COMMUTING RINGS OF SIMPLE $A(k)$-MODULES

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Abstract

For the Weyl algebra $A(k)$ and each finite dimensional division ring $D$ over $k$, there exists a simple $A(k)$-module whose commuting ring is $D$.

It has been known for some time that if $A(k)$ denotes the Weyl algebra over a field $k$ of characteristic zero, the commuting ring of a simple $A(k)$-module is a division algebra finite dimensional over $k$ (see the introduction of [1]). Which division algebras actually appear? Quebbemann [1] showed that if $D$ is a finite dimensional division algebra whose center is $k$, then it occurs as a commuting ring. We complete this circle of ideas by showing that any $D$ appears: a division algebra over $k$ appears as the commuting ring of a simple $A(k)$-module if and only if it is finite dimensional over $k$.


The construction

In what follows $k$ is a field of characteristic zero and $D$ is any division algebra finite dimensional over $k$. The Weyl algebra $A(k)$ is $k[x, y]$ subject to $yx - xy = 1$ and $A(D)$ denotes $D \otimes_k A(k)$.

We review Quebbemann's construction [1]. A polynomial $p \in D[x]$ is fixed and an action of $A(D)$ is defined on $D[x]$ where $x$, as well as elements of $D$, act by left multiplication and

$$y \cdot \pi = \pi' + \pi p$$

for $\pi \in D[x]$.

Quebbemann proves that $D[x]$ is a simple $A(D)$-module.

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The crux of this note is to calculate $C$, the centralizer of the restricted $A(k)$-action on $D[x]$, for a carefully chosen $p$. Here we modify a trick of Quebbemann. The center $K$ of $D$ is a finite field extension of $k$; choose a primitive element so that $K = k(\theta)$. Take any basis $c_1, \ldots, c_n$ for $D$ over $k$ with $c_n = \theta$ and set $p = \sum_{i=1}^{n} c_i x^i$. We begin with a general ring theoretic lemma which is undoubtedly well known.

**Lemma 1.** Let $D$ be a division ring finite dimensional over its center $K$. Suppose $S$ is a ring containing $D$ and containing a nonzero additive subgroup $L$ such that $DLD \subseteq L$ and $K \subseteq C_S(L)$. Then $C_L(D) \neq 0$ where $C_S(L)$ and $C_L(D)$ denote the centralizer of $L$ in $S$ and $D$ respectively.

**Proof.** By assumption $L$ is a $D \otimes_K D^\varphi$-module. Since $D \otimes_K D^\varphi$ is a central simple $K$-algebra, it has a unique simple module up to isomorphism—namely $D$. Inside $D$,

$$(d \otimes 1 - 1 \otimes d)(1) = 0.$$  

All $D \otimes_K D^\varphi$-modules are semisimple, so $L$ contains a copy of $D$. Consequently there is an element $g \in L$ with $dg - gd = 0$.

The centralizer $C$ consists of those members of $\text{End}_k(D[x])$ which commute with the actions by $x$ and $y$. The $k[x]$-module endomorphisms of $D[x]$ can be identified with $E[x]$ where $E = \text{End}_k D$. Thus $C$ is the centralizer of the $y$ action in $E[x]$. Notice that the map sending $\pi \in D[x]$ to $\pi p$ is the element $\tilde{p} = \sum_{i=1}^{n} \tilde{c}_i x^i \in E[x]$, where $\tilde{c}_j$ denotes right multiplication by $c_j$ on $D$.

**Lemma 2.** Suppose $f = f_T x^T + \sum_{j<T} f_j x^j \in E[x]$.

(i) $f \in C$ if and only if $f' = f \tilde{p} - \tilde{p} f$.

(ii) If $f \in C$ then $f_T$ commutes with multiplication by elements in $K$, the center of $D$.

**Proof.** $f \in C$ means $(\pi f - f \pi) \cdot \pi = 0$ for all $\pi \in D[x]$. Expanding,

$$(f(\pi))' + f(\pi) \cdot p - f(\pi') - f(\pi p) = 0.$$  

But $(f(\pi))' = f'(\pi) + f(\pi')$. Hence

$$f'(\pi) = f(\pi p) - f(\pi)p.$$  

We immediately obtain (i).

Look at the coefficient of $x^{n+T}$ in equation (i). On the left it is zero and on the right it is $f_T \theta - \theta f_T$. The lemma follows because $K = k(\theta)$. 

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Lemma 3. $C = D$. (The centralizer consists of left multiplications by elements in $D$.)

Proof. Since left and right multiplication by elements of $D$ are commuting maps, Lemma 2(i) yields $D \subseteq C$. Consequently, if we set

$$L_T = \left\{ f_T \in E \mid f_T x^T + \sum_{j < T} f_j x^j \in C \right\}$$

then $DL_T D \subseteq L_T$. By Lemma 2(ii), elements of $K$ centralize $L_T$. Lemma 1 now applies: if $L_T \neq 0$ there exists a nonzero $g \in L_T$ with $dg = gd$ for all multiplications $d \in D$. However, the members of $\text{End}_K D$ which centralize all such left multiplications are precisely the right multiplications by elements of $D$. We summarize:

$L_T \neq 0$ implies $L_T$ contains a nonzero right multiplication.

We next claim that $C$ is algebraic over $k$. One way to see this is to observe that Lemma 2(i) implies that nonzero elements of $C$ have nonzero constant terms. (Don't forget that char $k = 0$.) Thus the map sending a polynomial in $C$ to its constant term in $E$ is an injective ring homomorphism. Since $E$ is finite dimensional over $k$, so is $C$.

Putting the last two paragraphs together, we see that if $L_T \neq 0$ there is a polynomial in $E[x]$ of degree $T$ which is algebraic and has as its leading coefficient "right multiplication" by a nonzero element in the division ring $D$. But a nonconstant algebraic polynomial has a leading coefficient which is nilpotent. Therefore $C \subseteq E$.

Now if $h \in C$ then Lemma 2(i) yields

$$0 = \sum_{i=1}^n (\bar{c}_i h - h\bar{c}_i) x^i.$$ 

Hence $\bar{c}_i h = h\bar{c}_i$ for $i = 1, \ldots, n$. Evaluate these $k$-endomorphisms on $1 \in D$.

$$h(1)c_i = h(c_i) \quad \text{for } i = 1, \ldots, n.$$ 

Since the $c_i$ span $D$ over $k$,

$$h(1)d = h(d) \quad \text{for all } d \in D.$$ 

As required, we have shown that $h$ is left multiplication by an element of $D$.

Theorem. $D[x]$ is a simple $A(k)$-module with commuting ring $D$.

Proof. The simplicity argument can be found in [1]. We sketch an alternate proof.
Since $D[x]$ is a simple $A(D)$-module, $D[x] = A(D) \cdot \pi$ for some $\pi$. Hence $D[x] = \sum_{i=1}^{n} c_i A(k)\pi$; $D[x]$ is a noetherian $A(k)$-module. If $V$ is a maximal submodule then $\cap c_i^{-1}V$ is an $A(D)$-submodule and so is zero.

Thus $D[x]$ contains a simple $A(k)$-module $W$. By simplicity, $D[x] = DW$ which, in turn, is a direct sum of copies of $W$ as an $A(k)$-module. Since Lemma 3 states that the commuting ring of $D[x]$ as an $A(k)$-module is a division ring, there is only one copy of $W$ in that sum.

References