ON THE HIRSCH–PLOTKIN RADICAL OF A FACTORIZED GROUP

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1. Introduction. Let the group $G = AB$ be the product of two subgroups $A$ and $B$. A normal subgroup $K$ of $G$ is said to be factorized if $K = (A \cap K)(B \cap K)$ and $A \cap B \leq K$, and this is well-known to be equivalent to the fact that $K = AK \cap BK$ (see [1]). Easy examples show that normal subgroups of a product of two groups need not, in general, be factorized. Therefore the determination of certain special factorized subgroups is of relevant interest in the investigation concerning the structure of a factorized group. In this direction E. Pennington [5] proved that the Fitting subgroup of a finite product of two nilpotent groups is factorized. This result was extended to infinite groups by B. Amberg and the authors, who proved in [2] that if the soluble group $G = AB$ with finite abelian section rank is the product of two locally nilpotent subgroups $A$ and $B$, then the Hirsch–Plotkin radical (i.e. the maximum locally nilpotent normal subgroup) of $G$ is factorized. If $G$ is a soluble $S_1$-group and the factors $A$ and $B$ are nilpotent, it was shown in [3] that also the Fitting subgroup of $G$ is factorized. However, Pennington’s theorem becomes false for finite soluble groups which are the product of two arbitrary subgroups. For instance, the symmetric group of degree 4 is the product of a subgroup isomorphic with the symmetric group of degree 3 and a cyclic subgroup of order 4, but its Fitting subgroup is not factorized.

The aim of this paper is to prove that even in the case of a group factorized by two arbitrary subgroups the Hirsch–Plotkin radical and the Fitting subgroup have some factorization properties.

THEOREM A. Let the soluble-by-finite group $G = AB$ with finite abelian section rank be the product of two subgroups $A$ and $B$, and let $H$ be the Hirsch–Plotkin radical of $G$. Then $H = A_0H \cap B_0H$, where $A_0$ and $B_0$ are the Hirsch–Plotkin radicals of $A$ and $B$, respectively.

Here the requirement that $G$ has finite abelian section rank cannot be removed, as Ya. P. Sysak [10] gave an example of a triply factorized group $G = AB = AK = BK$, where $A$, $B$ and $K$ are torsion-free abelian subgroups and $K$ is normal in $G$, but $G$ is not locally nilpotent.

In the hypotheses of Theorem A, if the subgroups $A$ and $B$ are locally nilpotent, one has in particular that the Hirsch–Plotkin radical of $G$ is factorized. Similarly, the factorization of the Fitting subgroup of a soluble $S_1$-group factorized by two nilpotent subgroups is a consequence of the following result.

THEOREM B. Let the soluble-by-finite $S_1$-group $G = AB$ be the product of two subgroups $A$ and $B$, and let $F$ be the Fitting subgroup of $G$. Then $F = A_0F \cap B_0F$, where $A_0$ and $B_0$ are the Fitting subgroups of $A$ and $B$, respectively.

Most of our notation is standard and can for instance be found in [6]. In particular:

If $G$ is a group, $\hat{Z}(G)$ is the hypercentre of $G$.
If $G$ is a group, $\pi(G)$ is the set of prime divisors of the orders of elements of $G$. 

A group $G$ has finite abelian section rank if it has no infinite elementary abelian $p$-sections for every prime $p$.

A group $G$ is an $\mathcal{F}_1$-group if it has finite abelian section rank and the set of primes $\pi(G)$ is finite.

If $Q$ is a group and $M$ is a $Q$-module, $H_n(Q, M)$ and $H^n(Q, M)$ are the $n$-th homology group and the $n$-th cohomology group of $Q$ with coefficients in $M$, respectively.

If $N$ is a normal subgroup of a factorized group $G = AB$, the factorizer of $N$ in $G$ is the subgroup $X(N) = AN \cap BN$.

2. Proof of the Theorems. Our first lemma shows that Theorems A and B hold in the finite case.

**Lemma 1.** Let the finite group $G = AB$ be the product of two subgroups $A$ and $B$, and let $F$ be the Fitting subgroup of $G$. Then $F = A_0F \cap B_0F$, where $A_0$ and $B_0$ are the Fitting subgroups of $A$ and $B$, respectively.

