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On the class of weak U-Dunfords-Pettis operators

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Abstract. **In this paper we define a new class of operators called weak U-Dunford-Pettis, which generalizes the U-Dunford-Pettis, weak Dunford-Pettis and order Dunford-Pettis classes, then we also give a characterization for this class, which we compare with some lattice properties, we then set out the conditions under which this class coincides with the U-Dunford-Pettis class, the weak Dunford-Pettis class and the order Dunford-Pettis class.**

1 Introduction

We recall from [1, Definition 3.1] that an operator $T : E \to Y$ from a Banach lattice E to a Banach space Y is called a U-Dunford-Pettis operator if $\|Tx_n\| \to 0$ for every order bounded weakly null sequence (x_n) in E. Alternatively, T maps order bounded weakly relatively compact subsets of E to relatively compact subsets of Y (see [1, Theorem 3.2]).

Furthermore, as defined in [3, Section 2, page 231], an operator T from the Banach space Z to the Banach space X is called weak Dunford-Pettis if for every weakly null sequence (x'_n) in X' and every weakly null sequence (z_n) in Z, it holds that $\lim_{n\to\infty} x'_n(Tz_n) = 0$. Alternatively, T maps a weakly compact subset of Z onto a Dunford-Pettis set in X.

From [6, section 1, page 186] we define an operator $T : E \to X$ to be order Dunford-Pettis if it maps order bounded subsets onto Dunford-Pettis sets in X. In this paper, we introduce a new class of operators from a Banach lattice to a Banach space, nominated " weak U-Dunford-Pettis operators " (abbreviated as weak U-DP), which generalizes the classes of U-Dunford-Pettis operators, weakly Dunford-Pettis operators (abbreviated as weak DP) and order Dunford-Pettis operators (abbreviated as order DP). This article is organized as follows: First, we defined the weak U-DP operator in Definition 3.1. Then we used some examples to show the strict inclusions between the class of weak U-DP operators and other classes of operators. In Theorem 3.4, we present a characterization of the new class. In the Corollaries 3.5 and 3.6, we give a sufficient and necessary condition on Banach lattices such that for every operator (or order bounded operator) is weak U-Dunford-Pettis. In Theorem 3.10, we establish sufficient conditions under which certain lattice properties in Banach lattices coincide. Finally, Propositions 3.14, 3.16, 3.18, 3.19 and Theorem3.20 provide

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necessary or sufficient conditions for a weak U-DP operator to be classified as U-DP, weak DP and order DP, respectively.

2 Preliminaries

A norm bounded subset A of a Banach space X is called a Dunford-Pettis set if, for every weakly convergent sequence (x'_n) in the dual space X', we have $\sup_{x\in A}|x'_n(x)| \to 0$ (see [2, Theorem 5.98 (Andrews)]). A Banach space X has the Dunford-Pettis property (or simply DPP), if every weakly compact operator T from X into another Banach space Y is Dunford-Pettis operator (Andrews [2]), if and only if for every weakly null sequence (y'_n) of X' and weakly null sequence (x_n) of X we have $y'_n(x_n) \xrightarrow{w} 0$ ([6, Proposition 2.2 and Proposition 2.3]). A Banach lattice E has the U-Dunford-Pettis property (or simply UDPP), if each weakly compact operator $T : E \to Y$ is U-Dunford-Pettis for each Banach space Y. To illustrate, c_0 has $UDPP$. In fact, for each weakly compact operator $T : c_0 \to Y$, by applying [11, Proposition 3.1] the operator $T = T \circ Id_{c_0}$ is U-Dunford-Pettis because Id_{c_0} is U-Dunford-Pettis. This is due to the fact that c_0 is discrete and its norm is order continuous, as outlined in [1, Proposition 3.1]. We note that $L^2[0, 1]$ does not have $UDPP$. This is due to the fact that $Id_{L^2[0,1]}$ is a weakly compact operator. Furthermore, as $L^2[0, 1]$ is not discrete, as per [1, Proposition 3.1], $Id_{L^2[0,1]}$ is not U-Dunford-Pettis.

