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BICOMMUTATORS OF COFAITHFUL, FULLY DIVISIBLE MODULES

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We define below a notion for modules which is dual to that of faithful, and a notion of "fully divisible" which generalizes that of injectivity. We show that the bicommutator of a cofaithful, fully divisible left R-module is isomorphic to a subring of $Q_{\max}(R)$, the complete ring of left quotients of R.

In recent papers, Goldman [**2**] and Lambek [**3**] investigated rings of left quotients of a ring R constructed with respect to torsion radicals. It is known that every ring of left quotients of R is isomorphic to the bicommutator of an appropriate injective left R-module. We investigate below subrings of rings of quotients which are determined by radicals rather than torsion radicals, and show that any such ring can be constructed as the bicommutator of a fully divisible left R-module.

In particular, if σ is a torsion radical (an idempotent kernel functor in the terminology of [**2**]) and K is the kernel of the homomorphism $R \to Q_\sigma(R)$, then a radical ρ such that $\rho \leqq \sigma$ and $\mathrm{rad}_\rho(R) = K$ determines a subring $Q_\rho(R)$ of $Q_\sigma(R)$, the ρ-closure of R/K in $E(R/K)$. (Here we use $E(M)$ to denote the R-injective envelope of an R-module M.) Furthermore, if S is any such subring of a ring of left quotients of R, then S is isomorphic to the bicommutator of $E(S) \oplus E(S)/S$. We give various conditions under which a subring of a ring of left quotients is of this form.

1. Cofaithful modules; fully divisible modules.

All rings under consideration will be assumed to be associative rings with identity element, and all modules will be assumed to be unital. A direct sum of modules $\{{}_RM_\alpha\}_{\alpha\in A}$ will be denoted M^A if each module M_α is isomorphic to a fixed module ${}_RM$. We introduce the following notation for convenience. (All homomorphisms are R-homomorphisms unless stated otherwise.)

1.1. *Definition.* If ${}_RM$ and ${}_RN$ are left R-modules, and for some index set A there exists a homomorphism from M^A onto N, we will write $M > N$. If $M > N$ and $N > M$, we will write $M \sim N$.

It can easily be seen that for modules ${}_RM$ and ${}_RN$ the following are equivalent:

(i) $M > N$;

(ii) for each $x \in N$ there exist elements $m_i \in M$ and $f_i \in \mathrm{Hom}_R(M, N)$, $i = 1, 2, \ldots, k$, such that $x = \sum_{i=1}^k f_i(m_i)$;

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(iii) for each non-zero homomorphism g: ${}_RN \to {}_RX$ there exists a homomorphism f: ${}_RM \to {}_RN$ such that $gf \neq 0$.

From condition (iii) it is evident that ${}_RM > {}_RN$ and ${}_RN > {}_RP$ imply ${}_RM > {}_RP$, and so the relation $\sim$ is transitive as well as reflexive and symmetric. We next investigate modules ${}_RM$ such that ${}_RM \sim {}_RI$, where ${}_RI$ is used to denote the injective envelope of the module ${}_RR$. This notation will remain fixed throughout the paper.

1.2. Proposition. *${}_RM > {}_RI$ if and only if for some positive integer n, M^n contains an R-submodule isomorphic to ${}_RR$.*

Proof. First, suppose that $M > I$. Then for the identity $1 \in R \subseteq I$ we must have $1 = \sum_{i=1}^n f_i(m_i)$ for $m_i \in M$, $f_i \in \mathrm{Hom}_R(M, I)$. If $r \in R$ and $rm_i = 0$ for all i, then

$$r = r1 = r\left(\sum_{i=1}^n f_i(m_i)\right) = \sum_{i=1}^n f_i(rm_i) = 0.$$

This shows that the homomorphism f: $R \to M^n$ defined by

$$f(r) = (rm_1, rm_2, \ldots, rm_n)$$

is a monomorphism.

Conversely, suppose that there exists a monomorphism f: $R \to M^n$ for some positive integer n. To show that $M > I$, let g: ${}_RI \to {}_RX$ be any non-zero homomorphism. Since a homomorphism h: $R \to I$ with $gh \neq 0$ can be found, and this can be extended to k: $M^n \to I$ by the injectivity of I, we must have $gk_i \neq 0$ for some component k_i: $M \to I$ of k. Thus $M > I$.

The above proof shows that $M > I$ if and only if there exist

$$\{m_1, m_2, \ldots, m_n\} \subseteq M$$

with $\mathrm{Ann}(\{m_1, \ldots, m_n\}) = 0$, and so M is faithful if $M > I$. Since $I > Q$ for all injective modules ${}_RQ$, $M > I$ if and only if $M > Q$ for all injective modules Q. Thus $M > I$ if and only if for each homomorphism $0 \neq g$: ${}_RQ \to {}_RX$, with Q injective, there exists a homomorphism f: $M \to Q$ with $gf \neq 0$. The dual of this statement characterizes faithful modules (see [**1**] for the particulars) and so this motivates the definition below.

1.3. *Definition.* The module ${}_RM$ is called cofaithful if $M > I$.

