [**3**, Theorem 1], K and L are one-to-one onto. $KL=H_mH_{1/m}=H_m(H_m)^{-1}=I$ and similarly $LK=I$. Thus the theorem is proved.

COROLLARY. *If* $h\in\mathcal{L}_{\mu,p}$, *where* $1\le p<\infty$, $\mu>0$, *equation* (1) *has a unique solution* $f\in\mathcal{L}_{\mu,p}$ *given by* (3); *if* $f\in\mathcal{L}_{\mu,p}$, *where* $1\le p<\infty$, $\mu>0$, *equation* (3) *has a unique solution* $h\in\mathcal{L}_{\mu,p}$ *given by* (1). *If* $1<p<\infty$, $\mu>0$, $w\in\mathfrak{A}_p$, *and* $h\in\mathcal{L}_{w,\mu,p}$, *the equations* $h=Kf$ *and* $h=Lf$ *have unique solutions* $f\in\mathcal{L}_{w,\mu,p}$ *given by* $f=Lh$ *and* $f=Kh$ *respectively.*

We conclude by remarking that when K and L are extended to $\mathcal{L}_{w,\mu,p}$, then Kf for $f\in\mathcal{L}_{w,\mu,p}$ is not necessarily represented by equation (1), and similarly Lh for $h\in\mathcal{L}_{w,\mu,p}$ is not necessarily represented by equation (3). By examining the adjoint of K representations of K can be found on $\mathcal{L}_{w,\mu,p}$ and similarly for L.

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UNIVERSITY OF TORONTO,
TORONTO, CANADA,
M5S 1A1.