A Banach lattice E is said to have the order U-Dunford-Pettis property (or simply o-UDPP) if every order weakly compact operator $T : E \to Y$ is U-Dunford-Pettis for every Banach space Y (see [11, Definition 3.2]). Note that if a Banach lattice is o-UDPP, it is necessarily UDPP, ($L^2[0.1]$ does not have o-UDPP, and l^∞ has o-UDPP). A Banach lattice E is said to have the AM-compactness property, if every weakly compact operator defined on E into a Banach space X is AM-compact [7, section 3, page 169]. The lattice operations of a Banach lattice E is said to be weakly sequentially continuous, if every weakly null sequence (x_α) in E we have $|x_\alpha| \xrightarrow{w} 0$, $x^+_\alpha \xrightarrow{w} 0$ and $x^-_\alpha \xrightarrow{w} 0$. An operator $T : X \to Y$ between two Banach spaces called a Dunford-Pettis, if $\|Tx_n\| \to 0$ for all weakly null sequence (x_n) in X (see [2, Section 5.4,page 340]). An operator $R : G \to X$ from a normed vector lattice G into a Banach space X is order weakly compact if, and only if, $\|S(z_n)\| \to 0$ as for every order bounded weak null sequence $(z_n)_n$in the positive cone G^+ (see [11, Corollary 3.4.9]).

We need to establish some notations and definitions. A Banach lattice is a Banach space $(E, \|\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. Note that if E is a Banach lattice, then its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. A norm $\|.\|$ of a Banach lattice G is order continuous if, for any generalized sequence (z_β) such that $z_\beta \downarrow 0$ in E, (z_β) converges to 0 for the norm $\|.\|$, where the notation $z_\beta \downarrow 0$ means that (z_β) is decreasing, its infimum exists, and $inf(z_\beta) = 0$. An operator $T : E \to F$ between two Banach lattices is a bounded linear mapping.

- It is positive if $0 \leq T(x)$ in F whenever $0 \leq x$ in E.
- It is order bounded if $T(A)$ is order bounded in F for every order bounded subset A in E.

- It is regular if there are two positive operators T_1, T_2 such that $T = T_1 - T_2$.

We note by:

- $DP_u(E, Y)$: The collection of all U-Dunford-Pettis operators from Banach lattice E to Banach space Y.
- $DP_o(E, Y)$: The collection of all order Dunford-Pettis operators from Banach lattice E to the Banach space Y.
- $wDP(E, Y)$: The collection of all weak Dunford-Pettis operators from Banach lattice E to the Banach space Y.

3 Main Results

3.1 The class of weak U-Dunford-Pettis

Definition 3.1 An operator $T : E \longrightarrow Y$ acting between the Banach lattice E and the Banach space Y is said to be weak U-Dunford-Pettis (abb. weak U-DP) if $\langle Tx_n, y'_n \rangle \to 0$ for every order bounded weakly null sequence (x_n) in E and for every weakly null sequence (y'_n) in Y'.

We denote the collection of all weak U-Dunford-Pettis operators from Banach lattice E to Banach space Y by $wDP_u(E, Y)$. We note that $DP_u(E, Y) \subset wDP_u(E, Y)$, but the reverse inclusion is not generally true. In fact, let (r_n) denote the sequence of Rademacher functions on $[0, 1]$, and let $S : L^1[0, 1] \longrightarrow l^\infty$ be the positive operator defined by

$$S(g) = \left(\int_0^1 g(x) r_1^+(x) dx, \int_0^1 g(x) r_2^+(x) dx, \dots \right).$$

Given that $r_n \xrightarrow{w} 0$, $(r_n) \subset [-1, 1]$, and $\|S(r_n)\| \geq \int_0^1 r_n(t) r_n^+(t) dt = \frac{1}{2}$, we observe that S is not a U-Dunford-Pettis operator. However, because $L^1[0, 1]$ possesses the Dunford-Pettis property, it follows that S is a weak Dunford-Pettis operator. Therefore, S is a weak U-Dunford-Pettis operator.

Additionally, we have that $DP_o(E, Y)$ is strictly included in $wDP_u(E, Y)$. Indeed, the natural embedding $J : L^\infty[0, 1] \longrightarrow L^2[0, 1]$ is a positive U-Dunford-Pettis operator but is not AM-compact (see [1, Remark 3.4]). Therefore, it is weak U-Dunford-Pettis (since $DP_u(E, Y) \subset wDP_u(E, Y)$). Moreover, J fails to be an order Dunford-Pettis operator. If the operator $J = Id_{L^2[0,1]} \circ J$ were AM-compact (see [6, Theorem 2.7]), it would lead to a contradiction.

