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SKELETON C^*-SUBALGEBRAS

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ABSTRACT. We study skeleton C^*-subalgebras of a given C^*-algebra. We show that if A is a unital (non-unital but σ-unital) simple C^*-algebra, $\mathcal{M}$ is any unital (nonunital) matroid C^*-algebra, then A contains a skeleton C^*-subalgebra B with a quotient which is isomorphic to $\mathcal{M}$. Other results for skeleton C^*-subalgebras are also obtained. Applications of these results to the structure of quasi-multipliers and perturbations of C^*-algebras are given.

1. **Introduction.** Matrix algebras $\mathcal{M}_n$, the C^*-algebras of $n \times n$ matrices over $\mathbb{C}$, and $\mathcal{K}$, the C^*-algebra of compact operators on an infinite dimensional, separable Hilbert space are often called elementary C^*-algebras for the obvious reasons. Matroid C^*-algebras may be viewed as a generalization of elementary C^*-algebras. Though non-elementary matroid C^*-algebras are quite different (they are antiliminal, for instance) from elementary ones, they inherit many properties from elementary C^*-algebras. They are "matroid". Next, of course, are (simple) AF C^*-algebras. The class of AF C^*-algebras is one of the best understood classes of C^*-algebras. They have a rich but managable structure of projections and provide many interesting and important examples. The reason that AF C^*-algebras are better understood is that they are approximately finite dimensional and therefore "matrix-like".

In [20] and [25], fundamental approximate identities were studied. For example, S. Zhang ([25]) showed that every σ-unital (non-unital) simple C^*-algebra with real rank zero has a fundamental approximate identity. The existence of such an approximate identity provides some "matrix-like" structure inside the C^*-algebra. For example, we showed in [20] that a C^*-algebra with fundamental approximate identity has a "skeleton" algebra with a quotient isomorphic to $\mathcal{K}$. In this note we introduce formally the concept of "skeleton":

DEFINITION 1.1. Let A be a C^*-algebra. A C^*-subalgebra B is called a *skeleton C^*-subalgebra* if the hereditary C^*-subalgebra generated by B is A.

It should be noted that if A is unital, A has a skeleton C^*-subalgebra which is isomorphic to $\mathbb{C}$. Therefore, we do not search for a trivial skeleton but for a rich skeleton with nice properties. We will show that if A is a σ-unital (non-unital) simple C^*-algebra then for any unital (non-unital) matroid C^*-algebra $\mathcal{M}$, there is a skeleton C^*-subalgebra B of A such that B has a quotient which is isomorphic to $\mathcal{M}$. This shows that every σ-unital

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simple C^*-algebra has a "matrix-like" structure. For a simple C^*-algebra A with real rank zero, stable rank one and unperforated $K_0(A)$, there is a simple AF skeleton B of A such that $K_0(A) = K_0(B)$ and for every projection p in A there is a projection q in B such that $p \sim q$ (in the sense of Murray and von Neumann). Applications of these results are given in Section 3.

1.2. Let A be a C^*-algebra, $a, b \in A$. We write (see [10]) $a \underset{\approx}{<} b$ if there are $x, y \in A$ such that $a = xby$. If $a, b \in A_+$, $a \underset{\approx}{<} b$, then, by [10, 1.7], there is $z \in A$ such that $z^*z = a$, $zz^* \in \mathrm{Her}(b)$, the hereditary C^*-algebra generated by b.

1.3. Given $\varepsilon > 0$, let f_ε be the continuous function on $\mathbb{R}$ defined by

$$f_\varepsilon(t) = \begin{cases} 0 & t \leq \frac{\varepsilon}{2} \\ \frac{2}{\varepsilon}(t - \frac{\varepsilon}{2}) & \frac{\varepsilon}{2} < t < \varepsilon \\ 1 & t \geq \varepsilon. \end{cases}$$

1.4. Given z in A with polar decomposition (in A^{**}) $z = u|z|$ and $\varepsilon > 0$ we know from [10, 1.3] that $uf_\varepsilon(|z|)$ is in A. For any $x \in \mathrm{Her}(|z|)$,

$$\|uf_\varepsilon(|z|)x - ux\| \leq \|f_\varepsilon(|z|)x - x\| \longrightarrow 0$$

as $\varepsilon \longrightarrow 0$. Therefore, $ux \in A$ for any $x \in \mathrm{Her}(|z|)$.

In fact the mapping φ defined by

$$\varphi(x) = uxu^*$$

is an isomorphism from $\mathrm{Her}(|z|)$ onto $\mathrm{Her}(|z^*|)$ (see [10, 1.7]). If $a, b \in A_+$, we write $a \sim_\phi b$ if there is $z \in A$ such that $z^*z = a$, $zz^* = b$. If $a \sim_\phi b$, then there is a partial isometry $u \in A^{**}$, where $z = u|z|$ is the polar decomposition, such that the mapping $\varphi = uxu^*$ is an isomorphism from $\mathrm{Her}(a)$ onto $\mathrm{Her}(b)$. Moreover, if $a' \in \mathrm{Her}(a)_+, b' \in \mathrm{Her}(b)_+$ are such that $\varphi(a') = b'$ then $[u(a')^{1/2}][u(a')^{1/2}]^* = b'$ and $[u(a')^{1/2}]^*[u(a')^{1/2}] = a'$. Therefore, $a' \sim_\phi b'$. We write $a \underset{\sim}{<}_\phi b$ if there is $b' \in \mathrm{Her}(b)$ such that $a \sim_\phi b'$. Clearly, the relation $\underset{\sim}{<}_\phi$ is transitive and the relation "$\sim_\phi$" is an equivalence relation.

1.5. There is another relation "$\sim_T$" introduced by G.K. Pedersen [20, 5.26]. (See also [13] for the case of infinite sums.) If $x, y \in A_+$, we write $x \sim_T y$ if there are $z_i \in A$, $i = 1, 2, \ldots, n$, such that $x = \sum_{i=1}^n z_i^* z_i$, $y = \sum_{i=1}^n z_i z_i^*$, and write $x \underset{\sim T}{<} y$ if there is $y' \in A_+$ such that $x \sim_T y'$, $y' \leq y$. If A has a trace τ, then $\tau(x) = \tau(y)$ if $x \sim_T y$. (See [20, 5.26] or [13].)

1.6. We will use the notation $\mathcal{P}(A)$ for the Pedersen ideal of the C^*-algebra A.

2. Scaling Approximate Identities and Skeleton C^*-subalgebras.

LEMMA 2.1. *Let A be a C^*-algebra, a and b two positive elements in $\mathcal{P}(A)$. If $\varepsilon > 0$ is such that $f_\varepsilon(b)$ generates $\mathcal{P}(A)$ as an ideal, then there are $a_1, a_2, \ldots, a_n \in P(A)_+$ such that*

$$a = \sum_{i=1}^{n} a_i, a_2 \lesssim_T a_3 \lesssim_T \cdots \lesssim_T f_{\varepsilon/2}(b),$$
$$a_1 \lesssim_\phi a_i, \quad i = 1, 2, \ldots, n \text{ and } a_1 \lesssim_\phi f_{\varepsilon/2}(b).$$

PROOF. There are $x_i, y_i \in A$, $i = 1, 2, \ldots, m$ such that

$$a = \sum_{i=1}^{m} x_i f_\varepsilon(b) y_i \leq \frac{1}{2}\Big(\sum_{i=1}^{m} x_i f_\varepsilon(b) x_i^* + \sum_{i=1}^{m} y_i^* f_\varepsilon(b) y_i\Big).$$

We may write $a \leq \sum_{i=1}^{n} r_i$, where $0 \leq r_i \underset{\approx}{<} f_\varepsilon(b)$. It follows from [22, 1.4.10] that there are $z_i \in A$ such that $a = \sum_{i=1}^{n} z_i^* z_i$ and $z_i z_i^* \leq r_i$, $i = 1, 2, \ldots, n$. Therefore we may write, by 1.4,

$$a = \sum_{i=1}^{n} a_i \text{ and } a_i \sim_\phi b_i,\ b_i \in \mathrm{Her}\big(f_\varepsilon(b)\big), \quad i = 1, 2, \ldots, n.$$

We will adjust the b_i's and a_i's so that

$$b_1 \leq b_1 \leq \cdots \leq b_n \leq f_{\varepsilon/2}(b),$$
$$a_i \sim_T b_i,\ a_1 \lesssim_\phi a_i,$$

$i = 2, 3, \ldots, n$ and $a_1 \sim_\phi b_1$.