A module ${}_RM$ is called divisible if $dM = M$ for all non-zero-divisors $d \in R$. It is well known that all injective modules are divisible, and that sums and quotients of divisible modules are divisible. In particular, ${}_RI$ is divisible, and hence if ${}_RI > {}_RM$, then M is divisible. The converse is not necessarily true. Proposition 1.5 below can be used to show that if ${}_RM$ is an essential extension of ${}_RR$ and $I > M$, then M is isomorphic to I. But if R has a classical ring of left quotients Q_{cl}, then ${}_RQ_{\mathrm{cl}}$ is divisible and essential over ${}_RR$, but not necessarily isomorphic to ${}_RI$.

1.4. *Definition.* The module ${}_RM$ will be called fully divisible if $I > M$.

1.5. PROPOSITION. *${}_RM$ is fully divisible if and only if for any monomorphism i: ${}_RP \rightarrow {}_RN$, with P finitely generated and projective, and any homomorphism f: $P \rightarrow M$, there exists an extension g: $N \rightarrow M$ with $f = gi$.*

Proof. First assume that ${}_RM$ is fully divisible. For some index set A, there exists an epimorphism p: $I^A \rightarrow M$, and if ${}_RP$ is finitely generated and projective, i: $P \rightarrow N$ is a monomorphism, and f: $P \rightarrow M$, then since P is projective, f can be lifted to f': $P \rightarrow I^A$, with $pf' = f$. Since P is finitely generated, $f'(P)$ is contained in the direct sum of finitely many copies of I, which is then injective; thus f' can be extended to g': $N \rightarrow I^A$, with $g'i = f'$, since i is a monomorphism. Thus $f = pf' = pg'i$, and $g = pg'$ yields the required extension of f.

Conversely, if M satisfies the given condition, then for each element $m \in M$, there exists a homomorphism f: $R \rightarrow M$ with $f(1) = m$, and this can be extended to g: $I \rightarrow M$ with $g(1) = m$. This shows that $I > M$.

From Definition 1.4 and the fact that ${}_RM > {}_RN$, ${}_RN > {}_RP$ imply ${}_RM > {}_RP$, it is immediate that if ${}_RM$ is fully divisible, then so is any homomorphic image of M. Furthermore, a direct sum of modules is fully divisible if and only if each summand is fully divisible. This implies that if submodules $M_\alpha \subseteq M$, $\alpha \in A$, are fully divisible, then $\sum_{\alpha \in A} M_\alpha$ is fully divisible. Proposition 1.5 shows that the notion of fully divisible is a generalization of injectivity. Using a proof similar to the one for injective modules, it is easy to show that a direct product of modules is fully divisible if and only if each factor is fully divisible.

A ring R is called left hereditary if each left ideal of R is projective, and this is true if and only if every homomorphic image of an injective left R-module is injective. It is well known that R is left Noetherian if and only if every direct sum of injective left R-modules is injective. Combining these results shows that every fully divisible left R-module is injective if and only if R is left hereditary and left Noetherian.

In the following proposition we let all endomorphisms of the module ${}_RM$ operate on the left. If $S = \mathrm{End}_R(M)$, then M is a left S-module.

1.6. PROPOSITION. *Let ${}_RM$ be a left R-module and $S = \mathrm{End}_R(M)$.*

(i) *If ${}_RM$ is finitely generated, then ${}_SM$ is cofaithful.*

(ii) *If ${}_RM$ is faithful and fully divisible, then ${}_RM$ is cofaithful if and only if ${}_SM$ is finitely generated.*

Proof. (i) Assume that ${}_RM$ is finitely generated and that $m_1, \ldots, m_n$ are generators for ${}_RM$. If $s \in S$ and $s(m_i) = 0$ for $i = 1, \ldots, n$, then $s(M) = 0$ and $s = 0$. As in the proof of Proposition 1.2, this shows that ${}_SM$ is cofaithful.

(ii) Assume first that ${}_SM$ is finitely generated, say with generators $m_1, \ldots, m_n$. For each $m \in M$,

$$m = \sum_{i=1}^{n} s_i(m_i) \quad \text{for } s_1, \ldots, s_n \in S.$$

If $r \in R$ and $rm_i = 0$ for all i, then

$$rm = r\left(\sum_{i=1}^{n} s_i(m_i)\right) = \sum_{i=1}^{n} s_i(rm_i) = 0,$$

and $\operatorname{Ann}(M) = \operatorname{Ann}(\{m_1, \ldots, m_n\})$. This shows that if $A = \operatorname{Ann}(M)$, then M is a cofaithful (R/A)-module. In particular, if ${}_SM$ is finitely generated and ${}_RM$ is faithful, then ${}_RM$ is cofaithful.

Conversely, if ${}_RM$ is cofaithful and fully divisible, let $m_1, \ldots, m_n$ be the components of $f(1)$ in an embedding $f: R \to M^n$. Given $m \in M$, there exists $g: R \to M$ with $g(1) = m$, and since M is fully divisible, this can be extended to $s: M^n \to M$, with $g = sf$. If s_i, $i = 1, \ldots, n$, are the components of s, then we must have $m = sf(1) = \sum_{i=1}^{n} s_i(m_i)$, and this shows that ${}_SM$ is finitely generated.