We also have that $wDP(E, Y) \subset wDP_u(E, Y)$ strictly. In fact, the operator Id_{l^2} is weak U-Dunford-Pettis. This is due to the fact that l^2 has the AM-compactness property, which implies that Id_{l^2} is order Dunford-Pettis (see in [9, Proposition 3.4]). Although Id_{l^2} is weak U-Dunford-Pettis, it is not weak Dunford-Pettis. If that's not the case, then Id_{l^2} is a Dunford-Pettis operator. This is because Id_{l^2} is weakly compact (since l^2 is reflexive) and satisfies $Id_{l^2} = Id_{l^2} \circ Id_{l^2}$ (as shown in [2, Theorem 5.99]). This means that the unit ball of l^2 would be relatively compact, which is a contradiction since l^2 is infinite-dimensional.

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Now we will mention some algebraic properties related to this class of operators.

***Proposition* 3.1** *Let E and F be two Banach lattices and X and Y be two Banach spaces. We have the following assertions:*

(1) If $T : E \longrightarrow X$ is a weak U-Dunford-Pettis operator, then $S \circ T$ is weak U-Dunford-Pettis operator for every operator $S : X \longrightarrow Y$.
(2) If $T : F \longrightarrow Y$ is a weak U-Dunford-Pettis operator and $S : E \longrightarrow F$ is an order bounded operator, then $T \circ S$ is weak U-Dunford-Pettis operator.

Proof The proof is straightforward. ■

We note that $L^2[0,1]$ does not have the AM-compactness property, and its lattice operations are not weakly sequentially continuous. The identity operator $Id_{L^2[0,1]}$, although positive, is not a weak U-Dunford-Pettis operator. However, since $L^2[0,1]$ is reflexive, the theorem 3.4 states that $Id_{L^2[0,1]} = Id_{L^2[0,1]} \circ Id_{L^2[0,1]}$ is U-Dunford-Pettis. This leads to the conclusion that $L^2[0,1]$ must be discrete (as shown in [12, Proposition 3.1]), which is a contradiction. We will now outline the sufficient conditions on lattices that ensure an operator is weak U-Dunford-Pettis.

***Proposition* 3.2** *Let E and F be two Banach lattices, and let Y be a Banach space.*

(1) If the lattice operations of E are weakly sequentially continuous, then every operator $T : E \to Y$ is weak U-DP.
(2) If the lattice operations of F are weakly sequentially continuous, then every order bounded operator $T : E \to F$ is weak U-DP. That is, $L_b(E,F) \subset wDP_u(E,F)$.
(3) If F has the AM-compactness property, then every regular operator $T : E \to F$ is weak U-Dunford-Pettis. Alternatively, $L_r(E,F) \subset wDP_u(E,F)$.

Proof (1) Let (x_n) be an order bounded weak null sequence of E such that E has weakly sequentially continuous lattice operations, and T be an operator from E to another Banach space Y. Then $x_n \xrightarrow{w} 0 \implies x_n \xrightarrow{|w|} 0$ which implies that $\{0, x_1, x_2, ...\}$ is order $|\sigma|$-weakly compact subset of E. Hence, by [6, Theorem 2.6], $A = \{0, x_1, x_2, ...\}$ is a Dunford-Pettis set of E. So $T(A) = \{0, T(x_1), T(x_2), ...\}$ is also a Dunford-Pettis set of Y.

(2) Let (x_n) be an order bounded weakly null sequence of E and $T : E \to F$ be an order bounded operator. Then $(T(x_n))$ is an order bounded weakly null sequence in F. Since F has weakly sequentially continuous lattice operations, then $B = \{0, T(x_1), T(x_2), ...\}$ is an order $|\sigma|$-weakly compact subset of F. Hence, by [6, Theorem 2.6], B is a Dunford-Pettis set of F.

(3) Let $T : E \to F$ be a regular operator. Then there exist two positive operators $T_1, T_2 : E \to F$ such that $T = T_1 - T_2$.

Now let (x_n) be an order bounded weakly null sequence in E. Then there exists $x \in E^+$ such that $(x_n) \subset [-x, x]$, and let (f_n) be a weakly null sequence in F'. We have

$$|f_n(Tx_n)| \leq |f_n|(|Tx_n|) \leq |f_n|(|T_1x_n|) + |f_n|(|T_2x_n|).$$

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Since F satisfies the AM-compactness property, by [6, Theorem 3.7], for all $y \in F$, $|f_n|(y) \longrightarrow 0$. This implies that

$$|f_n|(|T_i x_n|) \leq |f_n|\,(|T_i|(|x_n|)) \leq |f_n|(T_i x) \longrightarrow 0, \quad i = 1, 2.$$

So $|f_n(Tx_n)| \longrightarrow 0$, showing that T is weak U-Dunford-Pettis.