We use induction on n. If $n = 2$, $a = a_1 + a_2$, $a_i \sim_\phi b_i$, and $b_i \in \mathrm{Her}\big(f_\varepsilon(b)\big)$, $i = 1, 2$. Since

$$b_1 \leq \big(f_{\varepsilon/2}(b) - b_2\big) + b_2,$$

applying [22, 1.4.10] we obtain c_1, c_1', d_1, d_1' such that $c_1 \sim_\phi d_1$, $c_1' \sim_\phi d_1'$, $b_1 = c_1 + c_1'$, $d_1 \leq f_{\varepsilon/2}(b) - b_1$, and $d_1' \leq d_2$. Set $b_1' = d_1'$ and $b_2' = b_2 + d_1$. Then $b_1' \leq b_2' \leq f_{\varepsilon/2}(b)$. Since $a_1 \sim_\phi b_1$, there are $t_1, t_1' \geq 0$ such that $a_1 = t_1 + t_1'$, $t_1 \sim_\phi c_1$, and $t_1' \sim_\phi c_1'$. Set $a_1' = t_1'$ and $a_2' = a_2 + t_1$. Then $a = a_1' + a_2'$, $a_1' \sim_\phi b_1'$, $a_2' \sim_T b_2'$.

Now assume that $a = \sum_{i=1}^{m} a_i$,

$$b_1 \leq b_3 \leq \cdots \leq b_n \leq f_{\varepsilon/2}(b),$$
$$a_i \sim_T b_i, \quad a_2 \lesssim_\phi a_i, \quad i = 3, 4, \ldots, n,$$
$$a_2 \sim_\phi b_2 \text{ and } b_1 \leq f_{\varepsilon/2}(b).$$

Since $b_1 \leq \big(f_{\varepsilon/2}(b) - b_n\big) + b_n$, applying [22, 1.4.10] we obtain $c_n, c_n', d_n, d_n' \geq 0$ such that

$$b_1 = c_n + c_n', \quad c_n \sim_\phi d_n, \quad c_n' \sim_\phi d_n',$$
$$d_n \leq f_{\varepsilon/2}(b) - b_n, \text{ and } d_n' \leq b_n.$$

Set $b_1' = d_n'$, $b_n' = b_n + d_n$. Then $b_1' \leq b_n' \leq f_{\varepsilon/2}(b)$. There are $t_n, t_n' \geq 0$ such that $t_n + t_n' = a_1$, $t_n \sim_\phi c_n$, $t_n' \sim_\phi c_n'$. Set $a_1' = t_n'$, $a_n' = a_n + t_n'$. Then $a_n' \sim_T b_n' \leq f_{\varepsilon/2}(b)$, $a_1' \sim_\phi b_1' \leq f_{\varepsilon/2}(b)$, and $a = a_1' + \sum_{i=2}^{n-1} a_i + (a_n + a_n')$ and

$$b_2 \leq b_3 \leq \cdots \leq b_n' = b_n + d_n, \quad b_1' \leq b_n'.$$

Repeating this argument with a_1', b_1' and b_{n-1}, we get $a_1' = t_{n-1} + t_{n_1}'$, $t_{n-1} \sim_\phi d_{n-1} \leq b_n - b_{n-1}$, $t_{n-1}' \sim_\phi d_{n-1}' \leq b_{n-1}, t_{n-1}, d_{n-1}, d_{n-1}' \geq 0$. Set $b_1'' = d_{n-1}'$, $b_{n-1}' = b_{n-1} + d_{n-1}$, $a_1'' = t_{n-1}'$, $a_{n-1}' = t_{n-1} + a_{n-1}$. Then

$$b_1'' \leq b_{n-1}' \leq b_n \leq b_n' \leq f_{\varepsilon/2}(b),$$
$$b_2' \leq b_3 \leq \cdots \leq b_{n-1} \leq b_{n-1}' \leq b_n \leq b_n',$$
$$a = a_1'' + \sum_{i=2}^{n-3} a_i + (a_{n-1} + t_{n-1}) + (a_n + t_n),$$

and $a_1'' \sim_\phi b_1''$, $a_{n-1}' = a_{n-1} + t_{n-1} \sim_T b_{n-1}'$, $a_n' \sim_\phi b_n'$.

Proceeding in this way, we can write

$$a_1 = \sum_{i=1}^{n} t_i,\ t_i \geq 0,\ t_i \sim_\phi d_i \leq b_{i+1} - b_i, \quad 2 \leq i \leq n$$

$\big(b_{n+1} = f_{\varepsilon/2}(b)\big)$, $t_1 \sim_\phi d_1' \leq b_2$, $b_i' = b_i + d_i$, $\quad 2 \leq i \leq n$. We have

$$a = t_1 + \sum_{i=2}^{n} (a_i + t_i) \text{ and } b_i' = b_i + d_i, \quad 2 \leq i \leq n.$$

Set $b_1' = d_1'$; then

$$b_1' \leq b_2' \leq \cdots \leq b_n' \leq f_{\varepsilon/2}(b)$$

and $a_i' = a_i + t_i \sim_T b_i'$, $2 \leq i \leq n$, $a_1' \sim_\phi b_1'$.

This completes the proof. ∎

DEFINITION 2.2. Let A be a σ-unital C^*-algebra and $\{e_n\}$ be an approximate identity for A. Denote $e_n - e_{n-1}$ by g_n ($e_0 = 0$). If there is a sequence of positive numbers $\{\varepsilon_k\}$ and a subsequence of positive integers $\{n(k)\}$ such that

(i) $f_{\varepsilon_k}(g_{n(k)}) \underset{\sim T}{>} g_n \underset{\sim T}{>} g_{n+1}$ for $n(k) < n < n(k+1)$,

(ii) $g_{n(k)} \underset{\sim}{<}_\phi g_n$ for $n(k-1) < n < n(k)$ and $g_{n(k)} \underset{\sim}{<}_\phi f_{\varepsilon_{k-1}}(g_{n(k-1)})$,

(iii) $g_{n(2k-1)} \perp g_n$ if $n > n(2k-1)$ or $n < n(2k-3)$,

(iv) $g_{n(2k)}\big(\sum_{n(2k-2)\leq i\leq n(k)} g_i\big) = \big(\sum_{n(2k-2)\leq i\leq n(2k)} g_i\big) g_{n(2k)} = g_{n(2k)}$,

where $k = 1, 2, \ldots$, then we say that $\{e_n\}$ is a scaling approximate identity.

It should be noted that if $\{e_n\}$ is a fundamental approximate identity, then $\{e_n\}$ is a scaling approximate identity.

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THEOREM 2.3. *If A is a σ-unital (non-unital) simple C^*-algebra, then A contains a scaling approximate identity.*

PROOF. Let a be a strictly positive element of A. By taking a proper sequence of continuous functions h_n, we can construct (by taking $e_n' = h_n(a)$) an approximate identity $\{e_n'\}$ such that for each n, there is $0 \le a_n \le e_n - e_{n+1}$ $(e_0 = 0)$, $a_n(e_n - e_{n-1}) = (e_n - e_{n-1})a_n = a_n$, $a_n \neq 0$, and $a_n \perp e_m - e_{m-1}$ if $n \neq m$. Moreover, $e_n \in P(A)$. Set $g_n = e_n - e_{n-1}$, $b_n = g_n - a_n$, $n = 1, 2, \ldots$. Applying Lemma 2.1, we obtain

$$b_2 = r_{2,1} + r_{2,2} + \ldots + r_{2,m(2)}$$

such that $0 \le r_{2,i+1} \underset{\sim T}{<} f_{\varepsilon_1}(a_1) \le f_{\varepsilon_1}(g_1)$, $r_{2,m(2)} \underset{\sim}{<}{}_{\phi}\, r_{2,i}$ and $r_{2,m(2)} \underset{\sim}{<}{}_{\phi} f_{\varepsilon_1}(a_1) \le f_{\varepsilon_1}(g_1)$, $i = 1, 2, \ldots, m(2) - 1$, for some $1 > \varepsilon_1 > 0$ and $r_{2,i} \neq 0$, $i = 1, 2, \ldots, m(2)$.

We also obtain

$$a_2 = r_{2,m(2)+1} + \cdots + r_{2,m(2)+m'(2)}$$

such that

$$\begin{aligned} 0 \le r_{2,m(2)+i+1} &\underset{\sim T}{<} r_{2,m(2)+i} \underset{\sim T}{<} f_{\varepsilon_2}(r_{2,m(2)}), \\ r_{2,m(2)+m'(2)} &\underset{\sim}{<}{}_{\phi}\, r_{2,m(2)+i}, \quad i = 1, \ldots, m'(2) - 1, \\ r_{2,m(2)+m'(2)} &\underset{\sim}{<}{}_{\phi} f_{\varepsilon_2}(r_{2,m(2)}), \end{aligned}$$

for some $1 > \varepsilon_2 > 0$, and $r_{2,m(2)+i} \neq 0$, $i = 1, 2, \ldots, m'(2)$.

