Let $N$ be a homomorphically closed class of associative rings. Put $N_1 = N^1 = N$ and, for ordinals $a \geq 2$, define $N_a (N^\alpha)$ to be the class of all associative rings $R$ such that every non-zero homomorphic image of $R$ contains a non-zero ideal (left ideal) in $N_\beta$ for some $\beta < \alpha$. In this way we obtain a chain $\{N_a\}$, the union of which is equal to the lower radical class $IN$ (lower left strong radical class $IsN$) determined by $N$. The chain $\{N_a\}$ is called Kurosh's chain of $N$. Sulinski, Anderson and Divinsky proved [7] that \(IN = \bigcup_{a=1}^{\infty} N_a\). Heinicke [3] constructed an example of $N$ for which $IN \neq N_k$ for $k = 1, 2, \ldots$. In [1] Beidar solved the main problem in the area showing that for every natural number $n \geq 1$ there exists a class $N$ such that $IN = N_{n+1} = N_n$. Some results concerning the termination of the chain $\{N^\alpha\}$ were obtained in [2, 4]. In this paper we present some classes $N$ with $N_a = N^\alpha$ for all $a$. Using this and Beidar's example we prove that for every natural number $n \geq 1$ there exists an $N$ such that $N_a = N^\alpha$ for all $\alpha$ and $N_n \neq N_{n+1} = N_{n+2}$. This in particular answers Question 6 of [4].

All rings in the paper are associative and $N$ is a homomorphically closed class of such rings. To denote that $I$ is an ideal (left ideal) of a ring $R$ we write $I < R$ ($I < L$). A subring $A$ of a ring $R$ is called accessible (left accessible) if there is a chain of subrings $A = A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_0 = R$ such that $A_i < A_{i-1}$ ($A_i < A_{i-1}$) for $i = 1, 2, \ldots, n$.

$\mathbb{Z}$ is used to denote the ring of integers, $\mathbb{Q}$ the field of rational numbers and $\mathbb{Q}(i)$ the field of complex numbers $a + bi$, where $a, b \in \mathbb{Q}$.

The fundamental definitions and properties of radicals may be found in [8] and those of strong radicals in [2]. The following proposition collects some well known properties of classes $N_\alpha$ and $N^\alpha$.

**Proposition 1.** (i) $N_\alpha \subseteq N^\alpha$ for every ordinal $\alpha$;
(ii) classes $N_\alpha$ and $N^\alpha$ are homomorphically closed for all $\alpha$;
(iii) ([2]) if $0 \neq R \in IsN$ then $R$ contains a non-zero left accessible subring in $N$;
(iv) ([7]) $R \in IN$ if and only if every non-zero homomorphic image of $R$ contains a non-zero accessible subring in $N$;
(v) ([2]) if $0 \neq R \in N^{\alpha+1}$, where $n$ is an integer $\geq 1$, then there are subrings $0 \neq L_n < \ldots < L_0 = R$ of $R$ such that $L_n \in N$;
(vi) [7] $R \in N_{n+1}$, where $n$ is an integer $\geq 1$, if and only if every non-zero homomorphic image $R'$ of $R$ contains subrings $0 \neq I_n < \ldots < I_0 = R'$ such that $I_n \in N$.

Recall that a radical class $S$ is called left stable if for every $L < R$, $S(L) \subseteq S(R)$. An example of a left stable radical class is the generalized nil radical $N_g$; this is the upper radical determined by the class of reduced rings i.e. rings without non-zero nilpotent elements.

**Theorem 1.** If $S$ is a left stable radical class containing $N_g$ then for $N = S \cup P$, where $P$ is a homomorphically closed class of commutative rings, $N_\alpha = N^\alpha$ for every ordinal $\alpha$. 

