Let $N$ be a homomorphically closed class of associative rings. Put $N_1 = N = N$ and, for ordinals $\alpha \geq 2$, define $N_\alpha$ ($N^\alpha$) to be the class of all associative rings $R$ such that every non-zero homomorphic image of $R$ contains a non-zero ideal (left ideal) in $N_\beta$ for some $\beta < \alpha$. In this way we obtain a chain $\{N_\alpha\}$ ($\{N^\alpha\}$), the union of which is equal to the lower radical class $lN$ (lower left strong radical class $lsN$) determined by $N$. The chain $\{N_\alpha\}$ is called Kurosh's chain of $N$. Sulinski, Anderson and Divinsky proved [7] that $lN = \bigcup_{n=1}^{\infty} N_n$. Heinicke [3] constructed an example of $N$ for which $lN \neq N_k$ for $k = 1, 2, \ldots$. In [1] Beidar solved the main problem in the area showing that for every natural number $n \geq 1$ there exists a class $N$ such that $lN = N_{n+1} \neq N_n$. Some results concerning the termination of the chain $\{N^\alpha\}$ were obtained in [2, 4]. In this paper we present some classes $N$ with $N_\alpha = N^\alpha$ for all $\alpha$. Using this and Beidar's example we prove that for every natural number $n \geq 1$ there exists an $N$ such that $N_\alpha = N^\alpha$ for all $\alpha$ and $N_n \neq N_{n+1} = N_{n+2}$. This in particular answers Question 6 of [4].

All rings in the paper are associative and $N$ is a homomorphically closed class of such rings. To denote that $I$ is an ideal (left ideal) of a ring $R$ we write $I \triangleleft R$ ($I < R$). A subring $A$ of a ring $R$ is called accessible (left accessible) if there is a chain of subrings $A = A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_0 = R$ such that $A_i \triangleleft A_{i-1}$ ($A_i < A_{i-1}$) for $i = 1, 2, \ldots, n$.

$\mathbb{Z}$ is used to denote the ring of integers, $\mathbb{Q}$ the field of rational numbers and $\mathbb{Q}(i)$ the field of complex numbers $a + bi$, where $a, b \in \mathbb{Q}$.

The fundamental definitions and properties of radicals may be found in [8] and those of strong radicals in [2]. The following proposition collects some well known properties of classes $N_\alpha$ and $N^\alpha$.

**Proposition 1.** (i) $N_\alpha \subseteq N^\alpha$ for every ordinal $\alpha$;
(ii) classes $N_\alpha$ and $N^\alpha$ are homomorphically closed for all $\alpha$;
(iii) ([2]) if $0 \neq R \in lsN$ then $R$ contains a non-zero left accessible subring in $N$;
(iv) ([7]) $R \in lN$ if and only if every non-zero homomorphic image of $R$ contains a non-zero accessible subring in $N$;
(v) ([2]) if $0 \neq R \in N^{n+1}$, where $n$ is an integer $\geq 1$, then there are subrings $0 \neq L_n < \ldots < L_0 = R$ of $R$ such that $L_n \in N$;
(vi) ([7]) $R \in N_{n+1}$, where $n$ is an integer $\geq 1$, if and only if every non-zero homomorphic image $R'$ of $R$ contains subrings $0 \neq I_n < \ldots < I_0 = R'$ such that $I_n \in N$.

Recall that a radical class $S$ is called left stable if for every $L < R$, $S(L) \subseteq S(R)$. An example of a left stable radical class is the generalized nil radical $N_g$; this is the upper radical determined by the class of reduced rings i.e. rings without non-zero nilpotent elements.