1.7. Proposition. *For a module ${}_RM$, the following are equivalent:*
(i) *M contains a faithful, fully divisible submodule;*
(ii) *${}_RI$ can be embedded in a direct product of copies of M.*

Proof. (i) $\Rightarrow$ (ii). If M contains a faithful, fully divisible submodule ${}_RN$, let $\prod_{\alpha \in A} N_\alpha$ be the direct product of A copies of N, where the index set A is N itself. Define an R-homomorphism $f: R \to \prod_{\alpha \in A} N_\alpha$ by $f_x(r) = rx$, where $x \in N = A$. Since N is faithful, this is a monomorphism. By assumption, N is fully divisible, and so $\prod_{\alpha \in A} N_\alpha$ is also fully divisible, and f may be extended to $g: I \to \prod_{\alpha \in A} N_\alpha$. Since I is an essential extension of R, g is also a monomorphism. Then $g: I \to \prod_{\alpha \in A} N_\alpha \subseteq \prod_{\alpha \in A} M_\alpha$ is the required embedding.

(ii) $\Rightarrow$ (i). Suppose that for some index set A there is a monomorphism $f: I \to \prod_{\alpha \in A} M_\alpha$, where $M_\alpha \approx M$ for all $\alpha \in A$. For each component f_α of f, $f_\alpha(I)$ is fully divisible, and if $N = \sum_{\alpha \in A} f_\alpha(I)$, then N is also fully divisible. If $0 \neq r \in R$, then $f(r) \neq 0$, and therefore $rf_\alpha(1) = f_\alpha(r) \neq 0$ for some $\alpha \in A$. This shows that N is faithful.

2. Bicommutators of cofaithful, fully divisible modules as subrings of $Q_{\max}(R)$. We first review some definitions and results from [**2**; **3**; **5**]. Using the terminology of Maranda, a radical of the category of left R-modules is a function ρ which assigns to each module ${}_RM$ a submodule $\operatorname{rad}_\rho(M)$ such that $\operatorname{rad}_\rho(M/\operatorname{rad}_\rho(M)) = 0$ and for any module ${}_RN$, $f(\operatorname{rad}_\rho(M)) \subseteq \operatorname{rad}_\rho(N)$ for all homomorphisms $f \in \operatorname{Hom}_R(M, N)$. If in addition

$$\operatorname{rad}_\rho(M_0) = M_0 \cap \operatorname{rad}_\rho(M)$$

for all submodules M_0 of M, then ρ is called a torsion radical. In the ter-

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minology of Goldman [**2**], a torsion radical is called an idempotent kernel functor. If ρ and σ are radicals and $\mathrm{rad}_\rho(M) \subseteq \mathrm{rad}_\sigma(M)$ for all modules ${}_RM$, then we write $\rho \leqq \sigma$.

With each module ${}_RM$ is associated a radical which we will denote by rad_M, defined by letting $\mathrm{rad}_M(N)$ be the intersection of all kernels of homomorphisms from N to M, for any module ${}_RN$. It is known that σ is a torsion radical if and only if there exists an injective module ${}_RM$ such that $\sigma = \mathrm{rad}_M$. We note that $\mathrm{rad}_M(R) = \mathrm{Ann}(M)$, and that $\mathrm{rad}_M(N) = 0 \Leftrightarrow \mathrm{rad}_M \leqq \mathrm{rad}_N$, for any modules ${}_RM$ and ${}_RN$.

If ρ is a radical and M_0 is a submodule of ${}_RM$, we define the ρ-closure $\hat{M}_0$ of M_0 in M as the inverse image in M of $\mathrm{rad}_\rho(M/M_0)$. Since ρ is a radical, the ρ-closure of $\hat{M}_0$ is just $\hat{M}_0$. In particular, if $\rho = \mathrm{rad}_N$ for some module ${}_RN$, then the N-closure of M_0 in M is

$$\{m \in M: f(m) = 0 \text{ for all } f \in \mathrm{Hom}_R(M, N) \text{ such that } f(M_0) = 0\}.$$

The following properties of this closure operation will be used throughout the remainder of the paper.

Let ρ be a radical and let ${}_RN$ and ${}_RM$ be modules with M_0 a submodule of M. Then:

(i) If $f \in \mathrm{Hom}_R(M, N)$ and $f(M_0) \subseteq N_0$ for some submodule N_0 of N, then for the respective ρ-closures we must have $f(\hat{M}_0) \subseteq \hat{N}_0$.

(ii) If $\mathrm{rad}_\rho(N) = 0$ and $f \in \mathrm{Hom}_R(M, N)$, then $f(M_0) = 0 \Rightarrow f(\hat{M}_0) = 0$.

(iii) If $\mathrm{rad}_\rho(N) = 0$ and $f, g \in \mathrm{Hom}_R(M, N)$, then if f and g agree on M_0 they also agree on $\hat{M}_0$.

We give a proof of (i), and (ii) and (iii) follow immediately from (i). Let $f \in \mathrm{Hom}_R(M, N)$ and $f(M_0) \subseteq N_0$. Then f induces a homomorphism $g: M/M_0 \to N/N_0$, and we must have $g(\mathrm{rad}_\rho(M/M_0)) \subseteq \mathrm{rad}_\rho(N/N_0)$ since ρ is a radical. Now since $\hat{M}_0$ and $\hat{N}_0$ are the inverse images in M and N, respectively, of $\mathrm{rad}_\rho(M/M_0)$ and $\mathrm{rad}_\rho(N/N_0)$, it follows that $f(\hat{M}_0) \subseteq \hat{N}_0$.