Proof. In view of Proposition 1 it suffices to prove that if $0 = L_n < \ldots < L_0 = R$ and $L_n \in N$ then there are $0 \neq I_n < \ldots < I_0 = R$ with $I_n \in N$. It is so if $S(R) \neq 0$. Hence, since $N_k \subseteq S$ and $S$ is left stable, we can assume that the ring $R$ is reduced and $L_n \in P$. Let $k$ be a minimal number such that $L_n$ is contained in the centre of $L_k$. Suppose that $k \geq 1$. Then there are $l \in L_n$ and $l' \in L_{k-1}$ such that $ll' - l'l \neq 0$. Now $(ll' - l'l)^2 = (l')^2 - l'2ll - l'2l' + (l'l)^2$. Since $l', (l')^2 \in L_k$ and $L_n$ is contained in the centre of $L_k$, we have $(l')^2 = l'2l', (l')^2l = (l')^2l$ and $(l')^2 = (l')^2l$. Thus $(ll' - l'l)^2 = 0$ and, since the ring $R$ is reduced, $ll' - l'l = 0$. This contradiction shows that $L_n$ is contained in the centre of $R$. On the other hand $RL_n \subseteq L_n$. Hence $L_n = L_n + RL_n < L_n + RL_n < \ldots < L_n + RL_n < R$ and the result follows.

Let $p$ be a prime of the form $4m + 3$ and, for $n \geq 0$, let $A_n$ be the subring of $\mathbb{Q}(i)$ generated by $p$ and $ip^n$. The following properties of the rings $A_n$ were established by Beidar in [1, Lemma 1].

**Proposition 2.** (i) $A_n \triangleleft A_m$ if and only if $n = m$ or $n = m + 1$;
(ii) proper homomorphic images of $A_n$ are finite;
(iii) the only subring of $\mathbb{Q}(i)$ isomorphic to $A_n$ is the ring $A_n$ itself;
(iv) if $B$ is a subring of $\mathbb{Q}(i)$ and $A_n \triangleleft B$ then $1 \in B$ or $B = A_n$ or $B = A_{n-1}$.

Now we prove the following theorem.

**Theorem 2.** If $N = N_k \cup T \cup \{A_n\}$, where $T$ is the class of rings whose additive groups are torsion and $(A_n)$ is the class of all isomorphic images of $A_n$ for an $n \geq 1$, then $N_1 = N^1 \not\subseteq N_2 = N^2 \not\subseteq \ldots \not\subseteq N_{n+1} = N^{n+1} = N_{n+2} = N^{n+2}$.

**Proof.** As an immediate consequence of Theorem 1 and Proposition 2(ii) one obtains that, for every ordinal $\alpha$, $N_\alpha = N^\alpha$. It follows from Proposition 1 (vi) and Proposition 2 that $A_0 \in N_{n+1}\setminus N_n$. Hence $N_1 \not\subseteq N_2 \not\subseteq \ldots \not\subseteq N_{n+1}$. It remains to prove that $lN = N_{n+1}$ or, equivalently, that every non-zero ring $R \in lN$ contains a non-zero ideal in $N_n$. Obviously we can assume that $R$ is semiprime and the additive group of $R$ is torsion-free. Then by Propositions 1 (iv) and 2 (ii), $R$ contains an accessible subring isomorphic to $A_n$. Let $t$ be the minimal integer $\geq 0$ such that there are $I_t < I_{t-1} < \ldots < I_0 = R$ with $I_t$ isomorphic to $A_k$ for some $0 < k \leq n$. We claim that $t \leq 1$. For, suppose that $t \geq 2$ and take $I$ the ideal of $I_{t-2}$ generated by $I_t$. By Andrunakievich's lemma, $I^2 \subseteq I_t$. This, semiprimeness of $R$ and properties of $A_n$ imply that $I$ is a prime ring without $1$. Since the additive group of $I$ is torsion-free, we can form the quotient ring $Z^{-1}I$. Now $Z^{-1}I_t \not\subseteq Z^{-1}I$ and $Z^{-1}I_t$, being isomorphic to $Q(i)$, contains 1. However the ring $Z^{-1}I$ is prime, so $Z^{-1}I_t = Z^{-1}I$. Thus $I$ is a ring without 1 isomorphic to a subring of $Q(i)$. By Proposition 2 (iii) and (iv), $I$ is isomorphic to $A_n$ or $A_{n-1}$. Moreover, if $I$ is isomorphic to $A_{n-1}$ then, since $I$ is a ring without 1, $n - 1 > 0$. This and the fact that the sequence $I < I_{t-2} < I_{t-3} < \ldots < I_0 = R$ is shorter than $I_t < I_{t-1} < \ldots < I_0 = R$ prove the claim. Thus $R$ contains a non-zero ideal isomorphic to $A_k$ for some $0 < k \leq n$. It is clear from Proposition 2 that $A_k \in N_n$. The result follows.

