A NOTE ON A SELF INJECTIVE RING

Kwangil Koh

(received June 3, 1964)

1. A ring R with unity is called right (left) self injective if the right (left) R-module R is injective [7]. The purpose of this note is to prove the following: Let R be a prime ring with a maximal annihilator right (left) ideal. If R is right (left) self injective then R is a primitive ring with a minimal one-sided ideal. If R satisfies the maximum condition on annihilator right (left) ideals and R is right (left) self injective then R is a simple ring with the minimum condition on one-sided ideals.

2. A right (left) ideal I of a ring R is called large if I has a non-zero intersection with each non-zero right (left) ideal of R. An element of R which annihilates some large left (right) ideal of R is called right (left) singular. The set of all right (left) singular elements in R is known to form a two sided ideal of R and this ideal is called the right (left) singular ideal of R [3].

We note here that if T is a subset of a ring R then T_r will denote the right annihilators in R of the set T. T_ℓ is defined as the set of left annihilators in R of the set T.

LEMMA 2.1. If R is a prime ring such that R contains a maximal annihilator right ideal then any right or left singular element of R is zero.

Proof. Let Z be the non-zero right singular ideal of R and I be a maximal annihilator right ideal of R. Then there is an element a in R such that $I = (a)_r$. $a.Z.a \neq (0)$ since R is a prime ring, hence there is an element $z \in Z$ such that $aza \neq 0$. $zaR \cap (a)_r \neq (0)$ since $(aza)_r = (a)_r$ and $(aza)_r$ is

Canad. Math. Bull. vol. 8, no. 1, February 1965

https://doi.org/10.4153/CMB-1965-005-6 Published online by Cambridge University Press

large. Thus there is $r \in R$ such that $zar \neq 0$ and $a(zar) = 0$. This is impossible since $a(zar) = 0 = (aza)r$ implies that $ar = 0$. Now assume that S be the non-zero left singular ideal of R and let s be an element in S such that $asa \neq 0$. $(a)_\ell \cap Ra \neq (0)$ since $(sa)_\ell$ is large. Hence there is $r \in R$ such that $ra \neq 0$ and $(ra)(sa) = 0$. This implies that $sa \in (ra)_r$. Notice that $(ra)_r = (a)_r$ since $(a)_r$ is a maximal annihilator right ideal and $ra \neq 0$. Thus $asa = 0$, which is impossible.

A ring R with unity is called <u>regular</u> if for any element x there is y with $xyx = x$. A right (left) ideal U of a ring R is called <u>uniform</u> if we have $J_1 \cap J_2 \neq 0$ for any non-zero right (left) ideals J_1 and J_2 such that $J_1, J_2 \leq U$.

LEMMA 2.2. If R is a regular ring and U is a uniform right (left) ideal of R then U is a minimal right (left) ideal of R.

<u>Proof</u>. Since R is a regular ring, the right (left) singular ideal of R is zero (see [5], p. 1386). Let u_o be an arbitrary non-zero element of U. Then there is $a \in R$ such that $u_o a u_o = u_o$. Hence $(1 - u_o a)_r \cap U \neq 0$ and $U \leq (1 - u_o a)_r$ by [1; Lemma 2.2]. Since $(1 - u_o a)_r = u_o R$, $U = u_o R$. This proves that U is a minimal right ideal of R.

LEMMA 2.3. Let R be a prime ring such that R contains a maximal annihilator right (left) ideal. If R is right (left) self injective then R is a primitive ring with a minimal one-sided ideal.