If σ is a torsion radical and $K = \mathrm{rad}_\sigma(R)$, let $Q_\sigma(R)$ be the σ-closure of R/K in its R-injective envelope $E(R/K)$. Since σ is a torsion radical and $\mathrm{rad}_\sigma(R/K) = 0$, it can be shown that $\mathrm{rad}_\sigma(E(R/K)) = 0$ and that $E(R/K)$ is an (R/K)-module. If $s \in R/K$, then right multiplication by s defines a homomorphism $f_s: R/K \to R/K$ with $f_s(1) = s$. This suggests a multiplication for $Q_\sigma(R)$. If $q \in Q_\sigma(R)$, let $f_q: R/K \to Q_\sigma(R)$ be the unique R-homomorphism such that $f_q(1) = q$. This can be extended to $h_q: E(R/K) \to E(R/K)$ by the injectivity of $E(R/K)$. From the properties of the closure operation associated with σ, $h_q(Q_\sigma(R)) \subseteq Q_\sigma(R)$, and so we may let $\phi_q: Q_\sigma(R) \to Q_\sigma(R)$ be the restriction of h_q. Furthermore, the extension of f_q to $Q_\sigma(R)$ must be unique, and so for $p, q \in Q_\sigma(R)$ define $p \cdot q = \phi_q(p)$. This gives $Q_\sigma(R)$ a ring structure which extends the action of R/K, and $Q_\sigma(R)$ is called the ring of left quotients with respect to σ. In the particular case when $\sigma = \mathrm{rad}_I$, $I = E(R)$, we obtain the complete ring of left quotients $Q_{\max}(R)$ of R.

This construction can be extended to certain radicals. We first prove a lemma describing subrings of $Q_{\max}(R)$.

2.1. LEMMA. *Let ${}_RS$ be an R-submodule of $Q_{\max}(R)$ such that $R \subseteq S \subseteq Q_{\max}(R)$. Then S is a subring of $Q_{\max}(R)$ if and only if for each $f \in \mathrm{End}_R(I)$, $f(R) \subseteq S$ implies $f(S) \subseteq S$.*

Proof. Suppose that ${}_RS$ is a submodule of $Q_{\max}(R)$ which satisfies the given conditions. To show that S is a subring of $Q_{\max}(R)$ it is only necessary to show that S is closed under the induced multiplication. Let $p, q \in S$. Then $p \cdot q = \phi_q(p)$, where $\phi_q(1) = q$ and ϕ_q: $Q_{\max}(R) \to Q_{\max}(R)$, and ϕ_q can be extended to an endomorphism h_q of I. Then $h_q(1) \in S$ implies by assumption that $h_q(S) \subseteq S$ and hence $p \cdot q = h_q(p) \in S$.

Conversely, suppose that S is a subring of $Q_{\max}(R)$ which contains R. It is well known that I is a $Q_{\max}(R)$-module and that every R-endomorphism of I is a $Q_{\max}(R)$-endomorphism. Thus if $f \in \mathrm{End}_R(I)$ and $f(R) \subseteq S$, it follows that $f(s) = sf(1) \in S$ for all $s \in S$.

2.2. PROPOSITION. *Let σ be a torsion radical. Then any radical ρ such that $\rho \leqq \sigma$ and* $\mathrm{rad}_\rho(R) = \mathrm{rad}_\sigma(R)$ *defines a subring $Q_\rho(R)$ of $Q_\sigma(R)$.*

Proof. Let $K = \mathrm{rad}_\rho(R) = \mathrm{rad}_\sigma(R)$, where ρ and σ satisfy the given conditions, and let $Q_\rho(R)$ be the ρ-closure of R/K in $E(R/K)$. Since $\rho \leqq \sigma$, we have $R/K \subseteq Q_\rho(R) \subseteq Q_\sigma(R)$. The R-module $E(R/K)$ is an (R/K)-module and as such $\mathrm{rad}_{E(R/K)}$ determines $Q_{\max}(R/K)$. It then follows from the fact that $\mathrm{rad}_\sigma(E(R/K)) = 0$ that $Q_\sigma(R) \subseteq Q_{\max}(R)$; thus actually we have $R/K \subseteq Q_\rho(R) \subseteq Q_{\max}(R/K)$. Every (R/K)-endomorphism of $E(R/K)$ is an R-endomorphism, and the fact that $Q_\rho(R)$ is a subring of $Q_\sigma(R)$ follows from Lemma 2.1 and the properties of the ρ-closure of a submodule.

2.3. THEOREM. *If ${}_RM$ contains a faithful, fully divisible submodule, then* rad_M *defines a subring $Q_M(R)$ of $Q_{\max}(R)$.*

Proof. If ${}_RM$ contains a faithful, fully divisible submodule, then by Proposition 1.7, ${}_RI$ can be embedded in a direct product of copies of M. Thus the intersection of kernels of R-homomorphisms from I to M is zero, and $\mathrm{rad}_M(I) = 0$. This shows both that $\mathrm{rad}_M \leqq \mathrm{rad}_I$ and that $\mathrm{rad}_M(R) = \mathrm{rad}_I(R) = 0$. By Proposition 2.2, the M-closure of R in $E(R)$ is a subring of $Q_{\max}(R)$, which we denote by $Q_M(R)$.

2.4. PROPOSITION. *Let ${}_RM$ be a module containing a faithful, fully divisible submodule, and let ${}_RN$ be fully divisible with* $\mathrm{rad}_M(N) = 0$.