Repeating this process, we get a sequence of nonzero positive elements as follows:

$$\begin{aligned} b_3 &= r_{3,1} + r_{3,2} + \cdots + r_{3,m(3)}, \\ a_3 &= r_{3,m(3)+1} + \cdots + r_{3,m(3)+m'(3)}, \\ &\cdots \\ b_k &= r_{k,1} + r_{k,2} + \cdots + r_{k,m(k)}, \\ a_k &= r_{k,m(k)+1} + \cdots + r_{k,m(k)+m'(k)}, \\ &\cdots \end{aligned}$$

such that

$$\begin{aligned} r_{k,i+1} &\underset{\sim T}{<} r_{k,i} \underset{\sim T}{<} f_{\varepsilon_{2k-1}}\big(r_{k-1,m(k-1)+m'(k-1)}\big), \\ r_{k,m(k)} &\underset{\sim}{<}{}_{\phi}\, r_{k,i}, \quad i = 1, 2, \ldots, m(k) - 1, \\ r_{k,m(k)} &\underset{\sim}{<}{}_{\phi} f_{\varepsilon_{k-1}}\big(r_{k-1,m(k-1)+m'(k-1)}\big) \end{aligned}$$

for some $1 > \varepsilon_{k-1} > 0$, and

$$\begin{aligned} r_{k,m(k)+i+1} &\underset{\sim T}{<} r_{k,m(k)+i} \underset{\sim T}{<} f_{2\varepsilon}(r_{k,m(k)}), \\ r_{k,m(k)+m'(k)} &\underset{\sim}{<}{}_{\phi}\, r_{k,m(k)+i}, \ i = 1, 2, \ldots, m'(k) - 1, \\ r_{k,m(k)+m'(k)} &\underset{\sim}{<}{}_{\phi} f_{\varepsilon_{2k}}(r_{k,m(k)}) \end{aligned}$$

for some $1 > \varepsilon_{2k} > 0$.

Now set

$$\begin{aligned}
&e_1 = e'_1,\ e_2 = e'_1 + r_{2,1},\ e_3 = e'_1 + r_{2,1} + r_{2,2},\\
&\dots\\
&e_{m(2)+1} = e'_1 + r_{2,1} + \cdots + r_{2,m(2)} = e'_1 + b_2,\\
&e_{m(2)+2} = e'_1 + b_2 + r_{2,m(2)+1},\\
&\dots\\
&e_{m(2)+m'(2)+1} = e'_1 + b_2 + r_{2,m(2)+1} + \cdots + r_{2,m(2)+m'(2)} = e'_1 + b_2 + a_2 = e'_2,\\
&\dots\\
&e_{1+m(k)+\sum_{n=2}^{k-1}(m(n)+m'(n))} = e'_{k-1} + \sum_{n=1}^{m(k)} r_{k,n} = e'_{k-1} + b_k,\\
&e_{2+m(k)+\sum_{n=2}^{k-1}(m(n)+m'(n))+1} = e'_{k-1} + b_k + r_{k,m(k)+1},\\
&\dots\\
&e_{1+\sum_{n=2}^{k}(m(n)+m'(n))} = e'_{k-1} + b_k + a_k = e'_k,\\
&\dots.
\end{aligned}$$

Take $n(1) = 1, n(2) = 1 + m(2), n(3) = 1 + m(2) + m'(2), \dots, n(2k) = 1 + m(2) + \cdots + m(k-1)$, and $n(2k+1) = 1 + m(2) + \cdots + m(k-1) + m'(k-1)$, $k = 1, 2, \dots$. From the construction one can check easily that $\{e_n\}$, $\{n_k\}$ and $\{\varepsilon_k\}$ satisfy the conditions (i) to (iv) in 2.2. ■

THEOREM 2.4. *Let $\mathcal{A}$ be a C^*-algebra with a scaling approximate identity $\{e_n\}$. Then $\mathcal{A}$ has a skeleton C^*-subalgebra B such that B has a quotient which is isomorphic to $\mathcal{K}$.*

PROOF. We will keep the notation of 2.2.

We first claim that there are $g_k^{(i)} \geq 0$, $1 \leq i \leq k$ and $u_k^{(i)}$, $1 \leq i \leq k-1$, $k = 1, 2, \dots$, such that

$$g_k^{(i)} \leq g_l^{(i)},\ \text{if } l \leq k, \quad g_k^{(i)} \in \mathrm{Her}\big(f_{\varepsilon_{2i}}\big(g_{n(2i)}\big)\big), \tag{1}$$

$$\big(u_k^{(i)}\big)\big(u_k^{(i)}\big)^* = g_k^{(i)},\ \big(u_k^{(i)}\big)^*\big(u_k^{(i)}\big) = f_{\sigma_{2i}}(g_{n(2i)}), \tag{2}$$

where $\sigma_k = \frac{1}{2}\delta_{2k}$, $k = 1, 2, \dots$.

We will prove the claim by induction on k. Assume that the claim is true for all $k' \leq k$. Since $g_{n(2(k+1))} \lesssim {}_\phi f_{\varepsilon_{2(k-1)}}(g_{2k})$, there is $u_{k+1}^{(k)}$ in A such that

$$\begin{aligned}
\big(u_{k+1}^{(k)}\big)^*\big(u_{k+1}^{(k)}\big) &= f_{\sigma_{k+1}}\big(g_{n(2(k+1))}\big),\\
\big(u_{k+1}^{(k)}\big)\big(u_{k+1}^{(k)}\big)^* &= g_{k+1}^{(k)} \in \mathrm{Her}\big(f_{\varepsilon_{2k}}\big(g_{n(2k)}\big)\big).
\end{aligned}$$

Define $u_{k+1}^{(i)} = u_k^{(i)} u_{k+1}^{(k)}$, $1 \leq i < k$. Then

$$\begin{aligned}
\big(u_{k+1}^{(i)}\big)^*\big(u_{k+1}^{(i)}\big) &= \big(u_{k+1}^{(k)}\big)^*\big(u_k^{(i)}\big)^*\big(u_k^{(i)}\big)\big(u_{k+1}^{(k)}\big),\\
&= \big(u_{k+1}^{(k)}\big)^* f_{\sigma_k}\big(g_{n(2(k))}\big)\big(u_{k+1}^{(k)}\big).
\end{aligned}$$

Since $f_{\sigma_k}\big(g_{n(2(k))}\big)$ is a unit for $\operatorname{Her}\big(f_{\varepsilon_{2k}}(g_{n(2(k))})\big)$,

$$\big(u_{k+1}^{(i)}\big)^*\big(u_{k+1}^{(i)}\big) = f_{\sigma_{k+1}}(g_{n(2(k+1))}).$$

Set

$$g_{k+1}^{(i)} = \big(u_{k+1}^{(i)}\big)\big(u_{k+1}^{(i)}\big)^* = \big(u_k^{(i)}\big)g_{k+1}^{(k)}\big(u_k^{(i)}\big)^*,$$

$1 \le i < k$; then $g_{k+1}^{(i)} \in \operatorname{Her}\big(f_{\varepsilon_{2i}}(g_{n(2i)})\big)$. This completes the proof of the claim.

Let $\chi_k(t)$ denote the characteristic function of the set $[0, \varepsilon_{2k}]$. Set $p_k^{(k-1)} = \chi_k(g_{n(2k)})$ and $p_k^{(i)} = u_k^{(i)}p_k^{(k-1)}\big(u_k^{(i)}\big)^*$. Then $p_k^{(i)}$ are closed projections (with respect to A) in A^{**}. Set $e'_k = \sum_{i=1}^k p_k^{(i)}$; e'_k is also a closed projection in A^{**}. Let B_2 be the C^*-subalgebra generated by $\big\{u_2^{(1)}, e_{n_2}\big\}, \dots, B_{k+1}$ the C^*-subalgebra generated by $\big\{B_k, u_{k+1}^{(k)}, e_{n(2(k+1))} - e_{n(2k)}\big\}$. Notice that e'_k commutes with $e_{n(2(i+1))} - e_{n(2i)}$ and $u_{i+1}^{(i)}$, $1 \le i \le k$. It is a routine exercise that e'_kB_k is isomorphic to $\mathcal{M}_k$ $(k \ge 2)$.

Now for fixed m, for $k \ge m$,

$$e_{n(2m)}e'_k = e'_ke_{n(2m)} = \sum_{i=1}^m p_k^{(i)}.$$

So $\big\{e_{n(2m)} \quad e'_k\big\}$ $(k \ge m)$ is a decreasing sequence of closed projections in A^{**}. So $\big\{e_{n(2m)}e'_k\big\}$ converges strongly to a positive element q_m in A^{**}. Hence q_m is an upper semi-continuous function or the quasi-state space of A (see [20, 3.11]). By a standard compactness argument, $q_m \ne 0$, and hence q_m is a nonzero projection in A^{**}. Now $\{q_m\}$ is an increasing sequence of projections, so $q_m \nearrow q$ for some nonzero projection q in A^{**}. Furthermore, $e'_k \longrightarrow q$ strongly.

