

A NEW EXAMPLE OF A MINIMAL NONFINITELY BASED SEMIGROUP

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Abstract

Only three of the 15 973 distinct six-element semigroups are presently known to be nonfinitely based. This paper introduces a fourth example.

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1. Introduction

A semigroup is *finitely based* if its identities are finitely axiomatizable. In the 1960s, Perkins proved that the six-element Brandt monoid

$$B_2^1 = \langle a, b, 1 \mid a^2 = b^2 = 0, aba = a, bab = b \rangle$$

is nonfinitely based as a semigroup [8]. In the decades that followed, two more similar semigroups were discovered: the six-element monoid

$$A_2^1 = \langle a, b, 1 \mid a^2 = aba = a, bab = b, b^2 = 0 \rangle$$

was independently shown by Sapir [10] and Trahtman [12] to be nonfinitely based, and the nonfinite basis property of the six-element semigroup

$$A_2^g = \left\langle a, b, g, z \mid \begin{array}{l} a^2 = aba = a, \quad ag = ga = bg = gb = zg = gz = g, \\ bab = b, \quad az = za = bz = zb = b^2 = g^2 = z^2 = z \end{array} \right\rangle$$

follows from results of either Mashevitskiĭ [7] or Volkov [14] (an explicit basis for A_2^g was recently given by Lee and Volkov [6]). Since all semigroups with five or fewer elements are finitely based [3, 13], the semigroups A_2^1 , A_2^g , and B_2^1 are minimal with respect to being nonfinitely based. Apart from these three examples, no other minimal nonfinitely based semigroup is known. This prompted Lee and Li [4] to ask the following fundamental question: is there any minimal nonfinitely based semigroup that is distinct from A_2^1 , A_2^g , and B_2^1 ? Recall that two semigroups are *distinct* if they

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are neither isomorphic nor anti-isomorphic. There are 15 973 distinct six-element semigroups [9], among which 1373 are monoids [2] and 14 600 are nonunital.

Lee and Li partially answered the above question by proving that among all distinct six-element monoids, only A_2^1 and B_2^1 are nonfinitely based [4]. After establishing this result, Lee continued investigating the finite basis problem for the remaining 14 600 nonunital six-element semigroups. In a private communication, he suggested that the authors examine the finite basis problem of several six-element semigroups. It turned out that one of these semigroups,

$$L = \langle d, e \mid d^2 = d, e^2 = e, ded = 0 \rangle,$$

is nonfinitely based. The main aim of this paper is to present a proof of this result. Consequently, the number of distinct minimal nonfinitely based semigroups is at least four.

This paper is organized as follows. In Section 3, it is shown that Lee's semigroup L has a basis that consists of identities with some special property. In Section 4, some restrictions on the identities satisfied by L are established. The semigroup L is then shown to be nonfinitely based in Section 5.

Recall that a semigroup S is *inherently nonfinitely based* if any locally finite variety containing S is nonfinitely based. Sapir proved that the semigroups A_2^1 and B_2^1 are inherently nonfinitely based [10]. On the other hand, a semigroup that satisfies a nontrivial identity of the form $xyx \approx \mathbf{w}$ is not inherently nonfinitely based [10]. Since the semigroups A_2^g and L satisfy the identities $xyx \approx xyxyx$ and $xyx \approx xyx^2$ respectively, they are not inherently nonfinitely based.

2. Preliminaries

Let X be a countably infinite alphabet throughout. For any subset \mathcal{Y} of X , let \mathcal{Y}^+ and \mathcal{Y}^* denote the free semigroup and free monoid over \mathcal{Y} respectively. Elements of X are called *letters* and elements of X^* are called *words*. A word is a *singleton* if it is a single letter. For any word \mathbf{w} :

- the *head* of \mathbf{w} , denoted by $h(\mathbf{w})$, is the first letter occurring in \mathbf{w} ;
- the *tail* of \mathbf{w} , denoted by $t(\mathbf{w})$, is the last letter occurring in \mathbf{w} ;
- the *content* of \mathbf{w} , denoted by $\text{con}(\mathbf{w})$, is the set of letters occurring in \mathbf{w} ;
- the number of occurrences of a letter x in \mathbf{w} is denoted by $\text{occ}(x, \mathbf{w})$;
- a letter x is *simple* in \mathbf{w} if $\text{occ}(x, \mathbf{w}) = 1$;
- the set of simple letters of \mathbf{w} is denoted by $\text{sim}(\mathbf{w})$;
- the set of nonsimple letters of \mathbf{w} is denoted by $\text{non}(\mathbf{w})$.