(i) *The R-structure of ${}_RN$ extends uniquely to give N the structure of a $Q_M(R)$-module.*

(ii) *Any R-submodule N_0 of N such that* $\mathrm{rad}_M(N/N_0) = 0$ *is a $Q_M(R)$-submodule of N.*

(iii) *If ${}_RP$ is also fully divisible with* $\mathrm{rad}_M(P) = 0$, *then any R-homomorphism from N to P is also a $Q_M(R)$-homomorphism.*

Proof. (i) If N is fully divisible, then for each $n \in N$ the R-homomorphism $f_n: R \to N$ defined by $f_n(r) = rn$, for all $r \in R$, can be extended to $\phi_n: Q_M(R) \to N$. The extension is unique because $\mathrm{rad}_M(N) = 0$ and $Q_M(R)$ is the M-closure of R in I. For $q \in Q_M(R)$ define $q \cdot n = \phi_n(q)$. This can easily be shown to give N a $Q_M(R)$-module structure.

(ii) If N_0 is a submodule of N with $\mathrm{rad}_M(N/N_0) = 0$, then N_0 is its own M-closure, and so for $n \in N_0$, $\phi_n(R) \subseteq N_0$ and therefore $\phi_n(Q_M(R)) \subseteq N_0$. Thus for all $q \in Q_M(R)$, $q \cdot n \in N_0$.

(iii) Suppose that ${}_RP$ is fully divisible and that $\mathrm{rad}_M(P) = 0$. If $f \in \mathrm{Hom}_R(N, P)$, we must show that $f(qn) = qf(n)$ for all $q \in Q_M(R)$, $n \in N$. Using the homomorphisms which define multiplication by elements of $Q_M(R)$, this reduces to showing that $f(\phi_n(q)) = \phi_{f(n)}(q)$, for all q and n. But $f(\phi_n(1)) = f(n) = \phi_{f(n)}(1)$; thus since these R-homomorphisms agree on R and $\mathrm{rad}_M(P) = 0$, it follows that they must agree on $Q_M(R)$. This completes the proof.

For any module ${}_RM$ we let $\mathrm{Bic}_R(M)$ denote the bicommutator of the image of R in $\mathrm{End}_Z(M)$ under the representation of R defined by the action of R on M. The commutator of the image of R is just all R-endomorphisms of M, and so $\mathrm{Bic}_R(M)$ consists of all Z-endomorphisms of M which commute with all R-endomorphisms of M.

If M_1 and M_2 are left R-modules, we may describe additive functions from $M_1 \oplus M_2$ into $M_1 \oplus M_2$ by using matrices of the form

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{11}: M_1 \to M_1$, $f_{12}: M_2 \to M_1$, $f_{21}: M_1 \to M_2$, and $f_{22}: M_2 \to M_2$ are all Z-homomorphisms and operate on the left. The commutator of the image of R in the Z-endomorphism ring of $M_1 \oplus M_2$ consists of all matrices whose entries are R-homomorphisms. The bicommutator consists of matrices of the form

$$\begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix},$$

where $q_{11} \in \mathrm{Bic}_R(M_1)$, $q_{22} \in \mathrm{Bic}_R(M_2)$, and moreover $q_{22}f_{21} = f_{21}q_{11}$ and $q_{11}f_{12} = f_{12}q_{22}$ for all $f_{21} \in \mathrm{Hom}_R(M_1, M_2)$ and $f_{12} \in \mathrm{Hom}_R(M_2, M_1)$.

2.5. LEMMA. *If* ${}_RM_1 > {}_RM_2$, *then the canonical ring homomorphism from* $\mathrm{Bic}_R(M_1 \oplus M_2)$ *into* $\mathrm{Bic}_R(M_1)$ *is a monomorphism.*

Proof. Let $\pi: \mathrm{Bic}_R(M_1 \oplus M_2) \to \mathrm{Bic}_R(M_1)$ be defined by setting

$$\pi\left(\begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix}\right) = q_{11}.$$

If $q_{11} = 0$, then for all $f_{21} \in \mathrm{Hom}_R(M_1, M_2)$ it follows that $q_{22}f_{21} = f_{21}q_{11} = 0$. Thus a condition sufficient to guarantee that π is one-to-one is that for each

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$0 \neq q_{22} \in \mathrm{Bic}_R(M_2)$ there exists $f_{21} \in \mathrm{Hom}_R(M_1, M_2)$ such that $q_{22}f_{21} \neq 0$. This condition is satisfied if $M_1 > M_2$. In fact, if $0 \neq q \in \mathrm{End}_Z(M_2)$, let $m \in M$ with $q(m) \neq 0$. If $M_1 > M_2$, then there exist elements $m_1, \ldots, m_n \in M_1$ and $f_1, \ldots, f_n \in \mathrm{Hom}_R(M_1, M_2)$ such that $m = \sum_{i=1}^{n} f_i(m_i)$. Since q is a Z-endomorphism, $q(m) \neq 0$ implies $qf_i(m_i) \neq 0$ for some index i.

2.6. Theorem. *If ${}_RM$ is cofaithful and fully divisible, that is, if ${}_RM \sim {}_RI$, then* $\mathrm{Bic}_R(M)$ *is isomorphic to* $Q_M(R)$.