Since e'_k commutes with every element of B_i, $2 \le i \le k$, we conclude that q commutes with every element of B_i, $2 \le i$. It is routine to check that qB_i is isomorphic to $\mathcal{M}_i$, $i \ge 2$. Denote by B the C^*-subalgebra generated by $\{B_i : i = 2, 3, \dots\}$; then q commutes with every element of B. Thus there is *-homomorphism from B onto Bq. Moreover, one can easily check that Bq is isomorphic to $\mathcal{K}$. ■

LEMMA 2.5. *Let A be a non-elementary simple C^*-algebra and a be a nonzero positive positive element of $P(A)$. Then for every k, there is a skeleton C^*-subalgebra B of* $\operatorname{Her}(a)$ *and a closed projection p in A^{**} such that p commutes with each element in B and such that pB is isomorphic to $\mathcal{M}_k$.*

PROOF. Since A is simple, so also is $\operatorname{Her}(a)$. If $\operatorname{sp}(a)$ is finite, then $\operatorname{Her}(a)$ has an identity e. There is a positive element b in $\operatorname{Her}(a)$ with infinitely many points in $\operatorname{sp}(b)$. So $\operatorname{sp}(e+b)$ has infinitely many points. Since $\operatorname{Her}(e+b) = \operatorname{Her}(a)$, we may assume that $\operatorname{sp}(a)$ has infinitely many points. There are continuous functions $h_1, h_2, \dots, h_k$ and $h'_1, h'_2, \dots, h'_k$ on $\operatorname{sp}(a)$ such that

(1) $$a \le \sum_{i=1}^k h_i(a),$$

(2) $$h'_i(a) \perp h_j(a) \text{ if } j \ne i,$$

(3) $$h'_i(a)h_i(a) = h_i(a)h'_i(a) = h'_i(a),$$
$$\|h'_i(a)\| = \|h_i(a)\| = 1, \quad i = 1, 2, \dots, k.$$

Repeated application of [10, 1.8] shows that there are nonzero elements $b'_i \in A_{h'_i(a)}$ such that

$$b'_1 \underset{\approx}{>} b'_2 \underset{\approx}{>} \cdots \underset{\approx}{>} b'_k$$

(see [10, 2.3]); we may assume that $0 \leq b'_i \leq 1$, and that $\|b'_i\| = 1$, $i = 1, 2, \ldots, k$. Take $b_1 = b'_k$ and apply [10, 1.7] repeatedly; we obtain $b_i \in A_{h'_i(a)}$, $\|b_i\| = 1$, and $z_i \in \mathrm{Her}(a)$ such that

$$z_i^* z_i = b_1,\ z_i z_i^* = b_i, \quad i = 2, \ldots, k.$$

There are $u_i \in \mathrm{Her}(a)$ such that

$$u_i^* u_i = f_{1/8}(b_1),$$
$$u_i u_i^* = f_{1/8}(b_i)$$

(see 1.4), $i = 1, \ldots, k$.

Let $\chi_{1/4}(t)$ denote the characteristic function of $[1/4, 1]$. Set $p_i = \chi_{1/4}(b_i)$ and $p = \sum_{i=1}^k p_i$. Then p is a closed projection in A^{**}. It is easy to see that p commutes with $h_i(a)$, $i = 1, 2, \ldots, k$, and commutes with u_i, $i = 1, \ldots, k$.

Let B denote the C^*-subalgebra generated by

$$\{h_i(a),\ i = 1, 2, \ldots, k,\ u_i,\ i = 2, 3, \ldots, k\}.$$

Then $\sum_{i=1}^k h_i(a) \geq a$. So B is a skeleton C^*-subalgebra of $\mathrm{Her}(a)$. Moreover, p commutes with each element of B. It is a routine exercise that pB is isomorphic to $\mathcal{M}_k$. ∎

THEOREM 2.6. *Let A be a σ-unital, non-unital, non-elementary simple C^*-algebra. Then for any non-unital matroid C^*-algebra $\mathcal{M}$, there is a skeleton C^*-subalgebra B of A such that B has a quotient isomorphic to $\mathcal{M}$.*

PROOF. As in the proof of 2.3, there is an approximate identity $\{e_n\}$ for A satisfying the following conditions:

(i) $e_n e_m = e_m e_n = e_m$, if $n > m$;

(ii) there are a_n in A such that $0 \leq a_n \leq e_n - e_{n-1} (e_0 = 0)$ and $a_n(e_m - e_{m-1}) = (e_m - e_{m-1})a_n = 0$ if $m \neq n$;

(iii) $(e_n - e_{n-1})(e_m - e_{m-1}) = 0$ if $|n - m| \geq 2$ and $\|e_n\| = 1$.

Suppose that

$$0 < q(1) < r(1) \leq q(2) < r(3) \leq \cdots$$

is a sequence of integers such that $\mathcal{M}$ is the following inductive limit:

$$\mathcal{M}_{q(1)} \xrightarrow{f_{q(1)r(1)}} \mathcal{M}_{r(1)} \xrightarrow{q_{r(1)q(2)}} \mathcal{M}_{q(2)} \xrightarrow{f_{q(2)r(2)}} \mathcal{M}_{r(2)} \xrightarrow{q_{r(2)q(3)}} \mathcal{M}_{q(3)} \longrightarrow \cdots.$$

Here $r(n) \mid q(n+1)$, and f_{mn} is the homomorphism consisting of adding $n - m$ rows and columns of zeros to each matrix in $\mathcal{M}_m$, and $g_{mn} = 1 \otimes 1_p$, i.e.

$$g_{mn}(x) = \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix}_{p \times p},$$

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where $p = \frac{n}{m}$. We set $s(n) = \frac{q(n+1)}{r(n)}$, $t(n) = r(n) - q(n)$ and $g_n = e_n - e_{n-1}$, $n = 1, 2, \ldots$.

As in Lemma 2.5, there are ${d_1}^{(1)}, {d_2}^{(1)}, \ldots, {d_{q(1)}}^{(1)}$ and $u_2, u_3, \ldots, u_{q(1)}$ in $\mathrm{Her}(a_1)$ such that $0 \le {d_i}^{(1)}$, $\|{d_i}^{(1)}\| = 1$, ${d_i}^{(1)} \perp {d_j}^{(1)}$ if $i \neq j$, and

$$u_i^{(1)*}u_i^{(1)} = f_{1/8}\big({d_1}^{(1)}\big),$$
$$u_i^{(1)}u_i^{(1)*} = f_{1/8}\big({d_i}^{(1)}\big).$$

Moreover, if we take $e_1' = \sum_{i=1}^{q(1)} p_i^{(1)}$, where $p_i^{(1)} = \chi_{1/4}\big({d_i}^{(1)}\big)$, and B_1 the C^*-subalgebra generated by $\{g_1, u_i, \quad i = 2, 3, \ldots, q(1)\}$, then e_1' commutes with each element of B_1 and $e_1'B_1$ is isomorphic to $\mathcal{M}_{q(1)}$.

Repeated application of [10, 1.8] shows that there are elements $d_i' \in \mathrm{Her}(a_{i+1})$, $\big(i = 1, 2, \ldots, t(1)\big)$ such that

$$f_{1/2}\big({d_1}^{(1)}\big) \gtrapprox d_1' \gtrapprox d_2' \gtrapprox \cdots \gtrapprox d_{t(1)}'$$

(see [10, 2.3]). As in the proof of Lemma 2.5, there are ${d_i}^{(2)} \in \mathrm{Her}(a_{i+1})$ with ${d_i}^{(2)} \ge 0$, $\|{d_i}^{(2)}\| = 1$, $i = 1, 2, \ldots, t(1)$, $d_{t(1)+1}^{(2)} \in \mathrm{Her}\big(f_{1/4}({d_1}^{(i)})\big)$ with $d_{t(1)+1}^{(2)} \le 0$, $\|d_{t(1)+1}^{(2)}\| = 1$, and $u_i^{(2)} \in A$ such that

$$\big(u_i^{(2)}\big)^*\big(u_i^{(2)}\big) = f_{1/8}\big({d_1}^{(2)}\big),$$
$$\big(u_i^{(2)}\big)\big(u_i^{(2)}\big)^* = f_{1/8}\big({d_i}^{(2)}\big),$$

$i = 2, 3, \ldots, t(1) + 1$. Set

$$u_{j+t(1)}^{(2)} = u_j^{(1)}u_{t(1)+1}^{(2)}, \quad j = 2, 3, \ldots, q(1),$$
$$p_i^{(2)} = \chi_{1/4}\big({d_i}^{(2)}\big), \quad i = 1, 2, \ldots, t(1) + 1.$$

Then

$$\big(u_i^{(2)}p_1^{(2)}\big)^*\big(u_i^{(2)}p_1^{(2)}\big) = p_1^{(2)},$$
$$\big(u_i^{(2)}p_1^{(2)}\big)\big(u_i^{(2)}p_1^{(2)}\big)^* = p_i^{(2)},$$