∎

Remark 3.3 In general, if a Banach lattice E has the AM-compactness property, its lattice operations are not necessarily weakly sequentially continuous. For example, the Banach lattice $L^1[0, 1]$ has the AM-compactness property, but its lattice operations are not necessarily weakly sequentially continuous. Conversely, the weakly sequentially continuity of the lattice operations of E does not imply that E admits the AM-compactness property. Indeed, the lattice operations of l^∞ are weakly sequentially continuous, but l^∞ does not have the AM-compactness property (see [5, Remark 2.2 (4)]).

The next result characterizes the weak U-Dunford-Pettis operator and it is an analogue of [2, Theorem 5.99].

***Theorem* 3.4** *For an operator $T : E \longrightarrow X$ from a Banach lattice E to a Banach space X, the following statements are equivalent:*

(1) T is a weak U-Dunford-Pettis operator.
(2) T carries order bounded relatively weakly compact subsets of E to Dunford-Pettis subsets of X.
(3) If S is a weakly compact operator from X to an arbitrary Banach space, then $S \circ T$ is an U-Dunford-Pettis operator.

Proof The proof is similar to that of [2, Theorem 5.99].

∎

In the following results, we give a some characterisation of of the weak U-DP class of operators, using the U-Dunford-Pettis property (UDPP) that it is defined in [10, Definition 3.2], as well as in corollaries 3.5 and 3.6.

***Corollary* 3.5** *Let E be a Banach lattice. The following assertions are equivalent:*

(1) Every operator $S : E \longrightarrow Y$ is weak U-Dunford-Pettis for every Banach space Y.
(2) Id_E is weak U-Dunford-Pettis.
(3) E has UDPP.

Proof (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1) Let $S : E \longrightarrow Y$ be an operator, since Id_E is weak U-DP operator then $S = S \circ Id_E$ is also a weak U-DP operator see Proposition 3.1.

(2) $\Longrightarrow$ (3) Let $S : E \longrightarrow Y$ be a weakly compact operator, since Id_E is a weak U-DP operator then $S = S \circ Id_E$ is also an U-Dunford-Pettis operator see Theorem 3.4 (1) $\Longrightarrow$ 3)), then E has UDPP.

$3 \implies 2$) Let $S : E \longrightarrow Y$ be a weakly compact operator and E has UDPP then S is an U-Dunford-Pettis operator, since $S = S \circ Id_E$ then by (3) $\implies$ 1)) of Theorem 3.4, Id_E is weak U-DP. ■

We have also similarly,

***Corollary* 3.6** *Let F be a Banach lattice. The following assertions are equivalent:*

(1) Every order bounded operator $S : E \longrightarrow F$ is weak U-Dunford-Pettis for every Banach lattice E.
(2) Id_F is weak U-Dunford-Pettis.
(3) F has UDPP.

By using the Corollary 3.5 and Definition 3.1, we can give an explicit version of the UDPP.

***Corollary* 3.7** *Let E be a Banach lattice. The following assertions are equivalent:*

(1) E has UDPP.
(2) For every order bounded weakly null sequence (x_n) in E, and every weakly null sequence (f_n) in E', we have $f_n(x_n) \longrightarrow 0$.
(3) For every order bounded weakly null sequence (x_n) in E, the set $\{x_1, x_2, ..., x_n, ..\}$ is a Dunford-Pettis set.

As consequence of corollary 3.5 and proposition 3.2 we have the following result,

***Corollary* 3.8** *Let E be a Banach lattice. If the lattice operations of E are weakly sequentially continuous or E has the AM compactness property, then E has the UDPP.*

Proof (1) If the lattice operations of E are weakly sequentially continuous, then the positive operator Id_E is weakly U-DP (see Proposition 3.2) and by the Corollary 3.5, E has UDPP.
(2) If E has the AM-compactness property, then Id_E is order Dunford-Pettis (see [9, Proposition 3.1]). Then it is weak U-Dunford-Pettis, so E has UDPP.

■

Remark 3.9 We note that if E is a Banach lattice with an order unit e, then $B_E = [-e, e]$. In this case, every order weakly compact operator from E to any Banach space Y is also weakly compact. Which implies that E has the UDPP, if and only if E has the o-UDPP). According to [1, Proposition 3.2], E has o-UDPP if and only if the lattice operations of E are weakly sequentially continuous, since every weakly null sequence in E is also an order bounded weakly null sequence.