**Theorem 1.** If $S$ is a left stable radical class containing $N_g$ then for $N = S \cup P$, where $P$ is a homomorphically closed class of commutative rings, $N_\alpha = N^\alpha$ for every ordinal $\alpha$.
Proof. In view of Proposition 1 it suffices to prove that if $0 \neq L_n < \ldots < L_0 = R$ and $L_n \in N$ then there are $0 \neq L_n < \ldots < L_0 = R$ with $L_n \in N$. It is so if $S(R) \neq 0$. Hence, since $N_n \subseteq S$ and $S$ is left stable, we can assume that the ring $R$ is reduced and $L_n \in P$. Let $k$ be a minimal number such that $L_n$ is contained in the centre of $L_k$. Suppose that $k \geq 1$. Then there are $l \in L_n$ and $l' \in L_{n-1}$ such that $ll' - l'l \neq 0$. Now $(ll' - l'l)^2 = (l')^2 - l(l')^2l - l'l^2l' + (l')^2l^2$. Since $l'l$, $(l')^2l \in L_k$ and $L_n$ is contained in the centre of $L_k$, we have $(ll')^2 = l'l^2l'$, $(l')^2l = (l')^2l^2$ and $(l')^2 = (l')^2l^2$. Thus $(ll' - l'l)^2 = 0$ and, since the ring $R$ is reduced, $ll' - l'l = 0$. This contradiction shows that $L_n$ is contained in the centre of $R$. On the other hand $RL_n \subseteq L_n$. Hence $L_n = L_n + RL_n < L_n + RL_{n-1} < \ldots < L_n + RL_n \subseteq R$ and the result follows.

Let $p$ be a prime of the form $4m + 3$ and, for $n \geq 0$, let $A_n$ be the subring of $\mathbb{Q}(i)$ generated by $p$ and $ip^n$. The following properties of the rings $A_n$ were established by Beidar in [1, Lemma 1].

**Proposition 2.** (i) $A_n \triangleleft A_m$ if and only if $n = m$ or $n = m + 1$;
(ii) proper homomorphic images of $A_n$ are finite;
(iii) the only subring of $\mathbb{Q}(i)$ isomorphic to $A_n$ is the ring $A_n$ itself;
(iv) if $B$ is a subring of $\mathbb{Q}(i)$ and $A_n \triangleleft B$ then $1 \in B$ or $B = A_n$ or $B = A_{n-1}$.

Now we prove the following theorem.

**Theorem 2.** If $N = N_k \cup T \cup \{A_n\}$, where $T$ is the class of rings whose additive groups are torsion and $(A_n)$ is the class of all isomorphic images of $A_n$ for an $n \geq 1$, then $N_1 = N \not\subseteq N_2 = N^2 \not\subseteq \ldots \not\subseteq N_{n+1} = N^{n+1} = N_{n+2} = N^{n+2}$.

**Proof.** As an immediate consequence of Theorem 1 and Proposition 2(ii) one obtains that, for every ordinal $\alpha$, $N_\alpha = N^\alpha$. It follows from Proposition 1 (vi) and Proposition 2 that $A_0 \in N_{n+1} \setminus N_n$. Hence $N_1 \not\subseteq N_2 \not\subseteq \ldots \not\subseteq N_{n+1}$. It remains to prove that $lN = N_{n+1}$ or, equivalently, that every non-zero ring $R \in lN$ contains a non-zero ideal in $N_n$. Obviously we can assume that $R$ is semiprime and the additive group of $R$ is torsion-free. Then by Propositions 1 (iv) and 2 (ii), $R$ contains an accessible subring isomorphic to $A_n$. Let $t$ be the minimal integer $t = 0$ such that there are $l < l_{t-1} < \ldots < l_0 = R$ with $l_t$ isomorphic to $Ak$ for some $0 < k \leq n$. We claim that $t \leq 1$. For, suppose that $t > 2$ and take $I$ the ideal of $l_{t-2}$ generated by $l_t$. By Andrunakievich's lemma, $I^2 \subseteq I$. This, semiprimes of $R$ and properties of $A_n$ imply that $I$ is a prime ring without 1. Since the additive group of $I$ is torsion-free, we can form the quotient ring $Z^{-1}I$. Now $Z^{-1}l_t \triangleleft Z^{-1}I$ and $Z^{-1}I_t$, being isomorphic to $Q(i)$, contains 1. However the ring $Z^{-1}I$ is prime, so $Z^{-1}l_t = Z^{-1}I$. Thus $I$ is a ring without 1 isomorphic to a subring of $Q(i)$. By Proposition 2 (iii) and (iv), $I$ is isomorphic to $A_n$ or $A_{n-1}$. Moreover, if $I$ is isomorphic to $A_{n-1}$ then, since $I$ is a ring without 1, $n - 1 > 0$. This and the fact that the sequence $I < l_{t-1} < l_{t-3} < \ldots < l_0 = R$ is shorter than $I < l_{t-1} < \ldots < l_0 = R$ prove the claim. Thus $R$ contains a non-zero ideal isomorphic to $A_k$ for some $0 < k \leq n$. It is clear from Proposition 2 that $A_k \in N_n$. The result follows.