$i = 2, 3, \ldots, t(1) + 1$, and

$$\big(u_j^{(2)}p_1^{(2)}\big)^*\big(u_j^{(2)}p_1^{(2)}\big) = p_1^{(2)}\big(u_{t(1)+1}^{(2)}\big)^*\big(u_{j-t(1)}^{(2)}\big)^*\big(u_{j-t(1)}^{(2)}\big)\big(u_{t(1)+1}\big)p_1^{(2)} = p_1^{(2)},$$
$$\big(u_j^{(2)}p_1^{(2)}\big)\big(u_j^{(2)}p_1^{(2)}\big)^* = p_j^{(2)} \le p_{j-t(1)}^{(1)}, j = t(1) + 2, \ldots, r(1),$$

where $p_j^{(2)}$ are closed projections. Let $e_2' = \sum_{j=1}^{r(1)} p_j^{(2)}$; then $e_2'p_{j-t(1)}^{(1)} = p_{j-t(1)}^{(1)}e_2' = p_j^{(2)}$, $j = t(1)+2, \ldots, r(1)$, $e_2'e_1' = e_1'e_2' = \sum_{j=t(1)+2}^{r(1)} p_j^{(2)}$ and e_2' commutes with each element of B_2, where B_2 is the C^*-subalgebra generated by $\{B_1, u_i^{(2)}, i = 1, 2, \ldots, t(1) + 1, e_{t(1)+1} - e_1\}$. It is a routine exercise to check that $e_2'B_2$ is isomorphic to $\mathcal{M}_{r(1)}$. Moreover, $e_2'B_1$ is isomorphic to $\mathcal{M}_{q(1)}$. If we identify $e_2'B_2$ with $\mathcal{M}_{r(1)}$, and $e_2'B_1$ with $\mathcal{M}_{q(1)}$, then the isomorphism $e_1'B_1 \longrightarrow e_2'e_1'B_1 = e_2'B_1$ gives the homomorphism $f_{q(1)r(1)}$ from $\mathcal{M}_{q(1)}$ into $\mathcal{M}_{r(1)}$, and we may write $e_1'B_1 \xrightarrow{f_{q(1)r(1)}} e_2'B_2$.

We assume that there are C^*-subalgebras $B_1, B_2, \ldots, B_m, \ldots$ satisfying the following:

(1) B_{2n} is generated by $\{B_{2n-1}, e_{\sum_{i=1}^{n} t(i)+1}, u_i^{(2n)}, i = 2,3,\ldots,t(n)+1\}$;

(2) there are $d_i^{(2n)} \geq 0$, $\|d_i^{(2n)}\| = 1$, $i = 1,2,\ldots,t(n)+1$, such that

$$d_i^{(2n)} \in \mathrm{Her}\big(a_{\sum_{j=1}^{n-1} t(j)+1+i}\big), \quad i = 1,2,\ldots,t(n),$$
$$d_{t(n)+1}^{(2n)} \in \mathrm{Her}\Big(f_{1/2}\big(d_1^{(2n-1)}\big)\Big),$$

$$\big(u_i^{(2n)}\big)^*\big(u_i^{(2n)}\big) = f_{1/8}\big(d_1^{(2n)}\big),$$
$$\big(u_i^{(2n)}\big)\big(u_i^{(2n)}\big)^* = f_{1/8}\big(d_i^{(2n)}\big),$$

$i = 2,3,\ldots,t(n)+1$;

(3) if we set $u_{j+t(n)}^{(2n)} = u_j^{(2n-1)}u_{t(n)+1}^{(2n)}$, $j = 2,3,\ldots,q(n)$ and $p_i^{(2n)} = \chi_{1/4}\big(d_i^{(2n)}\big)$, $i = 1,2,\ldots,t(n)+1$, then

$$\big(u_i^{(2n)}p_1^{(2n)}\big)^*\big(u_i^{(2n)}p_1^{(2n)}\big) = p_1^{(2n)},$$
$$\big(u_i^{(2n)}p_1^{(2n)}\big)\big(u_i^{(2n)}p_1^{(2n)}\big)^* = p_i^{(2n)},$$

$i = 2,3,\ldots,t(n)+1$, and

$$\big(u_j^{(2n)}p_1^{(2n)}\big)^*\big(u_j^{(2n)}p_1^{(2n)}\big) = p_1^{(2n)},$$
$$\big(u_j^{(2n)}p_1^{(2n)}\big)\big(u_j^{(2n)}p_1^{(2n)}\big)^* = p_j^{(2n)}$$

where $p_j^{(2n)} \leq p_{j-q(2)}^{(2n-1)}$ are closed projections, $j = t(n)+2,\ldots,r(n)$, and $p_1^{(2n)} \leq p_1^{(2n-1)}$;

(4) if we set $e'_{2n} = \sum_{j=1}^{r(n)} p_j^{(2n)}$ then $e'_{2n}e'_{2n-1} = e'_{2n-1}e'_{2n} = \sum_{j=t(n)+1}^{r(n)} p_j^{(2n)}$ and e'_{2n} commutes with each element of B_{2n}, $e'_{2n}B_{2n}$ is isomorphic to $\mathcal{M}_{r(n)}$ and $e'_{2n}B_{2n-1} = e'_{2n}e'_{2n-1}B_{2n-1}$ is isomorphic to $\mathcal{M}_{q(n)}$;

(5) if we identify $e'_{2n}B_{2n}$ with $\mathcal{M}_{r(n)}$ and $e'_{2n-1}B_{2n-1}$ with $\mathcal{M}_{q(n)}$, the isomorphism $e'_{2n-1}B_{2n-1}$ to $e'_{2n}B_{2n}$ given by $x \longrightarrow e'_{2n}x$ ($x \in e'_{2n-1}B_{2n-1}$) gives the homomorphism:

$$\mathcal{M}_{q(n)} \xrightarrow{f_{q(n)r(n)}} \mathcal{M}_{r(n)};$$

(6) B_{2n+1} is generated by $\{B_{2n}, u_i^{(2n+1)}, i = 2,3,\ldots,s(n)\}$;

(7) $u_i^{(2n+1)} \in \mathrm{Her}\Big(f_{1/2}\big(d_1^{(2n)}\big)\Big)$ $i = 2,\ldots,s(n)$, and there are

$$d_i^{(2n+1)} \in \mathrm{Her}\Big(f_{1/2}\big(d_1^{(2n)}\big)\Big),$$
$$d_i^{(2n+1)} \geq 0, \|d_i^{(2n+1)}\| = 1,\ d_i^{(2n+1)} \perp d_j^{(2n+1)},$$

$i \neq j, i,j = 1,2,\ldots,s(n)$, such that

$$\big(u_i^{(2n+1)}\big)^*\big(u_i^{2n+1)}\big) = f_{1/8}\big(d_1^{(2n+1)}\big),$$
$$\big(u_i^{(2n+1)}\big)\big(u_i^{(2n+1)}\big)^* = f_{1/8}\big(d_i^{(2n+1)}\big),$$

$i = 2,3,\ldots,2(n)$;

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(8) if we set

$$u_{j+(k-1)r(n)}^{(2n+1)} = u_k^{(2n)} u_j^{(2n-1)}, \quad j = 2,3,\ldots,s(n), k = 2,3,\ldots,s(n),$$
$$p_i^{(2n+1)} = \chi_{1/4}\big(d_i^{(2n+1)}\big), \quad i = 1,2,\ldots,s(n),$$

then

$$\big(u_i^{(2n+1)} p_1^{(2n+1)}\big)^* \big(u_i^{(2n+1)} p_1^{(2n+1)}\big) = p_1^{(2n+1)},$$
$$\big(u_i^{(2n+1)} p_1^{(2n+1)}\big) \big(u_i^{(2n+1)} p_1^{(2n+1)}\big)^* = p_i^{(2n+1)},$$

$i = 2,3,\ldots,s(n)$, and

$$\big(u_j^{(2n+1)} p_1^{(2n+1)}\big)^* \big(u_j^{(2n+1)} p_1^{(2n+1)}\big) = p_1^{(2n+1)},$$
$$\big(u_j^{(2n+1)} p_1^{(2n+1)}\big) \big(u_j^{(2n+1)} p_1^{(2n+1)}\big)^* = p_j^{(2n+1)},$$

$j = s(n)+1,\ldots,q(n+1)$, where $p_j^{(2n+1)}$ are closed projections;

(9) if we set $e'_{2n+1} = \sum p_j^{(2n+1)}$, then $e'_{2n+1}e'_{2n} = e'_{2n}e'_{2n+1} = e'_{2n+1}$, e'_{2n+1} commutes with each element of B_{2n+1}. Moreover, $e'_{2n+1}B_{2n+1}$ is isomorphic to $M_{q(n+1)}$ and $e'_{2n+1}B_{2n}$ is isomorphic to $M_{r(n)}$.