It is obvious that if E has o-UDPP, then it has UDPP. But the converse is not generally true. In fact, if we take $E = L^1[0, 1]$, then $E' = L^\infty[0, 1]$, since the lattice operations of E' are weakly sequentially continuous, then E has AM-compactness property according to [3, Theorem 3.3]), which implies that Id_E is order DP (see [9, Proposition 3.4]), so it is weak U-DP, thus E has UDPP (see Corollary 3.5). In addition, E does not have o-UDPP. In fact, the operator $S : L^1[0,1] \to l^\infty$ defined in subsection 3.1 is an order weakly compact operator, but not an U-DP operator. Note that the norm of $(L^1[0,1])' = L^\infty[0,1]$ is not order continuous. Now we will give some other important characterizations of the UDPP on a Banach lattice.

***Theorem* 3.10** *Let E be a Banach lattice such that the norm of E and the norm of E' are order continuous, then the following statements are equivalent:*

(1) E has the UDPP.
(2) E has the AM-compactness property .
(3) E has the o-UDPP.
(4) E is discrete.
(5) The lattice operations of E are weakly sequentially continuous.

Proof • (1) $\Longrightarrow$ (2) if E has UDPP, then Id_E is weak U-DP, since the norm of E is order continuous(see Proposition 3.20), then Id_E is order DP, implying that E has the AM compactness property (see [9, Proposition 3.4]).

- (2) $\Longrightarrow$ (1) (see Corollary 3.8)
- (1) $\Longrightarrow$ (3) Let us demonstrate this by contradiction: let us assume that E has the UDPP but not the o-UDPP. According to [1, Proposition 3.2], there exists a weakly null and order bounded sequence (x_n) in E such that $|x_n| \stackrel{w}{\nrightarrow} 0$. This implies the existence of a positive functional $f \in E'^{+}$ such that $f(|x_n|) \nrightarrow 0$. Consequently, for any $\epsilon > 0$, there exists a subsequence (y_n) of $((x_n))$ such that $f(|y_n|) > \epsilon$ for all $n \in \mathbb{N}^\star$.
  We have:
  $f(|y_n|) = \sup\{|g(y_n)| : g \in E', |g| \le f\}$
  which implies that there exist functional $g_m \in E'$ with $|g_m| \le f$ such that:

  $$f(|y_n|) \le |g_n(y_n)| + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}^\star.$$

  Since the norm of E' is order continuous and ($|g_n| \le f$), the sequence $((g_n))$ must have a weakly convergent subsequence $(g_{\phi(n)})$, converging to some g (i.e., $(g_{\phi(n)} \xrightarrow{w} g)$). Thus, we also have:

  $$\epsilon < f(|y_{\phi(n)}|) \le |g_{\phi(n)}(y_{\phi(n)})| + \frac{1}{\phi(n)} \le |(g_{\phi(n)} - g)(y_{\phi(n)})| + |g(y_{\phi(n)})| + \frac{1}{\phi(n)} \to 0.$$

  (This conclusion follows because E has the UDPP, and since $(g_{\phi(n)} - g) \xrightarrow{w} 0$ in (E') and $y_{\phi(n)} \xrightarrow{w} 0$ (which means $(|g(y_{\phi(n)})| \to 0)$), in addition $(y_{\phi(n)})$ is order bounded in E, which implies $(g_{\phi(n)} - g)(y_{\phi(n)}) \to 0$ see corollary 3.7).
  However, this is a contradiction, as we initially assumed that the sequence $(x_1, x_2, \ldots)$ forms a Dunford-Pettis set (see [6, Proposition 2.2]).
- (3) $\Longrightarrow$ (1) (obvious).

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- (3) $\Longrightarrow$ (4): Given that the norm of E is order continuous, it follows that Id_E is order weakly compact. As E has o-UDPP, we can conclude that Id_E is U-DP, which in fact implies that E is discrete (see [12, Proposition 3.1]).
- (4) $\Longrightarrow$ (5) If E is discrete, then E' is discrete (see [3, Corollary 2.3]), which implies that the lattice operations of E are weakly sequentially continuous (see [8, Corollary 2.2]).
- (5) $\Longrightarrow$ (3): Let T be an order weakly compact operator from E to an arbitrary Banach space X. Since the lattice operations of E are weakly sequentially continuous, T is an U-DP operator. In fact, for any order bounded weakly null sequence (x_n), we have

$$\|Tx_n\| = \|Tx_n^+ - Tx_n^-\| \leq \|Tx_n^+\| + \|Tx_n^-\| \to 0.$$

This follows from [11, Corollary 3.4.9], which shows that T is U-DP, and thus E has o-UDPP.