Note that $\text{con}(\mathbf{w}) = \text{sim}(\mathbf{w}) \cup \text{non}(\mathbf{w})$ and $\text{sim}(\mathbf{w}) \cap \text{non}(\mathbf{w}) = \emptyset$. Two words \mathbf{u} and \mathbf{v} are *disjoint* if $\text{con}(\mathbf{u}) \cap \text{con}(\mathbf{v}) = \emptyset$.

An identity is written as $\mathbf{u} \approx \mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in X^+$. An identity $\mathbf{u} \approx \mathbf{v}$ is *nontrivial* if $\mathbf{u} \neq \mathbf{v}$. A semigroup S *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if for any substitution φ from X into S , the elements $\mathbf{u}\varphi$ and $\mathbf{v}\varphi$ of S are equal. A set Σ of identities satisfied by a semigroup S is a *basis* for S if every identity of S can be deduced from Σ . A semigroup is *finitely*

based if it has a finite basis. For any concept of universal algebra that appears but is not defined in this paper, refer to Burris and Sankappanavar [1] and Shevrin and Volkov [11].

By writing $a = ede$, $b = de$, and $c = ed$ in the semigroup L , the multiplication table for L is given by

L	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	a	0	c	a
d	0	0	b	0	d	b
e	0	a	a	c	c	e

LEMMA 1. *Suppose that the semigroup L satisfies an identity $\mathbf{u} \approx \mathbf{v}$. Then $\text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v})$ and $\text{non}(\mathbf{u}) = \text{non}(\mathbf{v})$.*

PROOF. Suppose that $x \in \text{con}(\mathbf{u})$ and $x \notin \text{con}(\mathbf{v})$. Then letting $\varphi : \mathcal{X} \rightarrow L$ be the substitution that maps x to 0 and any other letter to e , we obtain the contradiction $\mathbf{u}\varphi = 0 \neq e = \mathbf{v}\varphi$. Thus $\text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$. Suppose that $x \in \text{sim}(\mathbf{u})$ and $x \notin \text{sim}(\mathbf{v})$. Then letting $\psi : \mathcal{X} \rightarrow L$ be the substitution that maps x to a and any other letter to e , we obtain the contradiction $\mathbf{u}\psi = a \neq 0 = \mathbf{v}\psi$. Therefore $\text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v})$, whence $\text{non}(\mathbf{u}) = \text{non}(\mathbf{v})$. □

3. Connected identities

A semigroup S is *idempotent-separable* if the following holds for any distinct elements x and y of S : $ex \neq ey$ for some idempotent e in S , and $xf \neq yf$ for some idempotent f in S . (The idempotents e and f need not be distinct.) It is easy to verify that the semigroup L is idempotent-separable by referring to its multiplication table in Section 2.

The *variety* generated by a semigroup S is the class of all semigroups that satisfy all identities of S . The four-element semigroup

$$A_0 = \langle a, b \mid a^2 = a, b^2 = b, ab = 0 \rangle$$

is required in the following result.

LEMMA 2 [5, Proposition 3.2]. *Let S be any idempotent-separable semigroup such that the variety generated by S contains the semigroup A_0 . Suppose that S satisfies an identity $\mathbf{u} \approx \mathbf{v}$ where the word \mathbf{u} can be written as $\mathbf{u} = \mathbf{u}'\mathbf{u}''$ for some disjoint words $\mathbf{u}', \mathbf{u}'' \in \mathcal{X}^+$. Then the word \mathbf{v} can be written as $\mathbf{v} = \mathbf{v}'\mathbf{v}''$ for some disjoint words $\mathbf{v}', \mathbf{v}'' \in \mathcal{X}^+$ such that $\text{con}(\mathbf{u}') = \text{con}(\mathbf{v}')$ and $\text{con}(\mathbf{u}'') = \text{con}(\mathbf{v}'')$. Further, S satisfies the identities $\mathbf{u}' \approx \mathbf{v}'$ and $\mathbf{u}'' \approx \mathbf{v}''$.*

A word of length at least two is *connected* if it cannot be written as a product of two disjoint nonempty words. A *connected* identity is an identity that is formed by a pair of connected words.