**Remark.** Let $N = N_k \cup T \cup \{A_{2n}: n = 1, 2, \ldots \}$. One can easily check using Propositions 1 and 2 that for every $0 < i < 2^n$, $n = 1, 2, \ldots$, $A_{2n+i} \in N_{2n+i+1}\setminus N_2n_i$. Hence by Theorem 1, $N_k = N^k \not\subseteq N_{k+1} = N^{k+1}$ for $k = 1, 2, \ldots$ and $lN = lS N = \bigcup_{k=1}^{\infty} N^k$. 

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In [4] it was proved that if the class \( N \) is hereditary (hereditary and contains nilpotent rings) then \( lsN = N^4 \) (\( lsN = N^3 \)) i.e. in those cases the chain \( \{ N^m \} \) terminates one step further than \( \{ N_a \} \). In [6] Stewart proved that if \( N \) is a class of zero rings then \( lnN = N_3 \). The following theorem shows in particular that in this case \( lsN = lnN = N^3 \). It can also be regarded as a generalization of the fact that the prime radical is strong.

**Theorem 3.** If \( N \) is a class of zero rings then \( S = lnN \) is a left strong radical class.

**Proof.** Suppose that \( L < R \) and \( L \in S \). Let \( U = \{ x \in L : Lx = 0 \} \). If \( U = L \) then \( L^2 = 0 \) and it is easy to check that \( LR^1 \in S \), where \( R^1 \) is the ring \( R \) with an unity adjoined. Suppose \( U \neq L \). Then \( 0 \neq L/U \in S \), so \( L/U \) contains a non-zero accessible subring \( A/U \in N \). Hence there are subrings \( A_0, \ldots, A_n \) of \( L \) such that \( A < A_n < \ldots < A_0 = L \). Now \( 0 \neq LAR^1 \in R \). Since \( (LAR^1)^m \subseteq (LA)^mR^1 \) for \( m = 1, 2, \ldots \) and the ideal of \( L \) generated by \( A \) is nilpotent, \( LAR^1 \) is a nilpotent ideal of \( R \). Suppose that \( (LAR^1)^{k+1} = 0 \), \( (LAR^1)^k \neq 0 \) and take \( t \in (LAR^1)^{k-1} \). Obviously \( LAR^1tR^1 \in R \). For every \( l \in L, x, y \in R^1 \), the map \( f : A \to lAxty \) given by \( f(a) = laxty \) is a ring epimorphism and \( lAxty \in LAR^1tR^1 \). Hence \( (LAR^1)^k \in S \). Therefore \( S(R) \neq 0 \) and the result follows.

Let us observe that if \( N \) is a class of \( N \)-nilpotent rings [5] and \( N_0 \) is the class of zero \( lnN \)-radical rings then \( lnN_0 = l(N_0 \cap N) \). Applying [5, Theorem 4] to \( \alpha = lnN \) one obtains that \( R^0 \in N \), where \( R^0 \) is the zero ring on the additive group of a ring \( R \in N \). The same theorem applied to \( \alpha = lnN_0 \) implies \( N \subseteq lnN_0 \). Hence \( lnN = LN_0 \) and Theorem 3 gives

**Corollary.** If \( N \) is a class of \( M \)-nilpotent rings then \( lnN = lsN = N_3 = N^3 \).

**References**