Proof. Let ${}_RM$ be cofaithful and fully divisible. By Proposition 1.2, there is an integer n and a monomorphism $f\colon R \to M^n$. By Proposition 1.5, f can be extended to $g\colon I \to M^n$, since M, and therefore M^n, is fully divisible. Since f is a monomorphism and I is an essential extension of R, g must also be a monomorphism, and so $g(I)$ is injective and therefore a direct summand of M^n. Thus $M^n \approx I \oplus N$, where N is also fully divisible.

Let $\theta\colon \mathrm{Bic}_R(I) \to Q_{\max}(R)$ be defined by $\theta(q) = q(1)$, for all $q \in \mathrm{Bic}_R(I)$. [**4**, p. 94, Proposition 1] shows that θ is a ring isomorphism. We use this to define a ring homomorphism $\Phi\colon \mathrm{Bic}_R(M) \to Q_{\max}(R)$ as the composition of the obvious ring homomorphisms $\eta\colon \mathrm{Bic}_R(M) \to \mathrm{Bic}_R(M^n) \to \mathrm{Bic}_R(I \oplus N)$, $\pi\colon \mathrm{Bic}_R(I \oplus N) \to \mathrm{Bic}_R(I)$, and θ. This leads to the following diagram:

$$
\begin{tikzcd}
\mathrm{Bic}_R(M) \arrow[rr, "\Phi"] \arrow[d] \arrow[dr, "\eta"] & & Q_{\max}(R) \\
\mathrm{Bic}_R(M^n) \arrow[r] & \mathrm{Bic}_R(I \oplus N) \arrow[r, "\pi"] & \mathrm{Bic}_R(I) \arrow[u, "\theta"']
\end{tikzcd}
$$

Both η and θ are isomorphisms. Since N is fully divisible, $I > N$ and Lemma 2.5 implies that π is a monomorphism. We will now show that the image of Φ in $Q_{\max}(R)$ is precisely $Q_M(R)$.

If $q \in \mathrm{Bic}_R(M)$, $\Phi(q) = \pi\eta q(1) = \eta q(1)$. Given an R-homomorphism $f\colon I \to M$ with $f(R) = 0$, we can define an R-endomorphism f^n of $I \oplus N$ by defining $f^n\colon I \oplus N \to M^n$ as follows: $f^n(x, y) = (f(x), \ldots, f(x))$ for $(x, y) \in I \oplus N$. The endomorphism ηq must commute with f^n, so that $f^n(\eta q(1)) = \eta q f^n(1) = 0$, and so $f(\eta q(1)) = 0$. This shows that $\eta q(1) \in Q_M(R)$, since $Q_M(R)$ can be characterized as

$$\{q \in I\colon f(q) = 0 \text{ for all } f \in \mathrm{Hom}_R(I, M) \text{ such that } f(R) = 0\}.$$

On the other hand, for $x \in Q_M(R)$, left multiplication by x defines a Z-endomorphism of M, since, by Proposition 2.4, M is a left $Q_M(R)$-module. That is to say, if we define $q(m) = xm$ for all $m \in M$, then $q \in \mathrm{Bic}_R(M)$. (This follows from Proposition 2.4 (iii).) Furthermore, $\Phi(q) = \eta q(1) = x \cdot 1 = x$, since each of the modules M, M^n, $I \oplus N$, and I are $Q_M(R)$-modules, and the R-homomorphisms $M \to M^n \to I \oplus N \to I$ which are used to define Φ are all $Q_M(R)$-homomorphisms.

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This shows that $\Phi(\mathrm{Bic}_R(M)) = Q_M(R) \subseteq Q_{\max}(R)$, and $\mathrm{Bic}_R(M)$ is isomorphic to a subring of $Q_{\max}(R)$. This mapping extends the identity on R, and so by [**4**, p. 99, Proposition 8], Φ is the only ring homomorphism with this property.

3. Subrings of rings of quotients determined by radicals. Proposition 2.2 shows that certain radicals determine subrings of rings of quotients. An example will be cited later which shows that not all such subrings are themselves rings of quotients of R. We begin by characterizing the subrings of $Q_{\max}(R)$ which are of this form. For any cofaithful and fully divisible module ${}_RM$ we identify $\mathrm{Bic}_R(M)$ and $Q_M(R)$.

3.1. PROPOSITION. *Let S be a subring of $Q_{\max}(R)$ such that $R \subseteq S \subseteq Q_{\max}(R)$. Then the following conditions are equivalent:*

(i) *$S = Q_\rho(R)$ for a radical ρ such that $\rho \leqq \mathrm{rad}_I$;*
(ii) *$\mathrm{Hom}_S(I, I/S) = \mathrm{Hom}_R(I, I/S)$;*
(iii) *For all $f \in \mathrm{Hom}_R(I, I/S)$, $f(R) = 0 \Rightarrow f(S) = 0$;*
(iv) *$S = \mathrm{Bic}_R(I \oplus I/S)$;*
(v) *$S = \mathrm{Bic}_R(M)$ for a cofaithful, fully divisible module ${}_RM$.*

Proof. (i) $\Rightarrow$ (ii). If $S = Q_\rho(R)$ for a radical ρ, then $\mathrm{rad}_\rho(I) = 0$ and $\mathrm{rad}_\rho(I/S) = 0$, by assumption. In a manner similar to that used in the proof of Proposition 2.4 (iii), it can be shown that every R-homomorphism from I to I/S is in fact an S-homomorphism.