∎

Remark 3.11 The order continuity condition of the norm of E is essential. In fact, we assume that $E = L^\infty[0,1]$. Note that $E' = (L^\infty[0,1])'$ is order continuous, but E is not. In this case, note that E has UDPP, o-UDPP and the lattice operations of E are weakly sequentially continuous. But E is not discrete and does not have AM-compactness property.

In the following result we study the domination problem for the class of weak U-Dunford-Pettis operators.

***Theorem* 3.12** *If a positive operator $S : E \to F$ is dominated by a weak U-Dunford-Pettis operator $T : E \to F$, then S is a weak U-Dunford-Pettis operator.*

Proof : Let $S, T : E \to F$ be two positive operators between Banach lattices such that $0 \leq S \leq T$, with T being a weak U-Dunford-Pettis operator. Let A be an order bounded relatively weakly compact subset of E.

By Theorem 3.4, $T(A)$ is a Dunford-Pettis set. Therefore, for every weakly null sequence (y'_n) in F', it converges uniformly to zero on the set $T(A)$ (as established in [10, Theorem 1]).

Additionally, since $S(A) \subset T(A)$ (because $0 \leq S \leq T$ implies $S[-x,x] \subset T[-x,x]$ for all $x \in E^+$), we have:

$$\sup_{x \in S(A)} |y'_n(x)| \leq \sup_{x \in T(A)} |y'_n(x)|.$$

This implies that (y'_n) converges uniformly to zero on the set $S(A)$. Therefore, we conclude that S is a weak U-Dunford-Pettis operator.

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3.2 The relationship between the class of weak U-Dunford-Pettis and other classes

We note that:

- $Id_{L^\infty[0,1]}$ is weak U-Dunford Pettis (because the lattice operations of $L^\infty[0, 1]$ are weakly sequentially continuous). However, $Id_{L^\infty[0,1]}$ is not an order weakly compact operator, and therefore it is not a weakly compact operator.
- Furthermore, $Id_{L^2[0,1]}$ is a weakly compact (respectively order-weakly compact) operator, but it is not weak U-Dunford-Pettis.

Similar to the DPP, UDPP and o-UDPP properties, we can define new properties on Banach lattices.

***Definition* 3.2** (1) A Banach lattice G is said to have the weak U-Dunford-Pettis property (wUDPP) if, for every weakly compact operator $S : G \to X$, the operator S is weak U-Dunford-Pettis for any Banach space X.

(2) A Banach lattice G is said to have the order weak U-Dunford-Pettis property (owUDPP) if, for every order weakly compact operator $S : G \to X$, the operator S is weak U-Dunford-Pettis for any Banach space X.

(3) A Banach lattice G is said to have the reciprocal weak U-Dunford-Pettis property (RwUDPP) if, for every weak U-Dunford-Pettis operator $S : G \to X$, the operator S is weakly compact for any Banach space X.

(4) A Banach lattice G is said to have the reciprocal order weak U-Dunford-Pettis property (RowUDPP) if, for every weak U-Dunford-Pettis operator $S : G \to X$, the operator S is order weakly compact for any Banach space X.

Remark 3.13 It is evident that if a Banach lattice E has o-UDPP, then it must also have owUDPP. For example, we may consider the Banach lattices l^p, where $1 \leq p < \infty$. These have owUDPP, however, $L^2[0, 1]$ does not possess this property. This is due to the fact that $Id_{L^2[0,1]}$ is order weakly compact and is not weak U-DP.

3.2.1 Comparison of the weak U-DP operator with the U-DP and the weak DP operators

Now, we give necessary and sufficient conditions for a weak U-Dunford-Pettis operator to be U-Dunford-Pettis.

***Proposition* 3.14** *If any positive weak U-Dunford-Pettis operator T from E to F is U-Dunford-Pettis, then one of the following assertion is valid:*

(1) E has o-UDPP.
(2) The norm of F is order continuous.

Proof If we assume that neither E has o-UDPP nor the norm of F is order continuous, then [1, Theorem 3.1] proves that there exist $S, T : E \to F$ such that $0 \leq S \leq T$, T is U-Dunford-Pettis and S is not U-Dunford-Pettis.