**Proof.** Assume that the lemma is false, and let $G = AB$ be a counterexample of minimal order. If $N_i$ and $N_2$ are distinct minimal normal subgroups of $G$, and $F_i/F_i$ is the Fitting subgroup of $G/N_i (i = 1, 2)$, it follows that $A_0F_i \cap B_0F_i = F_i$, since the result holds for the factor group $G/N_i$. Then

$$A_0F \cap B_0F \leq F_1 \cap F_2 = F,$$

and $F = A_0F \cap B_0F$. This contradiction shows that $G$ has a unique minimal normal subgroup $N$, and hence $F$ is a $p$-group for some prime $p$. Put $F_0 = A_0F \cap B_0F$. Since $F \leq F_0 \leq A_0F$, the subgroup $F_0$ is subnormal in $AF$, and similarly it is subnormal in $BF$. Then it follows from Satz 1 of [11] that $F_0$ is subnormal also in the factorized group $G = (AF)(BF)$. Therefore $F_0$ is not nilpotent, and there exists a prime $q \neq p$ dividing the order of $F_0$. The Sylow $q$-subgroup $Q_1$ of $A_0$ is clearly also a Sylow $q$-subgroup of $A_0F$, and hence $Q = Q_1 \cap F_0$ is a Sylow $q$-subgroup of $F_0$. Moreover $Q$ lies in $A_0$, and so is subnormal in $A$. Let $Q_2$ be the Sylow $q$-subgroup of $B_0$. Then $Q_2$ is a Sylow $q$-subgroup of $B_0F$, and thus there exists $x \in G$ such that

$$Q \leq Q_2^x \leq B_0^x.$$

As $B_0^x$ is the Fitting subgroup of $B^x$, we obtain that $Q$ is subnormal in $B^x$, and Satz 1 of [11] yields that $Q$ is subnormal in $G = AB^x$. Since $F$ is a $p$-group, it follows that $Q = 1$, and this contradiction proves the lemma.

**Lemma 2.** Let the group $G = AB = AK = BK$ be the product of two subgroups $A$ and $B$ and a radicable abelian normal $p$-subgroup $K$ satisfying the minimal condition. If $A_0$ and $B_0$ are nilpotent normal subgroups of $A$ and $B$, respectively, then the subgroup $A_0K \cap B_0K$ is nilpotent.

**Proof.** Assume that the lemma is false, and choose a counterexample

$$G = AB = AK = BK$$

such that $K$ has minimal Prüfer rank. Clearly the subgroups $A_0K$ and $B_0K$ are normal in $G$, and hence also $K_0 = A_0K \cap B_0K$ is a normal subgroup of $G$. Moreover $K_0/K \leq$
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$A_0K/K$ is obviously nilpotent. Suppose that $K_0$ is finite-by-nilpotent. Then there exists a positive integer $r$ such that the index $|K_0: Z_r(K_0)|$ is finite (see [6] Part 1, Theorem 4.25), so that $K \leq Z_r(K_0)$ and $K_0$ is nilpotent. This contradiction shows that $K_0$ is not finite-by-nilpotent. Let $L$ be an infinite $G$-invariant subgroup of $K$ with minimal Prüfer rank. Then $L$ is radicable and all its proper $G$-invariant subgroups are finite. By the minimality of the rank of $K$ the result holds for the factor group $G/L$, and hence $K_0/L$ is nilpotent. It follows that $[L, K_0] \neq 1$, and so $[L, K_0] = L$, since $[L, K_0]$ is radicable and $L$ has no infinite proper $G$-invariant subgroups. This means that $H_0(K_0/L, L) = 0$, and Theorem C of [8] yields that $H^2(G/L, L)$ has finite exponent. Therefore there exists a subgroup $J$ of $G$ such that $G = LJ$ and $L \cap J$ is finite. As $L \cap J$ is normal in $G$ and $K_0$ is not finite-by-nilpotent, also the factor group $G/(L \cap J)$ is a counterexample, and hence we may suppose that $L \cap J = 1$. Thus $K = L \times (J \cap K)$ and $J \cap K = K/L$ is a radicable normal subgroup of $G$. If $J \cap K \neq 1$, the result holds for the factor group $G/(J \cap K)$, and so $K_0/(J \cap K)$ is nilpotent. It follows that $K_0$ is nilpotent, and this contradiction proves that $J \cap K = 1$. Therefore $K = L$, and $K$ has no infinite proper $G$-invariant subgroups. Assume that $A \triangleleft K$ is infinite. As $A \triangleleft K$ is normal in $G = AK$, we obtain that $A \cap K = K$ and $K \leq A$. Then $A_0K$ is nilpotent, so that also $K_0$ is nilpotent. This contradiction shows that $A \cap K$ is finite, and similarly $B \cap K$ is finite. Thus the normal subgroup $N = (A \cap K)(B \cap K)$ of $G$ is also finite, and as above the factor group $G/N$ is a counterexample. Hence we may suppose that $A \cap K = B \cap K = 1$. If $A_1$ and $B_1$ are the Fitting subgroups of $A$ and $B$, respectively, it follows that $A_1K = B_1K$ is a normal subgroup of $G$ containing $K_0$. Since $H_0(K_0/K, K) = 0$, application of Theorem C of [8] yields that $H^1(A_1K/K, K)$ has finite exponent. But $K$ is a radicable abelian $p$-group of finite rank, and hence there exists a finite characteristic subgroup $E$ of $K$ such that the complements of $K/E$ in $A_1K/E$ are conjugate (see [7]). The factor group $G/E$ is also a counterexample, so that we may suppose that the complements of $K$ in $A_1K$ are conjugate. As $A_1$ and $B_1$ are both complements of $K$ in $A_1K$, there exists $x \in G$ such that $A_1^x = B_1$. Write $x = ab$, where $a \in A$ and $b \in B$. Then