(ii) $\Rightarrow$ (iii). Let $f \in \mathrm{Hom}_R(I, I/S)$ with $f(R) = 0$. If f is an S-homomorphism, then for all $s \in S$, $f(s) = sf(1) = 0$.

(iii) $\Rightarrow$ (iv). The R-module $I \oplus I/S$ is cofaithful since it contains a submodule isomorphic to ${}_RR$ and fully divisible since ${}_RI$ is injective and I/S is a quotient of an injective R-module. We have identified $\mathrm{Bic}_R(I \oplus I/S)$ with $\{q \in I: f(q) = 0$ for all $f \in \mathrm{Hom}_R(I, I \oplus I/S)$ such that $f(R) = 0\}$. We have assumed that $S \subseteq Q_{\max}(R)$, and so part of this condition is redundant. In fact, $\mathrm{Bic}_R(I \oplus I/S)$ can be identified with $\{q \in I: f(q) = 0$ for all $f \in \mathrm{Hom}_R(I, I/S)$ such that $f(R) = 0\}$. If $q \in I$ and $q \notin S$, the projection $p: I \to I/S$ yields an R-homomorphism such that $p(R) = 0$ but $p(q) \neq 0$. This shows that $\mathrm{Bic}_R(I \oplus I/S) \subseteq S$ for all subrings S of $Q_{\max}(R)$ which contain R. The assumption that (iii) holds is precisely what is needed to guarantee equality.

The implications (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) are immediate.

Using condition (i) of Proposition 3.1, it is not difficult to show that the intersection of the subrings which satisfy the conditions of Proposition 3.1 satisfies the conditions of Proposition 3.1. If $q \in I$ we let Rq^{-1} denote $\{r \in R: rq \in R\}$.

3.2. PROPOSITION. *Let S be a subring of $Q_{\max}(R)$ which contains R. If $S(Rs^{-1}) = S$ for all $s \in S$, then S satisfies the conditions of Proposition* 3.1.

Proof. We will verify that condition (iii) of Proposition 3.1 is satisfied. If $f \in \mathrm{Hom}_R(I, I/S)$ and $f(R) = 0$, let $s \in S$. By assumption, $S(Rs^{-1}) = S$, and so

$$1 = \sum_{i=1}^{n} s_i r_i \quad \text{for } s_i \in S \text{ and } r_i \in Rs^{-1}.$$

Now I/S is a left S-module, and so we must have

$$1 \cdot f(s) = \left(\sum_{i=1}^{n} s_i r_i\right) f(s) = \sum_{i=1}^{n} s_i f(r_i s) = 0,$$

because f is an R-homomorphism, $r_i s \in R$ for all i, and $f(R) = 0$. Thus $f(S) = 0$, and condition (iii) of Proposition 3.1 is satisfied.

3.3. COROLLARY. *Let S be a subring of $Q_{\max}(R)$ which contains R. If each element $s \in S$ can be expressed in the form $s = b^{-1}a$, where $a, b \in R$ and $b^{-1} \in S$, then S satisfies the conditions of Proposition* 3.1.

Proof. Let $s \in S$. Then $s = b^{-1}a$, where $a, b \in R$ and $b^{-1} \in S$. Therefore $bs = b(b^{-1}a) = a \in R$, and so $b \in Rs^{-1}$. Since $b^{-1} \in S$, this shows that $1 = b^{-1}b \in S(Rs^{-1})$, and the conclusion follows from Proposition 3.2.

If K is an ideal of R such that $K = \mathrm{rad}_\sigma(R)$ for some torsion radical σ, Lambek [**3**] calls K a torsion ideal. We have already observed that in this case $Q_\sigma(R)$ is a subring of $Q_{\max}(R/K)$. We next generalize Proposition 3.1 to the case of subrings lying between R/K and $Q_{\max}(R/K)$ for torsion ideals K of R. Let ${}_R\mathfrak{M}$ be the category of left R-modules.

3.4. THEOREM. *Let K be a torsion ideal and let S be a subring of $Q_{\max}(R/K)$ such that $R/K \subseteq S \subseteq Q_{\max}(R/K)$. The following conditions are equivalent:*

(i) *$S = Q_\rho(R)$ for a radical ρ of ${}_R\mathfrak{M}$ such that $\mathrm{rad}_\rho(R) = K$ and $\rho \leqq \mathrm{rad}_{E(R/K)}$;*

(ii) *$\mathrm{Hom}_S(E(R/K), E(R/K)/S) = \mathrm{Hom}_R(E(R/K), E(R/K)/S)$;*

(iii) *For all $f \in \mathrm{Hom}_R(E(R/K), E(R/K)/S)$, $f(R/K) = 0 \Rightarrow f(S) = 0$;*

(iv) *$S = \mathrm{Bic}_R(E(S) \oplus E(S)/S)$;*

(v) *$S = \mathrm{Bic}_R(M)$ for a cofaithful and fully divisible (R/K)-module M.*

Proof. (i) $\Rightarrow$ (iii). This is immediate from the construction of $Q_\rho(R)$.

Conditions (ii) and (iii) can easily be checked to be equivalent.