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Since T is an U-Dunford-Pettis operator, it is also weak U-Dunford-Pettis. By applying Theorem 3.12, we can also conclude that S is weak U-Dunford-Pettis. However, our hypothesis implies that S is U-Dunford-Pettis, which is a contradiction. ∎

Remark 3.15 • Condition 2) of Proposition 3.14 is not sufficient. Indeed, for $E = F = L^1[0, 1]$, we have that $Id_{L^1[0,1]}$ is weak U-Dunford-Pettis, since it admits UDPP. However, it is not U-Dunford-Pettis because $L^1[0, 1]$ is not discrete (see [12, Proposition 3.1]), although the norm of $L^1[0, 1]$ is order continuous.

• Similarly, condition 1) of Proposition 3.14 is not sufficient. For example, if we take $E = F = c$, we find that the identity operator Id_c is weak U-Dunford-Pettis. This is because the space c has the U-Dunford-Pettis property UDPP due to the continuity of its lattice operations. Consequently, it also satisfies the o-UDPP, which implies it has the UDPP. However, c does not possess the RowUDP property, since Id_c is not order weakly compact, as its norm is not order continuous. Additionally, Id_c is not U-Dunford-Pettis either, because its norm fails to be order continuous (see [12, Proposition 3.1]).

***Proposition* 3.16** *Let E and F be two Banach lattices.*

(1) If E has the o-UDPP and the RowUDPP, then every weak U-DP operator T from E to every Banach space X is U-DP.

(2) If F is discrete and its norm is order continuous, then for every order bounded weak U-Dunford-Pettis operator $T : G \longrightarrow F$, it follows that T is U-Dunford-Pettis for every Banach lattice G.

Proof (1) Let $T : E \longrightarrow X$ be a weak U-Dunford-Pettis operator. Given that E satisfies the de Row UDPP condition, it follows that T is an order weakly compact. Moreover, since E also satisfies the o-UDPP condition, we conclude that T is an U-Dunford-Pettis operator.

(2) Let $T : G \longrightarrow F$ be an order bounded weak U-Dunford-Pettis operator. If F is a discrete space with an order continuous norm, then the identity operator Id_F is U-Dunford-Pettis (as shown in [12, Proposition 3.2]). Since $T = Id_F \circ T$, we can apply [1, Proposition 3.1] to conclude that T is also U-Dunford-Pettis.

∎

Other sufficient condition,

***Proposition* 3.17** *Let be E a Banach lattice and Y a Banach space, if Y is reflexive, then every weak U-Dunford-Pettis operator $T : E \longrightarrow Y$ is U-Dunford-Pettis .*

Proof obvious ($T = Id_Y \circ T$ and Id_Y is weakly compact, by Proposition 3.1 T is U-DP). ∎

We now provide sufficient conditions for weak U-Dunford-Pettis operators to be weak Dunford-Pettis operators in the following result.

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***Proposition* 3.18** *Let $S : E \longrightarrow Z$ be a weak U-Dunford-Pettis operator, where E is a Banach lattice that admits an order unit, and Z is a Banach space, then S is a weak Dunford-Pettis operator.*

Proof Assume that a Banach lattice E has an order unit e, and let S be a weak U-Dunford-Pettis operator from E to a Banach space Z. We need to show that S is a weak Dunford-Pettis operator, which is equivalent to proving that for any relatively weakly compact subset A of E, the image $S(A)$ is a Dunford-Pettis set in Z. Since A is norm-bounded in E, it is also order-bounded. This is because there exists $r > 0$ such that $A \subset rB_E = r[-e, e]$. Given that S is a weak U-Dunford-Pettis operator, it follows that $S(A)$ is a Dunford-Pettis set in Z (see Theorem 3.4).

■

3.2.2 Comparison of the weak U-DP operator with the order DP operator

We give sufficient conditions for a weak U-DP operator from Banach lattice to Banach space to be an order DP operator.

***Proposition* 3.19** *Let E be a Banach lattice and Y a Banach space. Then, every weak U-Dunford-Pettis operator $T : E \longrightarrow Y$ is an order Dunford-Pettis operator. Whenever one of the following assertions is valid:*

(1) the norm of E is order continuous.
(2) E has the AM-compactness property.
(3) Y has the DPP and is reflexive.

Proof Let be $T : E \longrightarrow Y$ a weak U-Dunford-Pettis operator.

(1) If the norm of E is order continuous, then for all $x \in E^+$ the $[-x, x]$ is an order bounded relatively weakly compact subset of E by the hypothesis that $T([-x, x])$ is a Dunford-Pettis set of F, so T is order Dunford-Pettis.
(2) If E has AM-compactness property then Id_E is order Dunford-Pettis operator (see [9, Proposition 3.4]) and since that $T = T \circ Id_E$ then T is order Dunford-Pettis (see [12, Proposition 3.1]).
(3) Let $u \in E^+$. Then $T[-u, u]$ is a norm-bounded set in Y, since Y is reflexive. Therefore, $T[-u, u]$ is relatively weakly compact (see [2, Theorem 3.32]) and by the hypothesis that Y has DPP, we can conclude that $T[-u, u]$ is a Dunford-Pettis set (see [6, Proposition 2.3]). Thus, T is an order Dunford-Pettis operator.