$$A_1 = A_1^x = B_1^{b^{-1}} = B_1,$$

so that $A_1 = B_1$ is normal in $G$, and $A_1K$ is nilpotent. This last contradiction completes the proof of the lemma.

LEMMA 3. Let $G$ be a group, and let $K$ be a periodic abelian normal subgroup of infinite exponent of $G$ whose proper $G$-invariant subgroups are finite. Then $K$ is contained in the centre of the Fitting subgroup of $G$. In particular, if $C_G(K) = K$, then $K$ is the Fitting subgroup of $G$.

Proof. Let $N$ be a nilpotent normal subgroup of $G$. Then $KN$ is also nilpotent, and hence $K \cap Z(KN)$ is infinite, since $K$ has infinite exponent (see for instance [6], Theorem 2.23). But $K \cap Z(KN)$ is normal in $G$, and $K$ has no infinite proper $G$-invariant subgroups, so that $K \cap Z(KN) = K$. Therefore $K \leq Z(KN)$ and $N \leq C_G(K)$. This proves that $K$ lies in the centre of the Fitting subgroup of $G$.

Proof of Theorem A. Assume that the result is false, and among all the counterexamples for which the soluble radical $S$ of $G$ has minimal index choose one $G = AB$ such that $S$ has minimal derived length. As the theorem is true for finite groups
by Lemma 1, the group $G$ is infinite, and hence its soluble radical is not trivial. It follows that $G$ contains an abelian normal subgroup $K$ such that the theorem holds for the factor group $G/K$. Write $M = A_0H \cap B_0H$. Then $M/K$ lies in the Hirsch–Plotkin radical of $G/K$, and hence $M$ is ascendant in $G$, as the Hirsch–Plotkin radical of $G/K$ is hypercentral. Since $H < M$, this proves that $M$ is not locally nilpotent. The factorizer $X(H)$ of $H$ in $G = AB$ has a triple factorization

$$X(H) = \tilde{A}B = \tilde{A}H = \tilde{B}H,$$

where $\tilde{A} = A \cap BH$ and $\tilde{B} = B \cap AH$. If $\tilde{A}_0 = A_0 \cap \tilde{A} = A_0 \cap BH$ and $\tilde{B}_0 = B_0 \cap \tilde{B} = B_0 \cap AH$, then $\tilde{A}_0$ and $\tilde{B}_0$ are contained in the Hirsch–Plotkin radicals of $\tilde{A}$ and $\tilde{B}$, respectively. Moreover

$$\tilde{A}_0H \cap \tilde{B}_0H = (A_0 \cap BH)H \cap (B_0 \cap AH)H = A_0H \cap B_0H = M,$$

so that $\tilde{A}_0H \cap \tilde{B}_0H$ is not locally nilpotent. Therefore $X(H) = \tilde{A}B$ is also a minimal counterexample, and without loss of generality we may suppose that $G$ has a triple factorization

$$G = AB = AH = BH.$$

Then the subgroups $A_0H$ and $B_0H$ are normal in $G$, and hence also $M$ is a normal subgroup of $G$. The structure of soluble groups with finite abelian section rank (see [6]) allows us to investigate only the following possible choices for $K$.