LEMMA 3. *The semigroup L has a basis that contains only connected identities.*

PROOF. By identifying the elements a and b of L with 0 , we obtain a semigroup isomorphic to A_0 . Hence the semigroup A_0 belongs to the variety generated by L . As observed earlier, the semigroup L is idempotent-separable, whence Lemma 2 is applicable to L .

Let $\mathbf{u} \approx \mathbf{v}$ be any nontrivial identity satisfied by the semigroup L . Write $\mathbf{u} = \mathbf{u}_1 \cdots \mathbf{u}_m$ where $\mathbf{u}_1, \dots, \mathbf{u}_m \in \mathcal{X}^+$ are pairwise disjoint words, each of which is either connected or a singleton. By repeatedly applying Lemma 2, we obtain $\mathbf{v} = \mathbf{v}_1 \cdots \mathbf{v}_m$ for some pairwise disjoint words $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathcal{X}^+$ such that for each i , the semigroup L satisfies the identity $\mathbf{u}_i \approx \mathbf{v}_i$ with $\text{con}(\mathbf{u}_i) = \text{con}(\mathbf{v}_i)$. For any i , it follows from Lemma 1 that if either \mathbf{u}_i or \mathbf{v}_i is a singleton, then they must be the same word, whence the identity $\mathbf{u}_i \approx \mathbf{v}_i$ is trivial.

Now suppose that, for some i , both \mathbf{u}_i and \mathbf{v}_i are not singletons and \mathbf{v}_i is not connected. Then $\mathbf{v}_i = \mathbf{v}'_i \mathbf{v}''_i$ for some disjoint words $\mathbf{v}'_i, \mathbf{v}''_i \in \mathcal{X}^+$. But then Lemma 2 implies that $\mathbf{u}_i = \mathbf{u}'_i \mathbf{u}''_i$ for some disjoint words $\mathbf{u}'_i, \mathbf{u}''_i \in \mathcal{X}^+$, contradicting the connectedness of \mathbf{u}_i . Hence \mathbf{v}_i must be connected.

We have just shown that the semigroup L satisfies the nontrivial identity $\mathbf{u} \approx \mathbf{v}$ if and only if it satisfies the identities $\mathbf{u}_1 \approx \mathbf{v}_1, \dots, \mathbf{u}_m \approx \mathbf{v}_m$; each of these m identities is either trivial or connected. Consequently, any nontrivial identity in a basis for L that is not connected can be replaced by identities that are connected. \square

4. Identities satisfied by L

It is routine to verify that the semigroup L satisfies the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx. \tag{1}$$

In this section, we present some restrictions on identities satisfied by L .

LEMMA 4. *The semigroup L does not satisfy the identity*

$$xy^2x \approx \mathbf{z} \tag{2}$$

for any $\mathbf{z} \in \{xyxyx, xyxy, yxyx, x^2y^2, y^2x^2\}$.

PROOF. Let $x = e$ and $y = d$. Then the left-hand side of (2) is $ed^2e = a$. But the right-hand side of (2) can only be $edede = ede = dede = 0$, $e^2d^2 = c$, or $d^2e^2 = b$. \square

For each $n \geq 2$, define the sets of words

$$\begin{aligned} \mathcal{P}_n &= \{x^{r_1}y_1^{s_1} \cdots y_n^{s_n}x^{r_2} \mid r_1, r_2 \geq 1, s_1, \dots, s_n \geq 2\}, \\ \mathcal{Q}_n &= \{x^{r_1}y_n^{s_n} \cdots y_1^{s_1}x^{r_2} \mid r_1, r_2 \geq 1, s_1, \dots, s_n \geq 2\}. \end{aligned}$$

Note that any word in \mathcal{P}_n written in reverse order belongs to \mathcal{Q}_n , and vice versa.