(10) if we identify $e'_{2n+1}B_{2n+1}$ with $M_{q(n+1)}$ and $e'_{2n}B_{2n}$ with $M_{r(n)}$, then the isomorphism from $e'_{2n}B_{2n}$ to $e'_{2n+1}B_{2n}$ given by $x \longrightarrow e'_{2n+1}x, x \in e'_{2n}B_{2n}$, gives the homomorphism:

$$e'_{2n}B_{2n} \overset{g_{r(n)q(n+1)}}{\longrightarrow} e'_{2n+1}B_{2n}.$$

If $m = 2n$, then, as in Lemma 2.5, there are $d_1^{(m+1)}, d_2^{(m+1)}, \ldots, d_{s(n)}^{(m+1)}$ and

$$u_i^{(m+1)} \in \mathrm{Her}\big(f_{1/2}\big(d_i^{(m)}\big)\big),$$

$i = 2,3,\ldots,s(n)$, such that $d_i^{(m+1)} \geq 0$, $\|d_i^{(m+1)}\| = 1$, $d_i^{(m+1)} \perp d_j^{(m+1)}$ if $i \neq j$ and

$$\big(u_i^{(m+1)}\big)^* \big(u_i^{(m+1)}\big) = f_{1/8}\big(d_1^{(m+1)}\big),$$
$$\big(u_i^{(m+1)}\big) \big(u_i^{(m+1)}\big)^* = f_{1/8}\big(d_i^{(m+1)}\big),$$

$i = 2,3,\ldots,s(b)$. Set B_{m+1} equal to the C^*-subalgebra generated by

$$\{B_m, u_i^{(m+1)}, \quad i = 2,3,\ldots,s(n)\}$$

and

$$p_i^{(m+1)} = \chi_{1/4}\big(d_i^{(m+1)}\big), \quad i = 1,2,\ldots,s(n),$$
$$u_{j+(k-1)r(n)}^{(m+1)} = u_k^{(m)} u_j^{(m+1)}, \quad j = 2,3,\ldots,s(n),\ k = 2,3,\ldots,s(n).$$

Then

$$\big(u_i^{(m+1)} p_1^{(m+1)}\big)^* \big(u_i^{(m+1)} p_1^{(m+1)}\big) = p_1^{(m+1)},$$
$$\big(u_i^{(m+1)} p_1^{(m+1)}\big) \big(u_i^{(m+1)} p_1^{(m+1)}\big)^* = p_i^{(m+1)},$$

$i = 2, 3, \ldots, s(n)$, and

$$\begin{aligned}\left(u^{(m+1)}_{j+(k-1)r(n)}p^{(m+1)}_1\right)^*\left(u^{(m+1)}_{j+(k-1)r(n)}p^{(m+1)}_1\right) \\ &= p^{(m+1)}_1\left(u^{(m+1)}_j\right)^*\left(u^{(m)}_k\right)^*\left(u^{(m)}_k\right)\left(u^{(m+1)}_j\right)p^{(m+1)}_1 \\ &= p^{(m+1)}_1,\end{aligned}$$

$$\begin{aligned}\left(u^{(m+1)}_{j+(k-1)r(n)}p^{(m+1)}_1\right)\left(u^{(m+1)}_{j+(k-1)r(n)}p^{(m+1)}_1\right)^* \\ &= u^{(m)}_k u^{(m+1)}_j p^{(m+1)}_1 p^{(m+1)}_1\left(u^{(m+1)}_j\right)^*\left(u^{(m)}_k\right)^* \\ &= u^{(m)}_k u^{(m+1)}_j p^{(m+1)}_1 p^{(m+1)}_1\left(u^{(m+1)}_j\right)^*\left(u^{(m)}_k\right)^* \\ &= p^{(m+1)}_{j+(k-1)r(n)},\end{aligned}$$

$j = 2, 3, \ldots, s(n)$, $k = 2, 3, \ldots, s(n)$, where $p^{(m+1)}_{j+(k-1)r(n)}$ are closed projections. If we set $e'_{m+1} = \sum_{j=1}^{q(n+1)} p^{(m+1)}_j$, then $e'_{m+1}e'_m = e'_m e'_{m+1} = e'_{m+1}$ and e'_{m+1} commutes with each element in B_{m+1}. It is a routine exercise to check that $e'_{m+1}B_{m+1}$ is isomorphic to $\mathcal{M}_{q(n+1)}$. Moreover, if we define a map φ from $e'_m B_m$ onto $e'_{m+1}B_m$ by $\varphi(x) = e'_{m+1}x$ then the map is an isomorphism. If we identify $e'_{m+1}B_{m+1}$ with $\mathcal{M}_{q(n+1)}$ and $e'_m B_m$ with $\mathcal{M}_{r(n)}$, the map φ gives the homomorphism:

$$e'_m B_m \xrightarrow{g_{r(n)q(n+1)}} e'_{m+1} B_{m+1}.$$

If $m = 2n - 1$, repeated application of [10, 1.8] shows that there are elements $d'_i \in \operatorname{Her}\left(a_{\sum_{j=1}^{n-1} t(j)+1+i}\right)$, $i = 1, 2, \ldots, t(n)$, such that

$$f_{1/2}\left(d^{(m)}_1\right) \underset{\approx}{>} d'_2 \underset{\approx}{>} d'_2 \underset{\approx}{>} \cdots \underset{\approx}{>} d'_{t(n)}$$

(see [10, 2.3]). As in the proof of Lemma 2.5, there are

$$d_i^{(m+1)} \in \operatorname{Her}\left(a_{\sum_{j=1}^{n-1} f(j)+1+i}\right),$$
$$d_{t(n)+1}{}^{(m+1)} \in \operatorname{Her}\left(f_{1/2}\left(d_1{}^{(m)}\right)\right),$$

$d_i{}^{(m+1)} \geq 0$, $\|d_i{}^{(m+1)}\| = 1$, $i = 1, 2, \ldots, t(n) + 1$, and $u^{(m+1)}_i \in A$ such that

$$\left(u^{(m+1)}_i\right)^*\left(u^{(m+1)}_i\right) = f_{1/8}\left(d_1{}^{(m+1)}\right)$$

and

$$\left(u^{(m+1)}_i\right)\left(u^{(m+1)}_i\right)^* = f_{1/8}\left(d_i{}^{(m+1)}\right),$$

$i = 2, 3, \ldots, t(n) + 1$.

Set

$$u^{(m+1)}_{j+t(n)} = u^{(m)}_j u^{(m+1)}_{t(n)+1}, \quad j = 2, 3, \ldots, q(n),$$

and

$$p^{(m+1)}_i = \chi_{1/4}\left(d^{(m+1)}_i\right), \quad i = 1, 2, \ldots, t(n) + 1.$$

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Then

$$\left(u_i^{(m+1)}p_1^{(m+1)}\right)^*\left(u_i^{(m+1)}p_1^{(m+1)}\right) = p_1^{(m+1)},$$
$$\left(u_i^{(m+1)}p_1^{(m+1)}\right)\left(u_i^{(m+1)}p_1^{(m+1)}\right)^* = p_i^{(m+1)}, \quad i = 2,3,\ldots,t(n)+1.$$

Moreover,

$$\begin{aligned}
&\left(u_{j+t(n)}^{(m+1)}p_1^{(m+1)}\right)^*\left(u_{j+t(n)}^{(m+1)}p_1^{(m+1)}\right)\\
&\quad = p_1^{(m+1)}\left(u_{t(n)+1}^{(m+1)}\right)^*\left(u_j^{(m)}\right)^*\left(u_j^{(m)}\right)\left(u_{t(n)+1}^{(m+1)}p_1^{(m+1)}\right)\\
&\quad = p_1^{(m+1)},\\
&\left(u_{j+t(n)}^{(m+1)}p_1^{(m+1)}\right)\left(u_{j+t(n)}^{(m+1)}p_1^{(m+1)}\right)^*\\
&\quad = u_j^{(m)}u_{t(n)+1}^{(m+1)}p_1^{(m+1)}p_1^{(m+1)}\left(u_{t(n)+1}^{(m+1)}\right)^*\left(u_j^{(m)}\right)^*\\
&\quad = p_{j+t(n)}^{(m+1)}, \quad j = 2,3,\ldots,q(n),
\end{aligned}$$

where $p_{j+t(n)}^{(m+1)}$ are closed projections and $p_{j+t(n)}^{(m+1)} \le p_j^{(m)}, \quad j = 1,2,\ldots,q(n)$.