**Case 1:** $K$ is finite. By induction on the order of $K$ be may suppose that $K$ is a minimal normal subgroup of $G$. As $M$ is not locally nilpotent, we have that $[K, M] \neq 1$ and hence $[K, M] = K$. Then $H_0(M/K, K) = 0$, and it follows from Theorem 3.4 of [9] that $H^2(G/K, K) = 0$. Therefore there exists a subgroup $J$ of $G$ such that $G = KJ$ and $K \cap J = 1$. The centralizer $C_J(K)$ is normal in $G$, and Lemma 1 shows that the theorem holds for the finite factor group $G/C_J(K)$. In particular $MC_J(K)/C_J(K)$ is locally nilpotent, and so $M$ is locally nilpotent since $K \cap C_J(K) = 1$. This contradiction proves that the subgroup $K$ cannot be finite.

**Case 2:** $K$ is periodic and residually finite. Each primary component $K_p$ of $K$ is finite, and so by Case 1 the group $M/K_p$ is locally nilpotent for every prime $p$. As the groups $K_p$ and $K/K_p$ are $G$-isomorphic, it follows that $K_p$ is hypercentrally embedded in $M$. Then $K$ is hypercentrally embedded in $M$, and $M$ is locally nilpotent, a contradiction.

**Case 3:** $K$ is a radicable $p$-group ($p$ prime). By induction on the rank of $K$ we may suppose that every proper $G$-invariant subgroup of $K$ is finite. In particular, as $K$ is not hypercentrally embedded in $M$, the intersection $\tilde{Z}(M) \cap K$ is finite. It follows from Case 1 that also the factor group $G/\tilde{Z}(M) \cap K$ is a counterexample, and hence it can be assumed that $Z(M) \cap K = 1$. Thus $H^0(M/K, K) = 0$. Moreover, $G/C_G(K)$ is isomorphic with an irreducible linear group by Lemma 5 of [4], and hence it is abelian-by-finite (see [6] Part 1, Theorem 3.21). Then $M/C_G(K)$ is FC-hypercentrally embedded in $G$, and Theorem 3.5 of [9] yields that $H^2(G/K, K) = 0$. Therefore there exists a subgroup $J$ of $G$ such that $G = KJ$ and $K \cap J = 1$. The centralizer $C_J(K)$ is normal in $G$, and $MC_J(K)/C_J(K)$ is not locally nilpotent. Put $\tilde{G} = G/C_J(K)$. As $K$ and $\tilde{K}$ are isomorphic $M$-modules, we obtain that $Z(\tilde{M}) \cap \tilde{K} = 1$. Moreover $C_G(\tilde{K}) = \tilde{K}$, and replacing $G$ by $\tilde{G}$ we may suppose that $C_G(K) = K$ and $Z(M) \cap K = 1$. In particular $K$ is the Fitting
subgroup of \( G \) by Lemma 3, and the factor group \( G/K \) is abelian-by-finite. Let \( L/K \) be an abelian normal subgroup of \( G/K \) such that \( G/L \) is finite. For each positive integer \( n \), the \( n \)-th term \( Z_n(H) \) of the upper central series of \( H \) is a nilpotent normal subgroup of \( G \), so that \( Z_n(H) \leq K \). On the other hand, \( K \) lies in \( Z_m(H) \), since \( H \) is hypercentral, and so \( K = Z_n(H) \). Assume that \( Z(A_0) \cap K \) contains a non-trivial element \( a \), and let \( m \) be the least positive integer such that \( a \in Z_m(H) \). Then \( Z_{m-1}(H) \) is properly contained in \( K \), and hence is finite. Write \( \tilde{G} = G/Z_m(H) \). Then \( \tilde{a} \) centralizes \( \tilde{A}_0 \) and \( \tilde{H} \), so \( \tilde{a} \in Z(\tilde{M}) \cap \tilde{K} \) and \( Z(\tilde{M}) \cap \tilde{K} \neq 1 \). As \( Z_{m-1}(H) \) is finite and \( Z(M) \cap K = 1 \), this contradicts Lemma 2.3 of [2]. Therefore \( Z(A_0) \cap K = 1 \) and hence also \( A_0 \cap K = 1 \). But \( A \cap K \) is contained in \( A_0 \), so that \( A \cap K = 1 \). The same argument shows that \( B \cap K = 1 \). Then the subgroups \( A \) and \( B \) are abelian-by-finite, and in particular the indices \( [A:A_0] \) and \( [B:B_0] \) are finite. The factorizer \( X = \tilde{X}(K) \) of \( K \) in \( G = AB \) has a triple factorization