LEMMA 5. *Let $\mathbf{u} \approx \mathbf{v}$ be any identity satisfied by the semigroup L such that $\mathbf{u} \in \mathcal{P}_n$. Then $\mathbf{v} \in \mathcal{P}_n \cup \mathcal{Q}_n$.*

PROOF. By assumption,

$$\mathbf{u} = x^{r_1} y_1^{s_1} \cdots y_n^{s_n} x^{r_2},$$

for some $r_1, r_2 \geq 1$ and $s_1, \dots, s_n \geq 2$. By Lemma 1, we have $\text{sim}(\mathbf{v}) = \text{sim}(\mathbf{u}) = \emptyset$ and $\text{non}(\mathbf{v}) = \text{non}(\mathbf{u}) = \{x, y_1, \dots, y_n\}$. Hence

(a) $\text{occ}(x, \mathbf{v}) = k$ and $\text{occ}(y_i, \mathbf{v}) = \ell_i$ for some $k, \ell_i \geq 2$.

Suppose that $h(\mathbf{v}) = y_j$ for some j . Then by letting φ be the substitution $y_i \mapsto y^2$ for all i , we obtain

$$xy^2x \stackrel{(1)}{\approx} (\mathbf{u}\varphi)x^2 \stackrel{L}{\approx} (\mathbf{v}\varphi)x^2 \in y^2\{x, y^2\}^+x^2 \stackrel{(1)}{\approx} \{y^2x^2, yxyx\}.$$

But this implies that the semigroup L satisfies an identity from (2), contradicting Lemma 4. Therefore $h(\mathbf{v}) \neq y_j$ for any j , whence $h(\mathbf{v}) = x$. By a symmetrical argument, we have $t(\mathbf{v}) = x$. Hence \mathbf{v} can be written as

(b) $\mathbf{v} = x^{k_1} \mathbf{w}x^{k_2}$ for some $k_1, k_2 \geq 1$ and $\mathbf{w} \in \{x, y_1, \dots, y_n\}^+$ with $h(\mathbf{w}) \neq x \neq t(\mathbf{w})$.

Suppose that $x \in \text{con}(\mathbf{w})$, say $\mathbf{w} = \mathbf{w}_1x\mathbf{w}_2$ for some $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}^+$. Then

$$xy^2x \stackrel{(1)}{\approx} \mathbf{u}\varphi \stackrel{L}{\approx} \mathbf{v}\varphi \stackrel{(1)}{\approx} xyxyx.$$

But this implies that the semigroup L satisfies the identity $xy^2x \approx xyxyx$ from (2), contradicting Lemma 4. Therefore $x \notin \text{con}(\mathbf{w})$, whence by (a) and (b) we have

(c) $\mathbf{v} = x^{k_1} \mathbf{w}x^{k_2}$ for some $\mathbf{w} \in \{y_1, \dots, y_n\}^+$ and $k_1, k_2 \geq 1$ with $k_1 + k_2 = k$.

Suppose that some y_i in \mathbf{w} is sandwiched between two occurrences of y_j with $i \neq j$, say $\mathbf{w} = \mathbf{w}_1y_j\mathbf{w}_2y_i\mathbf{w}_3y_j\mathbf{w}_4$ for some $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 \in \mathcal{X}^*$. Then by letting χ be the substitution $z \mapsto x^2$ for all $z \neq y_j$, we obtain

$$xy_j^2x \stackrel{(1)}{\approx} \mathbf{u}\chi \stackrel{L}{\approx} \mathbf{v}\chi \stackrel{(1)}{\approx} xy_jxy_jx.$$

But this implies that the semigroup L satisfies the identity $xy^2x \approx xyxyx$ from (2), contradicting Lemma 4. Hence no letter in \mathbf{w} is sandwiched between two same occurrences of some other letter. It then follows from (a) and (c) that

(d) $\mathbf{v} = x^{k_1} y_{1\pi}^{\ell_{1\pi}} \cdots y_{n\pi}^{\ell_{n\pi}} x^{k_2}$ for some integers $k_1, k_2 \geq 1$ and $\ell_{1\pi}, \dots, \ell_{n\pi} \geq 2$ and some permutation π of $\{1, \dots, n\}$.

