BULL. AUSTRAL. MATH. SOC. MOS 2030
VOL. 2 (1970), 275-276.

A note on types

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A characterization is obtained of the types τ for which $[G/G(\tau)](\tau) = 0$ for all torsion-free abelian groups G .

In a torsion-free abelian group G , for any type τ , the elements x of G with types $T(x)$ such that $T(x) \geq \tau$ form a subgroup $G(\tau)$. (We shall use the notation of [1], Chapter VII.) The aim of this note is to prove

THEOREM. *Let* τ *be a type. Then* $[G/G(\tau)](\tau) = 0$ *for all torsion-free abelian groups* G *if and only if* τ *is the type of a height* $(h_1, h_2, \ldots, h_n, \ldots)$ *where* h_n *takes only the values* 0 *and* ∞ .

Suppose τ is the type of a height $(h_1, h_2, \ldots, h_n, \ldots)$ where $h_n = 0$ or ∞ for each n , and let m be such that $h_m = \infty$. If for some torsion-free G there is an element g such that $T(g + G(\tau)) \geq \tau$, then for each positive integer i , there exists $g_i \in G$ for which $p_m^i g_i - g \in G(\tau)$. But then $p_m^i g_i - g$ is divisible by p_m^i , so g is also. Thus $T(g) \geq \tau$, so $[G/G(\tau)](\tau) = 0$.

The converse is a consequence of

PROPOSITION. *Let* τ *be the type of a height* $(h_1, h_2, \ldots, h_n, \ldots)$ *where* $0 < h_n < \infty$ *for infinitely many values of* n . *Then there is a torsion-free abelian group* G *with the following properties:*

(i) G *has rank* 2 ,

(ii) $G(\tau)$ *has rank* 1 ,

Received 21 February, 1970.

(iii) $[G/G(\tau)](\tau) = G/G(\tau)$.

Proof. Let $M = \{n | h_n = \infty\}$. Let $(k_1, k_2, \ldots, k_n, \ldots)$ be the subsequence of positive finite terms of $(h_1, h_2, \ldots, h_n, \ldots)$ and re-label the associated primes as $q_1, q_2, \ldots$. Let $\{x, y\}$ be a basis for a two-dimensional rational vector space V and G the subgroup of V generated by

$$\left\{p^{-n}x,\ p^{-n}y,\ q_n^{-k_n n}x,\ q_n^{-k_n n}\left(q_n^{-1}x+y\right) \middle| p \in M\ ,\ n = 1, 2, \ldots\right\} .$$

A routine argument using the linear independence of x and y shows that the height of x is $(h_1, h_2, \ldots, h_n, \ldots)$, so that $T(x) = \tau$. Suppose y is divisible by $q_n^{k_n}$ for some n . Since the same is true of $q_n^{-1}x + y$, x has q_n-height $k_n + 1$ at least, which is impossible. Thus $T(y) < \tau$. Clearly $[x]_* \subseteq G(\tau)$ where $[x]_*$ is the smallest pure subgroup of G containing x . If this inclusion is proper, then $G(\tau)$, being a pure subgroup, must have rank 2 and therefore coincide with G . But $y \notin G(\tau)$ and hence $G(\tau) = [x]_*$.

Let π denote the natural homomorphism from G to $G/[x]_*$. $G/[x]_*$ is generated by

$$\left\{p^{-n}\pi(y),\ q_n^{-k_n n}\pi(y) \middle| p \in M,\ n = 1, 2, \ldots\right\}$$

and so is rational of type τ . Thus

$$[G/G(\tau)](\tau) = \left[G/[x]_*\right](\tau) = G/[x]_* = G/G(\tau) .$$

Reference

[1] L. Fuchs, *Abelian groups* (Publishing House of the Hungarian Academy of Sciences, Budapest, 1958).

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https://doi.org/10.1017/S0004972700041903 Published online by Cambridge University Press