<u>Proof</u>. Since the right singular ideal is zero by Lemma 2.1, R is a semi-simple ring by [6; Lemma 8]. Hence, by [7; Theorem 1], R is a regular ring. Let I be a maximal annihilator right ideal of R. Then there is an element $a \in R$ such that $I = (a)_r$. Let J be a right ideal of R such that $I \lneq J$. If $x \in R$ such that $x \cdot J = (0)$ then $x = 0$. Thus, if f is a R-homomorphism of J into R such that $f(J) = (0)$ then f is zero mapping. Hence by [2; Proposition 2.2] J is a large right

https://doi.org/10.4153/CMB-1965-005-6 Published online by Cambridge University Press

ideal of R. Thus I is a maximal complement right ideal of R. Let U be a right ideal of R which is a complement of I. Then U is uniform. Thus, by Lemma 2.2, U is a minimal right ideal of R.

LEMMA 2.4. If R is right (left) self injective then a complement right (left) ideal is an annihilator right ideal.

Proof. Let C be a complement right ideal of R. Then there is a right ideal I of R such that $C \cap I = (0)$, and if J is a right ideal such that $J \gneq C$ then $J \cap I \neq 0$. Define a mapping f such that $f(i) = i$ for all $i \in I$ and $f(c) = 0$ for all $c \in C$. Then there is $a \in R$ such that $f = a$. Thus $C \leq (a)_r$. If $C \lneq (a)_r$ then $(a)_r \cap I \neq (0)$ hence there is $i \in I$, $i \neq 0$, such that $f(i) = ai = 0$. This is impossible. Thus $C = (a)_r$.

THEOREM. If R is a prime ring which satisfies the maximum condition on annihilator right ideals and is right injective, then R is a simple ring with the minimum condition on right ideals.

Proof. By Lemma 2.3, R is a primitive ring with a minimal right ideal, say M. Let $D = \mathrm{Hom}_R(M, M)$; then D is a division ring. Since R satisfies the maximum condition on annihilator right ideals, by Lemma 2.4 the maximum condition on complement right ideals holds in R. Hence the left D-space M is finite dimensional by [4; Theorem 3.1]. Thus R is a simple ring with the minimum condition on right ideals.

Remark. In general, a right self-injective prime ring which contains a maximal annihilator right ideal is not necessarily a simple ring with the minimum condition on right ideals. For example, let ∇ be an infinite dimensional, left vector space over a division ring D. Let $T_o(D, \nabla)$ be the ring of all linear transformations of ∇ over D which are of finite rank. It is known that $T_o(D, \nabla)$ is a simple ring with a minimal right ideal, and if $x \in T_o(D, \nabla)$ then $(x)_r \neq (0)$. Let Q be the maximal right quotient ring (in the sense of R. E. Johnson) of $T_o(D, \nabla)$.

https://doi.org/10.4153/CMB-1965-005-6 Published online by Cambridge University Press

Then Q is a primitive ring with a minimal right ideal [see 5; 3.1], and Q is right-self injective. However, Q is not a simple ring with the minimum conditions on right ideals, for, if it were, $T_o(D, \nabla)$ would contain an element x such that $(x)_r = 0$ [see 5; 2.6 and p. 1390].

REFERENCES

1. A. W. Goldie, The structure of prime ring, Proc. London Math. Soc. (3), 8(1958), 589-608.

2. G. D. Findlay and J. Lambek, A generalized ring of quotients, I, Canad. Math. Bull., 1(1958), 77-85.

3. R. E. Johnson, "The extended centralizer of a ring over a module, Proc. Amer. Math. Soc., 2(1951), 891-895.

4. R. E. Johnson and E. T. Wong, Quasi-injective modules and irreducible rings, J. London Math. Soc., 36(1961), 260-268.

5. R. E. Johnson, Quotient rings of rings with zero singular ideal, Pacific J. Math. 11, No. 4(1961), 1385-1392.

6. Y. Utumi, On a theorem on modular lattices, Proc. Japan Acad. 35(1959), 16-21.

7. ________, On continuous regular rings and semi-simple self injective rings, Canad. J. Math. 12(1960), 597-605.

North Carolina State
of the University of North Carolina
at Raleigh, North Carolina

https://doi.org/10.4153/CMB-1965-005-6 Published online by Cambridge University Press