Suppose that there exists some i such that neither $y_i^{\ell_i} y_{i+1}^{\ell_{i+1}}$ nor $y_{i+1}^{\ell_{i+1}} y_i^{\ell_i}$ is a factor of \mathbf{v} . Then by letting ψ be the substitution that maps both y_i and y_{i+1} to y and any other letter to x , we obtain

$$xy^2x \stackrel{(1)}{\approx} \mathbf{u}\psi \stackrel{L}{\approx} \mathbf{v}\psi \stackrel{(1)}{\approx} xyxyx.$$

But this implies that the semigroup L satisfies the identity $xy^2x \approx xyxyx$ from (2), contradicting Lemma 4. Therefore

(e) for each i , either $y_i^{\ell_i} y_{i+1}^{\ell_{i+1}}$ or $y_{i+1}^{\ell_{i+1}} y_i^{\ell_i}$ is a factor of \mathbf{v} .

It is now easy to show by (d) and (e) that either π is a trivial permutation or $(1\pi, 2\pi, \dots, n\pi) = (n, n - 1, \dots, 1)$, whence either $\mathbf{v} \in \mathcal{P}_n$ or $\mathbf{v} \in \mathcal{Q}_n$. □

LEMMA 6. *Let $\mathbf{u} \approx \mathbf{v}$ be any connected identity satisfied by the semigroup L such that $|\text{con}(\mathbf{u})|, |\text{con}(\mathbf{v})| \leq n$. Suppose that there exist words $\mathbf{e}, \mathbf{f} \in \mathcal{X}^*$ and an endomorphism φ of \mathcal{X}^+ such that $\mathbf{e}(\mathbf{u}\varphi)\mathbf{f} \in \mathcal{P}_n$. Then $\mathbf{e}(\mathbf{v}\varphi)\mathbf{f} \in \mathcal{P}_n$.*

PROOF. By assumption, $\mathbf{e}(\mathbf{u}\varphi)\mathbf{f} = \mathbf{x}$ where

$$\mathbf{x} = x^{r_1} y_1^{s_1} \cdots y_n^{s_n} x^{r_2}$$

for some $r_1, r_2 \geq 1$ and $s_1, \dots, s_n \geq 2$. Since the word \mathbf{u} is connected, the image $\mathbf{u}\varphi$ is a connected factor of \mathbf{x} . The connected factors of \mathbf{x} are exhausted by the following:

(F1) the factors of $x^{r_1}, y_1^{s_1}, \dots, y_n^{s_n}, x^{r_2}$ of length at least two;

(F2) $x^{q_1} y_1^{s_1} \cdots y_n^{s_n} x^{q_2}$ where $1 \leq q_1 \leq r_1$ and $1 \leq q_2 \leq r_2$.

Case 1. $\mathbf{u}\varphi$ belongs to (F1). It suffices to assume that $\mathbf{u}\varphi$ is a factor of some $y_i^{s_i}$, since the argument is very similar when $\mathbf{u}\varphi$ is a factor of either x^{r_1} or x^{r_2} . Then $\mathbf{u}\varphi = y_i^k$ for some $k \in \{2, \dots, s_i\}$. Hence

$$\mathbf{x} = \underbrace{x^{r_1} y_1^{s_1} \cdots y_{i-1}^{s_{i-1}} y_i^{k'}}_{\mathbf{e}} \cdot \underbrace{y_i^k}_{\mathbf{u}\varphi} \cdot \underbrace{y_i^{k''} y_{i+1}^{s_{i+1}} \cdots y_n^{s_n} x^{r_2}}_{\mathbf{f}} \tag{3}$$

for some $k', k'' \geq 0$ such that $k' + k + k'' = s_i$. Since the semigroup L satisfies the identity $\mathbf{u} \approx \mathbf{v}$, we have $\text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v})$ and $\text{non}(\mathbf{u}) = \text{non}(\mathbf{v})$ by Lemma 1. It follows that $\mathbf{v}\varphi = y_i^\ell$ for some $\ell \geq 2$, whence $\mathbf{e}(\mathbf{v}\varphi)\mathbf{f} \in \mathcal{P}_n$ in view of (3).