**Remark.** Let $N = N_k \cup T \cup \{A_{2n} : n = 1, 2, \ldots\}$. One can easily check using Propositions 1 and 2 that for every $0 < i < 2^n$, $n = 1, 2, \ldots, A_{2n+i} \in N_{2n+i+1} \setminus N_{2n+i}$. Hence by Theorem 1, $N_k = N^k \not\subseteq N_{k+1} = N^{k+1}$ for $k = 1, 2, \ldots$ and $lN = lS N = \bigcup_{k=1}^\infty N^k$. 


In [4] it was proved that if the class $N$ is hereditary (hereditary and contains nilpotent rings) then $lsN = N^4$ ($lsN = N^3$) i.e. in those cases the chain $\{N^a\}$ terminates one step further than $\{N_a\}$. In [6] Stewart proved that if $N$ is a class of zero rings then $lN = N_3$. The following theorem shows in particular that in this case $lsN = lN = N_3$. It can also be regarded as a generalization of the fact that the prime radical is strong.

**THEOREM 3.** If $N$ is a class of zero rings then $S = lN$ is a left strong radical class.

**Proof.** Suppose that $L < R$ and $L \in S$. Let $U = \{x \in L : Lx = 0\}$. If $U = L$ then $L^2 = 0$ and it is easy to check that $LR^1 \in S$, where $R^1$ is the ring $R$ with an unity adjoined. Suppose $U \neq L$. Then $0 \neq L/U \in S$, so $L/U$ contains a non-zero accessible subring $A/U \in N$. Hence there are subrings $A_0, \ldots, A_n$ of $L$ such that $A < A_n < \ldots < A_0 = L$. Now $0 \neq LAR^1 \triangleleft R$. Since $(LAR^1)^m \subseteq (LA)^mR^1$ for $m = 1, 2, \ldots$ and the ideal of $L$ generated by $A$ is nilpotent, $LAR^1$ is a nilpotent ideal of $R$. Suppose that $(LAR^1)^{k+1} = 0$, $(LAR^1)^{k} \neq 0$ and take $t \in (LAR^1)^{k-1}$. Obviously $LAR^1tR^1 \triangleleft R$. For every $l \in L$, $x, y \in R^1$, the map $f : A \rightarrow lAxty$ given by $f(a) = laxty$ is a ring epimorphism and $lAxty \triangleleft LAR^1tR^1$. Hence $(LAR^1)^k \in S$. Therefore $S(R) \neq 0$ and the result follows.

Let us observe that if $N$ is a class of $N$-nilpotent rings [5] and $N_0$ is the class of zero $lN$-radical rings then $lN_0 = l(N_0 \cap N)$. Applying [5, Theorem 4] to $\alpha = lN$ one obtains that $R^0 \in N$, where $R^0$ is the zero ring on the additive group of a ring $R \in N$. The same theorem applied to $\alpha = lN_0$ implies $N \subseteq lN_0$. Hence $lN = lN_0$ and Theorem 3 gives

**COROLLARY.** If $N$ is a class of $M$-nilpotent rings then $lN = lsN = N_3 = N^3$.

**REFERENCES**