\[
X = A^*B^* = A^*K = B^*K,
\]

where \( A^* = A \cap BK \) and \( B^* = B \cap AK \). It follows from Lemma 2 that \( A_0K \cap B_0K = (A_0 \cap BK)K \cap (B_0 \cap AK)K \) is nilpotent-by-finite and hence \( X \) is also. Thus the Fitting subgroup \( Y \) of \( X \) is nilpotent and \( X/Y \) is finite. As \( K \leq Y \leq L \), we have that \( Y \leq L \) is a nilpotent normal subgroup of \( L \). Clearly \( K \) is the Fitting subgroup of \( L \), so that \( Y \leq L \) is a singular group and \( K \) has finite index in \( X \). But \( A^* \cap K = B^* \cap K = 1 \), so that \( A^* \) and \( B^* \) are finite, and \( X = A^*B^* \) is also finite. This contradiction completes the proof of this case.

**Case 4:** \( K \) is a periodic radicable group. Each primary component \( K_p \) of \( K \) is radicable, so that Case 3 shows that \( M/K_p \) is locally nilpotent for every prime \( p \). Then \( K/K_p \) is hypercentrally embedded in \( M \), and hence \( K_p \) lies in the hypercentre of \( M \). It follows that \( K \) is hypercentrally embedded in \( M \), and \( M \) is locally nilpotent.

**Case 5:** \( K \) is torsion-free. Let \( T \) be the maximum periodic normal subgroup of \( G \). As \( K \cap T = 1 \), we have that \( MT/T \) is not locally nilpotent, and hence the factor group \( G/T \) is also a counterexample. Thus we may suppose that \( G \) has no non-trivial periodic normal subgroups, so that in particular the set of primes \( \pi(G) \) is finite (see [6] Part 2, Lemma 9.34). It follows that \( G \) is nilpotent-by-polycyclic-by-finite (see [6] Part 2, Theorem 10.33). If \( F \) is the Fitting subgroup of \( G \), then \( K \cap Z(F) \neq 1 \). Consider a non-trivial element \( x \) of \( K \cap Z(F) \), and let \( N \) be the normal closure of \( x \) in \( G \). Thus \( N \) is a cyclic module over the polycyclic-by-finite group \( G/F \), and hence it contains a free abelian subgroup \( E \) such that \( N/E \) is a \( \pi \)-group, where \( \pi \) is a finite set of primes (see [6] Part 2, Corollary 1 to Lemma 9.53).

\[
\left( \bigcap_{p \notin \pi} N^p \right) \cap E = \bigcap_{p \notin \pi} (N^p \cap E) = \bigcap_{p \notin \pi} E^p = 1,
\]

so that \( \bigcap_{p \notin \pi} N^p \) is periodic, and \( \bigcap_{p \notin \pi} N^p = 1 \) since \( N \leq K \) is torsion-free. Let \( p \) be any prime which does not belong to \( \pi \). As \( N^p \neq 1 \), by induction on the torsion-free rank of \( G \) we may suppose that the theorem holds for \( G/N^p \). Therefore \( M/N^p \) is locally nilpotent. Let \( r \) be the Früher rank of \( N \). Then \( |N/N^p| = p^r \), so that \( N/N^p \) lies in the \( r \)-th term of the upper central series of \( M/N^p \). It follows that

\[
[N, M, \ldots, M] \leq \bigcap_{p \notin \pi} N^p = 1,
\]
and so $N \leq Z_r(M)$. Thus $M$ is locally nilpotent, and this last contradiction completes the proof of Theorem A.