Case 2. $\mathbf{u}\varphi$ belongs to (F2). Then

$$\mathbf{x} = \underbrace{x^{r_1 - q_1}}_{\mathbf{e}} \cdot \underbrace{x^{q_1} y_1^{s_1} \cdots y_n^{s_n} x^{q_2}}_{\mathbf{u}\varphi} \cdot \underbrace{x^{r_2 - q_2}}_{\mathbf{f}}.$$

By assumption, $|\text{con}(\mathbf{u})| < n + 1 = |\text{con}(\mathbf{u}\varphi)|$. Hence there exists some letter z of \mathbf{u} such that the factor $z\varphi$ of $\mathbf{u}\varphi$ contains at least two distinct letters. Therefore the word $z\varphi$ contains one of the following factors: $xy_1, y_1y_2, \dots, y_{n-1}y_n, y_nx$. Now one of these is a factor of $\mathbf{v}\varphi$ so that $\mathbf{e}(\mathbf{v}\varphi)\mathbf{f} \notin \mathcal{Q}_n$. Since $\mathbf{e}(\mathbf{u}\varphi)\mathbf{f} \in \mathcal{P}_n$ and the identity $\mathbf{e}(\mathbf{u}\varphi)\mathbf{f} \approx \mathbf{e}(\mathbf{v}\varphi)\mathbf{f}$ is satisfied by the semigroup L , we have $\mathbf{e}(\mathbf{v}\varphi)\mathbf{f} \in \mathcal{P}_n \cup \mathcal{Q}_n$ by Lemma 5. Consequently, $\mathbf{e}(\mathbf{v}\varphi)\mathbf{f} \in \mathcal{P}_n$. □

5. Main result

LEMMA 7. *For each $n \geq 2$, the semigroup L satisfies the identity $\mathbf{p}_{(n)} \approx \mathbf{q}_{(n)}$ where*

$$\mathbf{p}_{(n)} = xy_1^2 \cdots y_n^2 x \quad \text{and} \quad \mathbf{q}_{(n)} = xy_n^2 \cdots y_1^2 x.$$

PROOF. Let φ be any substitution into L . It is easy to check that $zLz = \{0\}$ and $z^2 = 0$ for any $z \in \{0, a, b, c\}$. Hence we have $\mathbf{p}_{(n)}\varphi = 0 = \mathbf{q}_{(n)}\varphi$ whenever $x\varphi \in \{0, a, b, c\}$ or $y_i\varphi \in \{0, a, b, c\}$ for some i . Therefore it suffices to assume that $x\varphi, y_1\varphi, \dots, y_n\varphi \in \{d, e\}$. It is also clear that the condition $y_1\varphi = \dots = y_n\varphi$ implies that $\mathbf{p}_{(n)}\varphi = \mathbf{q}_{(n)}\varphi$. Hence further assume that $\{y_1\varphi, \dots, y_n\varphi\} = \{d, e\}$. Then there are two cases depending on how the elements d and e appear in the sequence $(y_1\varphi, \dots, y_n\varphi)$.

Case 1. Some e in the sequence $(y_1\varphi, \dots, y_n\varphi)$ is sandwiched by two or more occurrences of d . Then it is easy to show that $(y_1^2 \cdots y_n^2)\varphi = 0 = (y_n^2 \cdots y_1^2)\varphi$, whence $\mathbf{p}_{(n)}\varphi = 0 = \mathbf{q}_{(n)}\varphi$.