■

In the following we characterize weak U-Dunford-Pettis operators between two Banach lattices as order Dunford-Pettis operators under equivalent conditions.

***Theorem* 3.20** *Let E and F be two Banach lattices such that E and F are σ-Dedekind complete and F has the DPP. The following assertions are equivalent:*

(1) Every weak U-Dunford-Pettis operator $T : E \longrightarrow F$ is order Dunford-Pettis operator.

(2) Every positive weak U-Dunford-Pettis operator $T : E \longrightarrow F$ is order Dunford-Pettis operator.
(3) One of this assertions is valid:

(a) The norm of E is order continuous.
(b) The norm of F is order continuous.

Proof It is obvious that (3) implies (1) and (1) implies (2).

To show that (2) implies (3), it is assumed that if the norm of E and the norm of F are not order continuous, then there exists a positive operator $T : E \longrightarrow F$ which is weak U-Dunford-Pettis and such that T is not order Dunford-Pettis. Since E and F are σ-Dedekind complete, it follows from Theorem 4.56 [2] that there exist two positive projections $P_1 : E \longrightarrow l^\infty$ and $P_2 : F \longrightarrow l^\infty$ and two canonical injections $i_1 : l^\infty \longrightarrow E$ and $i_2 : l^\infty \longrightarrow F$ which are all weak U-Dunford-Pettis because the lattice operations of l^∞ are sequentially continuous (applying the Proposition 3.2). Then the composition operators $T = i_2 \circ P_1 : E \longrightarrow F$ is weak U-Dunford-Pettis see Proposition 3.1, but not order Dunford-Pettis (see [5, Remark 2.2(4)] and [9, Proposition 3.4]). This is clear from the fact that $Id_{l^\infty} = P_2 \circ T \circ i_1$ is not order Dunford-Pettis. ∎

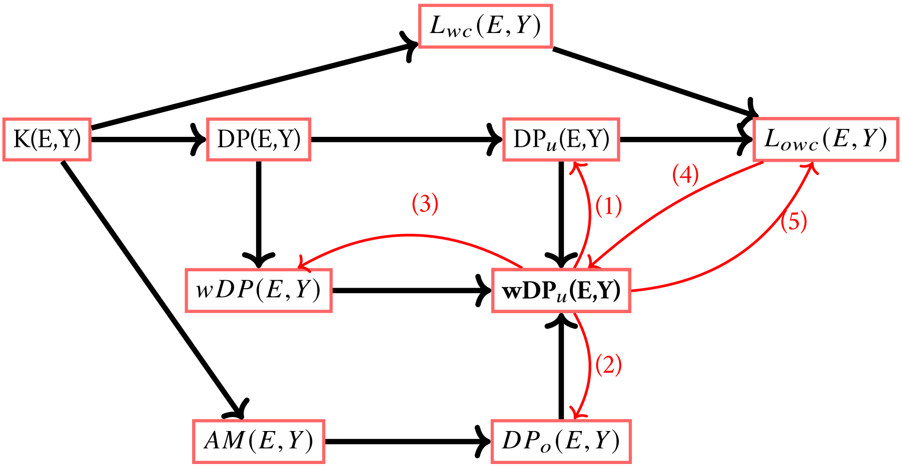
The proposition 3.20 states that if $E = F$ we get the following result.

Corollary 3.21 *Let E be a σ-Dedekind complete Banach lattice. Then the following assertions are equivalent:*

(1) Every weak U-Dunford-Pettis operator $T : E \longrightarrow E$ is an order Dunford-Pettis. operator.
(2) Every positive weak U-Dunford-Pettis operator $T : E \longrightarrow E$ is an order Dunford-Pettis. operator.
(3) the norm of E is order continuous.

3.3 Conclusion

The following diagram provides a summary of the relationships between the different classes of operators that have been the subject of this work:



- (1): E has o-UDPP and RowUDPP or Y is reflexive.
- (2): E is order continuous or E has AM-compactness or Y is reflexive and has DPP.
- (3): E has order unit.
- (4): E has owUDPP.
- (5): E has RowUDPP.

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