(iii) $\Rightarrow$ (iv). Both $E(R/K)$ and $E(R/K)/S$ are (R/K)-modules. As such,

$$\mathrm{Hom}_{R/K}(E(R/K), E(R/K)/S) = \mathrm{Hom}_R(E(R/K), E(R/K)/S).$$

Thus S satisfies the conditions of Proposition 3.1 as an extension of R/K. The result follows by observing that $E(S)$, the R-injective envelope of S, is just $E(R/K)$, and that $\mathrm{Bic}_{R/K}(E(S) \oplus E(S)/S) = \mathrm{Bic}_R(E(S) \oplus E(S)/S)$.

(iv) $\Rightarrow$ (v). $E(S)$ is injective as an (R/K)-module, and so it follows that $E(S) \oplus E(S)/S$ is fully divisible as an (R/K)-module. It is immediate that $E(S) \oplus E(S)/S$ is cofaithful as an (R/K)-module.

(v) $\Rightarrow$ (i). If $S = \mathrm{Bic}_R(M)$, where M is cofaithful and fully divisible as an (R/K)-module, then $S = Q_M(R/K)$. If we consider the R-module M, then $\mathrm{rad}_M(R) = \mathrm{Ann}(M) = K$ and $\mathrm{rad}_M \leqq \mathrm{rad}_{E(R/K)}$ since $\mathrm{rad}_M(E(R/K)) = 0$. Since $\mathrm{Hom}_R(E(R/K), M) = \mathrm{Hom}_{R/K}(E(R/K), M)$, it is clear that $S = Q_M(R/K) = Q_M(R)$.

This shows that if K is a torsion ideal of R, then R/K itself satisfies the conditions of Theorem 3.4. Lambek [**3**] gives an example due to Hans Storrer [**3**, Example 7, § 2] which shows that R/K may not be a ring of left quotients even if K is a torsion ideal. On the other hand, of course, any ring of left quotients satisfies the conditions of Theorem 3.4.

If K is a torsion ideal, then $E(R/K)$ is the (R/K)-injective envelope of R/K. Thus any (R/K)-fully divisible module is also fully divisible as an R-module. Applying Theorem 3.4 (v) and Proposition 1.6 (ii), we see that a ring satisfying the conditions of Theorem 3.4 is isomorphic to the bicommutator of a fully divisible R-module M which is finitely generated over the ring $\mathrm{End}_R(M)$. This generalizes the known fact that any ring of left quotients of R is isomorphic to the bicommutator of an injective R-module M which is finitely generated over its ring of endomorphisms $\mathrm{End}_R(M)$. (See [**3**, Proposition 2.8].)

3.5. COROLLARY. *If R is left hereditary, then every ring satisfying the conditions of Theorem* 3.4 *is a ring of left quotients of R.*

Proof. Let K be a torsion ideal of R and $R/K \subseteq S \subseteq Q_{\max}(R/K)$. If R is left hereditary, then $E(S) \oplus E(S)/S$ is R-injective, since every homomorphic image of an R-injective module is R-injective. Thus if S satisfies the conditions of Theorem 3.4, $S = Q_M(R)$ for the R-injective module $M = E(S) \oplus E(S)/S$. This shows that S is a ring of left quotients of R.

3.6. COROLLARY. *Let K be a torsion ideal of R and let S be a subring of $Q_{\max}(R/K)$ such that $R/K \subseteq S \subseteq Q_{\max}(R/K)$. Either of the following conditions is sufficient to guarantee that S satisfies the conditions of Theorem* 3.4:

(i) *$R \rightarrow S$ is an epimorphism in the category of rings;*

(ii) *S is contained in an R-projective submodule of $E(R/K)$.*

Proof. (i) It is well known that if $R \rightarrow S$ is an epimorphism in the category of rings, then every R-homomorphism between S-modules is in fact an S-homomorphism. By assumption, both $E(R/K)$ and $E(R/K)/S$ are S-modules, and so S must satisfy condition (ii) of Theorem 3.4.

(ii) Suppose that ${}_RS \subseteq {}_RP \subseteq {}_RE(R/K)$ for an R-projective module ${}_RP$. For convenience we let $E(R/K) = E$. We will show that S satisfies condition (iii) of Theorem 3.4. Let $f \in \mathrm{Hom}_R(E, E/S)$ such that $f(R/K) = 0$. If p: $E \rightarrow E/S$ is the projection, then since P is projective, the restriction of f to P can be lifted to g: $P \rightarrow E$ with $pg = fi$, where i: $P \rightarrow E$ is the inclusion. Since E is injective, g can be extended to h: $E \rightarrow E$ with $g = hi$. (All homomorphisms are R-homomorphisms.) Now $fi = phi$, and so f and ph agree on P, and consequently $ph(R/K) = f(R/K) = 0$. This implies that $h(R/K) \subseteq S$, and since h is an (R/K)-endomorphism of E, Lemma 2.1 implies that $h(S) \subseteq S$. But then $f(S) = ph(S) = 0$, and this establishes condition (iii) of Theorem 3.4.

We note that Corollaries 3.5 and 3.6 can be combined to show that if R is left hereditary and $Q_\sigma(R)$ is a ring of left quotients of R with $Q_\sigma(R)$ R-projective, then any subring S of $Q_\sigma(R)$ such that $R/\mathrm{rad}_\sigma(R) \subseteq S$ is a ring of left quotients of R.

Added in proof. Hans Storrer has pointed out to me that a ring S which satisfies the conditions of Proposition 3.2 in fact satisfies condition (i) of Corollary 3.6, so that Proposition 3.2 follows from Corollary 3.6.

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