Set $e'_{m+1} = \sum_{j=1}^{r(n)} p_j^{(m+1)}$; then

$$e'_{m+1}p_j^{(m)} = p_j^{(m)}e'_{m+1} = p_{j+t(n)}^{(m+1)}, \quad j = 1,2,\ldots,q(n),$$
$$e'_{m+1}e'_m = e'_m e'_{m+1} = \sum_{j=t(1)+1}^{r(1)} p_j^{(m+1)}$$

and e'_{m+1} commutes with each element in B_{m+1}, where B_{m+1} is the C^*-subalgebra generated by

$$\left\{B_m, u_i^{(m+1)}, \quad i = 1,2,\ldots,t(n)+1, \quad e_{\sum_{j=1}^n t(j)+1} - e_1\right\}.$$

It is a routine exercise to check that $e'_{m+1}B_{m+1}$ is isomorphic to $\mathcal{M}_{r(n)}$. Moreover, if we define a map φ from $e'_m B_m$ onto $e'_{m+1}B_m$ by $\varphi(x) = e'_{m+1}x$ then the map is an isomorphism. If we identify $e'_{m+1}B_{m+1}$ with $\mathcal{M}_{r(n)}$ and $e'_m B_m$ with $\mathcal{M}_{q(n)}$, the map φ gives the homomorphism:

$$e'_m B_m \xrightarrow{f_{q(n)r(n)}} e'_{m+1}B_{m+1}.$$

For fixed n, $\{e'_n e'_m\}$ is a decreasing sequence of closed projections ($m \ge n$). So $\{e'_n e'_m\}$ converges strongly to a positive element q_m in A^{**}. Hence q_m is an upper semicontinuous function on the quasi-state space of A (see [20, 3.11]). By a standard compactness argument, $q_m \ne 0$, and hence q_m is a nonzero projection in A^{**}. Now $\{q_m\}$ is an increasing sequence of projections, and so $q_m \nearrow q$ for some nonzero projection q in A^{**}. Furthermore, $e'_m \longrightarrow q$ strongly.

Since e'_m commutes with every element in B_i, $1 \le i \le m$, we conclude that q commutes with every element of B_m. It is then easy to see that qB_m is isomorphic to $e'_m B_m$. If B denotes the C^*-subalgebra generated by $\{B_m, m = 1,2,\ldots,m\}$, then q commutes each element of B. So there is a homomorphism from B onto qB. By the construction of $\{B_m\}$, it is easily checked that qB is the norm closure of the following inductive limit:

$$qB_1 \xrightarrow{f_{q(1)r(1)}} qB_2 \xrightarrow{g_{r(1)q(2)}} qB_3 \xrightarrow{f_{q(2)r(2)}} qB_4 \xrightarrow{g_{r(2)q(3)}} qB_5 \longrightarrow \cdots.$$

Therefore qB is isomorphic to $\mathcal{M}$. Since $\sum_{k=1}^{n} t(k) \longrightarrow \infty$ as $n \longrightarrow \infty$, and $e_{\sum_{k=1}^{n} t(k)+1} \in B$, B is a skeleton C^*-subalgebra of A. This completes the proof. ∎

We also have the following:

THEOREM 2.7. *Let A be a unital and non-elementary simple C^*-algebra. Then for any unital matroid C^*-algebra $\mathcal{M}$, there is a skeleton C^*-subalgebra B of A such that B has a quotient which is isomorphic to $\mathcal{M}$.*

2.8. Real rank of a C^*-algebra has been defined by L. G. Brown and G. K. Pedersen in [5]. A C^*-algebra is said to have real rank zero if the invertible selfadjoint elements are norm dense in $A_{s.a.}$. A C^*-algebra has real rank zero if and only if the elements in $A_{s.a.}$ with finite spectra are dense in $A_{s.a.}$, and if and only if A has (HP), i.e. every hereditary C^*-subalgebra of A has an approximate identity consisting of projections (see [5, 2.6]). Trivial examples of C^*-algebras with real rank zero are von Neumann algebras and AF C^*-algebras.

THEOREM 2.9. *Let A be a separable C^*-algebra with real rank zero and stable rank one. If $K_0(A)$ is unperforated, then there is a skeleton C^*-subalgebra B of A such that B is an AF C^*-algebra with $K_0(B) = K_0(A)$. Moreover, for every projection p in A, there is a projection q in B such that p is equivalent (in the sense of Murray and von Neumann) to q.*

PROOF. $K_0(A)$ is a countable, unperforated ordered group. It follows from [2, 6.5.1] and [24, 1.6] that $K_0(A)$ has the Riesz interpolation property (see [15, A3.1]). Therefore, $K_0(A)$ is a dimension group ([15, 3.1]). In other words,

$$K_0(A) = \varinjlim\{\mathbb{Z}^{(r_n)}, \varphi_n\}.$$

Suppose that $\{e_n\}$ is an approximate identity for A consisting of projections. Set

$$p_1 = e_1, \quad p_n = e_n - e_{n-1}, \quad n = 2, 3, \ldots.$$

If A is unital, we assume that $p_1 = 1$, $p_n = 0$ if $n > 1$. Without loss of generality, we may assume that $[p_1] \in \mathbb{Z}^{(r_1)}$ and $[p_1] = (k(1), k(2), \ldots, k(r_1))$, where $k(i)$ is a nonzero integer. Suppose that $[q_1] = (1, 0, \ldots, 0)$. Then $[q_1] \le [p_1]$. Since A has cancellation (see [2, 6.5.1]), $p_1 \underset{\sim}{>} q_1$ (in the sense of Murray and von Neumann). Therefore there is a projection $q_{1,1}^{(1)} \le p_1$ such that

$$q_{1,1}^{(1)} \in \left[q_{1,1}^{(1)}\right] \text{ and } \left[p_1 - q_{1,1}^{(1)}\right] = (k(1) - 1, k(2), \ldots, k(r_1)).$$

Recursively, we can construct projections

$$q_{i,j}^{(1)} \le p_1,$$

$1 \le j \le k(i)$, $i = 1, 2, \ldots, r(1)$, such that

$$q_{i',j'}^{(1)} \perp q_{i,j}^{(1)} \text{ if } i \ne i' \text{ or } j \ne j',$$

$$q_{i,j}^{(1)} \sim q_{i,j'}^{(1)} \text{ and } \left[q_{i,j}^{(1)}\right] = (\overbrace{0, \ldots, 0, 1}^{i}, 0, \ldots, 0).$$

Moreover, $\sum_{i,j} q_{i,j}^{(1)} = p_1$. Let $v_{i,j}^{(1)}$ be partial isometries in A such that $\left(v_{i,j}^{(1)}\right)^*\left(v_{i,j}^{(1)}\right) = q_{i,1}^{(1)}$ and $\left(v_{i,j}^{(1)}\right)\left(v_{i,j}^{(1)}\right)^* = q_{i,j}^{(1)}, \quad 2 \le j \le k(i), \quad i = 1, 2, \ldots, r_1$.

It is routine to check that the C^*-subalgebra B_1 generated by $\{v_{i,j}^{(1)},\ 2 \le j \le k(i),\ i = 1, 2, \ldots, r_1\}$ is isomorphic to

$$\mathcal{M}_{k(1)} \oplus \mathcal{M}_{k(2)} \oplus \cdots \oplus \mathcal{M}_{k(r_1)},$$

and $K_0(B_1) \cong \mathbb{Z}^{(r_1)}$. We may assume that $[e_2] = [p_1 + p_2] = [p_1] + [p_2] \in \mathbb{Z}^{(r_2)}$ and

$$[p_1 + p_2] = (m(1), m(2), \ldots, m(r_2)).$$

Suppose that, in $\mathbb{Z}^{(r_2)}$,

$$\left[q_{1,1}^{(1)}\right] = (k_{11}, k_{21}, \ldots, k_{s(1),1}, 0, \ldots, 0).$$

Repeating the above argument, we can construct projections $q_{i,j}^{(2)} \le q_{1,1}^{(1)}, \quad 1 \le j \le k(i),\ i = 1, 2, \ldots, s(1)$, such that

$$q_{i',j'}^{(2)} \perp q_{i,j}^{(2)} \text{ if } i \ne i' \text{ or } j \ne j',$$

$$q_{i,j}^{(2)} \sim q_{i,j'}^{(2)} \text{ and } \left[q_{ij}^{(2)}\right] = \big(\overbrace{0, \ldots, 0, 1}^{i}, 0, \ldots, 0\big).$$

Let us do the same for each $q_{i,j}^{(1)}$. If $[p_2 - p_1] = (s(1), s(2), \ldots, s(r_2))$ (some of $s(i)$ may be zero.) Let us add $s(i)$ orthogonal but equivalent projections $q_{i,j}^{(2)}$ in $(p_2 - p_1)A(p_2 - p_1)$ for each i. Suppose that $v_{i,j}^{(2)}$ are partial isometries in A such that $\left(v_{i,j}^{(2)}\right)^*\left(v_{i,j}^{(2)}\right) = q_{i,1}^{(2)}$ and $\left(v_{i,j}^{(2)}\right)\left(v_{i,j}^{(2)}\right)^* = q_{i,j}^{(2)}$, $2 \le j \le m(i)$, $i = 1, 2, \ldots, r_2$.