**Proof of Theorem B.** Assume that the result is false, and choose a counterexample $G = AB$ such that the radicable part $R$ of the maximum periodic normal subgroup of $G$ has minimal total rank. Put $F_0 = A_0 F \cap B_0 F$. Then Theorem A proves that $F_0$ lies in the Hirsch–Plotkin radical of $G$, and hence is locally nilpotent. The periodic subgroups of the factor group $G/R$ are finite, so that the Hirsch–Plotkin radical and the Fitting subgroup of $G/R$ coincide (see [6] Part 2, p. 35), and it follows again from Theorem A that $F_0/R$ is contained in the Fitting subgroup of $G/R$. As the Fitting subgroup of an $S^r$-group is nilpotent, we obtain that $F_0$ is subnormal in $G$ and $F_0/R$ is nilpotent. Also, in an $S^r$-group each nilpotent subnormal subgroup lies in the Fitting subgroup, so $F_0$ is not nilpotent and $R \neq 1$. Since $F_0$ is locally nilpotent, we have also that $F_0$ is not finite-by-nilpotent. Let $S$ be an infinite $G$-invariant subgroup of $R$ with minimal total rank. Then $S$ is a radicable abelian $p$-group for some prime $p$, and all its proper $G$-invariant subgroups are finite. Thus $G/C_S(S)$ is isomorphic with an irreducible linear group by Lemma 5 of [4], and hence it is abelian-by-finite. Moreover $G/S$ is an $S^r$-group, so that its Fitting subgroup $F_0/S$ is nilpotent and $F_0 \leq F_1$ by the minimal choice of $G$. Therefore $[S, F_1] \neq 1$, and hence $[S, F_1] = S$. Thus $H_0(F_1/S, S) = 0$, and Theorem C of [8] yields that $H^2(G/S, S)$ has finite exponent. Then there exists a subgroup $J$ of $G$ such that $G = SJ$ and $S \cap J$ is finite. The subgroup $S \cap J$ is normal in $G$, and the factor group $G/(S \cap J)$ is also a counterexample, since $F_0$ is not finite-by-nilpotent. Therefore we may suppose that $S \cap J = 1$, so that $R = S \times (J \cap R)$, where $J \cap R$ is a radicable normal subgroup of $G$. Clearly $F_0(J \cap R)/(J \cap R)$ is not nilpotent, so $J \cap R = 1$ and $R$ has no infinite proper $G$-invariant subgroups. The centralizer $C_J(R)$ is normal in $G$, and the periodic subgroups of $J/C_J(R)$ are finite (see [6] Part 1, Corollary to Lemma 3.28), so that $G/C_J(R)$ is an $S^r$-group. As $C_J(R) \cap R = 1$, the group $F_0/C_J(R)(C_J(R)$ is not nilpotent, and the theorem is false for the group $G/C_J(R)$. Clearly $R$ is $G$-isomorphic with the radicable part of the maximum periodic normal subgroup of $G/C_J(R)$, so that $G/C_J(R)$ is also a minimal counterexample. Moreover

$$C_{J/C_J(R)}(RC_J(R)/C_J(R)) = 1,$$

and hence we may suppose that $C_J(R) = 1$ and $C_G(R) = R$. Thus it follows from Lemma 3 that $R$ is the Fitting subgroup of $G$. The factorizer $X = X(R)$ of $R$ in $G$ has the triple factorization

$$X = A^* B^* = A^* R = B^* R,$$

where $A^* = A \cap BR$ and $B^* = B \cap AR$. Write $A_0^* = A_0 \cap BR$ and $B_0^* = B_0 \cap AR$. Then $A_0^*$ and $B_0^*$ are nilpotent normal subgroups of $A^*$ and $B^*$, respectively, and Lemma 2 shows that $A_0^* R \cap B_0^* R$ is nilpotent. Since

$$A_0^* R \cap B_0^* R = (A_0 \cap BR)R \cap (B_0 \cap AR)R = A_0 R \cap B_0 R = A_0 F \cap B_0 F = F_0,$$

we have that $F_0$ is nilpotent. This contradiction completes the proof of Theorem B.
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