Case 2. Each e in the sequence $(y_1\varphi, \dots, y_n\varphi)$ is not sandwiched by any two occurrences of d . Then the sequence $(y_1\varphi, \dots, y_n\varphi)$ is one of the following:

$$(d, \dots, d, e, \dots, e), \quad (e, \dots, e, d, \dots, d), \quad (e, \dots, e, d, \dots, d, e, \dots, e).$$

Recall that $x\varphi \in \{e, d\}$. Therefore $(y_1\varphi, \dots, y_n\varphi) = (d, \dots, d, e, \dots, e)$ implies that

$$\mathbf{p}_{(n)}\varphi = (x\varphi)de(x\varphi) = (x\varphi)ed(x\varphi) = \mathbf{q}_{(n)}\varphi,$$

$(y_1\varphi, \dots, y_n\varphi) = (e, \dots, e, d, \dots, d)$ implies that

$$\mathbf{p}_{(n)}\varphi = (x\varphi)ed(x\varphi) = (x\varphi)de(x\varphi) = \mathbf{q}_{(n)}\varphi,$$

and $(y_1\varphi, \dots, y_n\varphi) = (e, \dots, e, d, \dots, d, e, \dots, e)$ implies that

$$\mathbf{p}_{(n)}\varphi = (x\varphi)ede(x\varphi) = \mathbf{q}_{(n)}\varphi.$$

This concludes the proof. □

THEOREM 8. *The semigroup L is nonfinitely based.*

PROOF. Working toward a contradiction, suppose that the semigroup L is finitely based. Then there exist a basis Σ for L and some fixed integer $n \geq 2$ such that

$$(\dagger) \quad \text{if } \mathbf{u} \approx \mathbf{v} \in \Sigma, \text{ then } |\text{con}(\mathbf{u})|, |\text{con}(\mathbf{v})| \leq n.$$

By Lemma 3, we may further assume that the identities in Σ are connected. Now by Lemma 7, the semigroup L satisfies the identity $\mathbf{p}_{(n)} \approx \mathbf{q}_{(n)}$. Therefore we have some deduction sequence

$$\mathbf{p}_{(n)} = \mathbf{w}_0 \Rightarrow \mathbf{w}_1 \Rightarrow \cdots \Rightarrow \mathbf{w}_m = \mathbf{q}_{(n)},$$

where each deduction $\mathbf{w}_i \Rightarrow \mathbf{w}_{i+1}$ indicates that there exist words $\mathbf{e}_i, \mathbf{f}_i \in \mathcal{X}^*$, an identity $\mathbf{u}_i \approx \mathbf{v}_i$ from Σ , and an endomorphism φ_i of \mathcal{X}^+ such that

$$\mathbf{w}_i = \mathbf{e}_i(\mathbf{u}_i\varphi_i)\mathbf{f}_i \quad \text{and} \quad \mathbf{w}_{i+1} = \mathbf{e}_i(\mathbf{v}_i\varphi_i)\mathbf{f}_i.$$

Since $\mathbf{w}_0 \in \mathcal{P}_n$ and the semigroup L satisfies the identities $\mathbf{w}_0 \approx \mathbf{w}_1 \approx \cdots \approx \mathbf{w}_m$, we have $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathcal{P}_n \cup \mathcal{Q}_n$ by Lemma 5. Suppose that $\mathbf{w}_k \in \mathcal{P}_n$. Then $\mathbf{w}_k = \mathbf{e}_k(\mathbf{u}_k\varphi_k)\mathbf{f}_k$ where $\mathbf{u}_k \approx \mathbf{v}_k \in \Sigma$. Since the identity $\mathbf{u}_k \approx \mathbf{v}_k$ is satisfied by L and $|\text{con}(\mathbf{u}_k)|, |\text{con}(\mathbf{v}_k)| \leq n$ by (\dagger) , we have $\mathbf{e}_k(\mathbf{v}_k\varphi_k)\mathbf{f}_k \in \mathcal{P}_n$ by Lemma 6, that is, $\mathbf{w}_{k+1} \in \mathcal{P}_n$. Therefore we have just shown by induction that $\mathbf{w}_0, \dots, \mathbf{w}_m \in \mathcal{P}_n$, which implies the contradiction $\mathbf{q}_{(n)} = \mathbf{w}_m \notin \mathcal{Q}_n$. Consequently, the finite basis Σ for the semigroup L does not exist. □

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