The C^*-subalgebra B_2 generated by $\{v_{i,j}^{(2)},\ 2 \le j \le m(i),\ i = 1, 2, \ldots, r_2\}$ is isomorphic to

$$\mathcal{M}_{m(1)} \oplus \mathcal{M}_{m(2)} \oplus \cdots \oplus \mathcal{M}_{m(r_2)},$$
$$B_1 \subset B_2,$$

and $K_0(B_2) \cong \mathbb{Z}^{(r_2)}$.

Continuing this way, we get a sequence of C^*-subalgebras $B_1 \subset B_2 \subset \cdots \subset B_n \subset B_{n+1} \subset \ldots$ such that

$$B_n \cong \mathcal{M}_{m^{(n)}(1)} \oplus \mathcal{M}_{m^{(n)}(2)} \oplus \cdots \oplus \mathcal{M}_{m^{(n)}(r_n)}$$

for some integers $m^{(n)}(i)$, $i = 1, 2, \ldots, r_n$, $K_0(B_n) = \mathbb{Z}^{(r_n)}$, and the embedding: $B_n \longrightarrow B_{n+1}$ gives a homomorphism:

$$\mathbb{Z}^{(r_n)} \xrightarrow{\varphi_n} \mathbb{Z}^{r_{n+1}}.$$

Let B be the C^*-subalgebra generated by $\{\cup_{n=1}^{\infty} B_n\}$. Then B is an AF C^*-algebra and $K_0(B) \cong \varinjlim \mathbb{Z}^{(r_n)}$. Since $e_n \in B$, B is a skeleton C^*-subalgebra of A. If p is a projection in A, we may assume that $[p] \in \mathbb{Z}^{(r_n)}$. Therefore there is $q \in B_n \subset B$ such that p is equivalent to q (in the sense of Murray and von Neumann). ∎

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REMARK 2.10. Separable AF C^*-algebras have real rank zero, stable rank one and unperforated $K_0(A)$. Theorem 2.9 shows that separable C^*-algebras with real rank zero, stable rank one and unperforated $K_0(A)$ are somewhat similar to separable AF C^*-algebras. However, a recent result of M. D. Choi and G. A. Elliott ([7]) provides examples (namely, irrational rotation C^*-algebras) of simple C^*-algebras with real rank zero, stable rank one and unperforated $K_0(A)$ which are not approximate finite-dimensional. (Note these simple C^*-algebras have cancellation [23]. Hence, by [2, 6.5.7], they have stable rank one.) The author would like to raise the following question:

Are separable nuclear (simple) C^*-algebras with real rank zero, stable rank one, unperforated K_0-groups and trivial K_1-flows (see [26]) approximate finite dimensional?

3. **Applications.**

3.1. Let A be a C^*-algebra and denote by A^{**} its enveloping von Neumann algebra. An element x in A^{**} is a multiplier if xa and ax are in A for all a in A, x is a left multiplier if xa is in A for all a in A, x is a right multiplier if ax is in A for all a in A, and x is a quasi-multiplier if axb is in A for all a and b in A. We denote the collections of multipliers, left multipliers, right multipliers and quasi-multipliers by $M(A)$, $LM(A)$, $RM(A)$ and $QM(A)$ respectively. If B is a skeleton C^*-subalgebra of A, then $M(B) \subset M(A)$, $LM(B) \subset LM(A)$, $RM(B) \subset RM(A)$ and $QM(B) \subset QM(A)$ (see [19, 3.7]). (It should be noted that the above inclusions do not hold if B is merely a C^*-subalgebra of A.) Therefore the results in § 2 may help us to determine the structure of $M(A)$, $LM(A)$, $RM(A)$ and $QM(A)$.

It is easy to see that $LM(A) + RM(A) \subset QM(A)$. The question whether $LM(A) + RM(A) = QM(A)$ was raised in [1]. The problem has been studied in [4], [19], [20], [21], among other articles. In this section we will give applications of the results in § 2 to this problem.

Recall that a C^*-algebra is scattered if it is type I and has scattered spectrum $\hat{A}$ (see [16]). Let X be a scattered topological space. Define $X_{[0]} = X, X_{[1]} = X \setminus \{$isolated point of $X\}$. If $X_{[\alpha]}$ is defined for some ordinal number α, define $X_{[\alpha+1]} = X \setminus \{$isolated points in $X\}$; if β is a limit ordinal, define $X_{[\beta]} = \cap_{\alpha<\beta} X_{[\alpha]}$. We set $\lambda(X) = \alpha$, where α is the least ordinal such that $X_{[\alpha]}$ is discrete.

The following is a generalization of [19, Theorem 6.3] (see [20, Theorem 3] also).

THEOREM 3.2. *Let A be a C^*-algebra with a scaling approximate identity and B a unital C^*-algebra. Then $QM(B\otimes A) = LM(B\otimes A)+RM(B\otimes A)$ implies that B is scattered and $\lambda(B) < \infty$.*

PROOF. It follows from 2.4 that there is a skeleton C^*-subalgebra A_0 of A such that there is a *-homomorphism from A_0 onto $\mathcal{K}$. Thus $B \otimes A_0$ is a skeleton C^*-subalgebra of $B \otimes A$ and there is a *-homomorphism φ such that $\varphi(B \otimes A_0) = B \otimes \mathcal{K}$. By [19, 3.1], if $QM(B\otimes A) = LM(B\otimes A)+RM(B\otimes A)$, then $QM(B\otimes A_0) = LM(B\otimes A_0)+RM(B\otimes A_0)$. It follows from [19, 4.13] that if $QM(B \otimes K) = LM(B \otimes K) + RM(B \otimes K)$, then, by [19, 6.3] (note that the "only if" part of [19, 6.3] works for σ-unital C^*-algebras), B is scattered and $\lambda(B) < \infty$. ∎

THEOREM 3.3. *Let A be a σ-unital simple C^*-algebra. Then $QM(A) = LM(A) + RM(A)$ if and only if A is elementary or A is unital.*

PROOF. Only the "only if" part needs a proof. Assume that A is non-unital and non-elementary. Take a non-elementary stable matroid C^*-algebra $\mathcal{M}$. By Theorem 2.6, there is a skeleton C^*-subalgebra B of A such that B has a quotient which is isomorphic to $\mathcal{M}$. If $QM(A) = LM(A) + RM(A)$, then, by [10, 3.1], $QM(B) = LM(B) + RM(B)$. Therefore, by [19, 4.3], $QM(M) = LM(M) + RM(M)$. This contradicts Theorem 6.3 in [19], since $\mathcal{M}$ is a stable matroid C^*-algebra. ∎

THEOREM 3.4. *Let A be a σ-unital C^*-algebra and B a σ-unital, non-unital and non-elementary simple C^*-algebra. Then*

$$QM(A \otimes B) \neq LM(A \otimes B) + RM(A \otimes B).$$

PROOF. Suppose that $\mathcal{M}$ is a non-elementary matroid C^*-algebra. It follows from Theorem 2.6 that there is a skeleton C^*-subalgebra B_0 of B such that B_0 has a quotient which is isomorphic to $\mathcal{M} \otimes \mathcal{K}$. Therefore $A \otimes B$ has a skeleton C^*-algebra $A \otimes B$ with a quotient isomorphic to $A \otimes \mathcal{M} \otimes K$. The conclusion then follows from the proof of 3.3. ∎

3.2. L. G. Brown in [4] showed the connection between the problem of whether $QM(A) = LM(A) + RM(A)$ and the problem of perturbations of C^*-algebras. Perturbations of C^*-algebras have been studied in several different ways (see [6], [8], [9], [17] and [18]). One of them is to ask whether an almost isometric ($\|\varphi\| - 1$ and $\|\varphi\| - 1$ are small) complete order automorphism φ of a C^*-algebra is close to an isometry.

THEOREM 3.6. *If A is a σ-unital, non-elementary simple C^*-algebra without identity, then there exists a sequence $\{\varphi_n\}$ of complete order automorphisms of A such that*

$$\lim_{n\to\infty} \|\varphi_n\| = 1,$$
$$\lim_{n\to\infty} \|\varphi_n^{-1}\| = 1,$$

but

$$\inf\{\|\theta - \varphi_n\| : n = 1, 2, \ldots, \theta \text{ automorphisms of } A\} > 0.$$

PROOF. By Theorem 3.3, $QM(A) \neq LM(A) + RM(A)$, and so Theorem 7 in [20] applies. ∎

ADDED IN PROOF. N. C. Phillips pointed out to us that there are examples of separable C^*-algebras with real rank zero, stable rank one, unperforated K_0-groups and trivial K_1-flows but not nuclear and so not AF. Thanks to his remark, we now add the condition of nuclearity to the original question in Remark 2.10.

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