PERTURBATION OF BANACH SPACE OPERATORS WITH A 
COMPLEMENTED RANGE

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Abstract. Let $C[\mathcal{X}]$ be any class of operators on a Banach space $\mathcal{X}$, and let $Holo^{-1}(C)$ denote the class of operators $A$ for which there exists a holomorphic function $f$ on a neighbourhood $\mathcal{N}$ of the spectrum $\sigma(A)$ of $A$ such that $f$ is non-constant on connected components of $\mathcal{N}$ and $f(A)$ lies in $C$. Let $R[\mathcal{X}]$ denote the class of Riesz operators in $B[\mathcal{X}]$. This paper considers perturbation of operators $A \in \Phi_+(\mathcal{X}) \cup \Phi_-(\mathcal{X})$ (the class of all upper or lower [semi] Fredholm operators) by commuting operators in $B \in Holo^{-1}(R[\mathcal{X}])$. We prove (amongst other results) that if $\pi_B(B) = \prod_{i=1}^{m} (B - \mu_i)$ is Riesz, then there exist decompositions $\mathcal{X} = \oplus_{i=1}^{m} \mathcal{X}_i$ and $B = \oplus_{i=1}^{m} B |_{\mathcal{X}_i}$ such that: (i) If $\lambda \neq 0$, then $\pi_B(A, \lambda) = \prod_{i=1}^{m} (A - \lambda \mu_i)^{\alpha_i} \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$) if and only if $A - \lambda B_0 - \lambda \mu_i \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$), and (ii) (case $\lambda = 0$) $A \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$) if and only if $A - B_0 \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$), where $B_0 = \oplus_{i=1}^{m} (B_0 - \mu_i)$; (iii) if $\pi_B(A, \lambda) \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$) for some $\lambda \neq 0$, then $A - \lambda B \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$).


1. Introduction. Given an infinite-dimensional complex Banach space $\mathcal{X}$, let $B[\mathcal{X}]$ denote the algebra of operators (equivalently, bounded linear transformations) of $\mathcal{X}$ into itself. Let $A^{-1}(0)$ and $A(\mathcal{X})$ denote, respectively, the null space and the range of an operator $A \in B[\mathcal{X}]$. The operator $A$ has an inner generalized inverse if there exists an operator $B \in B[\mathcal{X}]$ such that $ABA = A$. It is easily seen that if $B$ is an inner generalized inverse of $A$, then $AB$ is a projection from $\mathcal{X}$ onto $A(\mathcal{X})$ and $I_{\mathcal{X}} - BA$ is a projection from $\mathcal{X}$ onto $A^{-1}(0)$; Indeed, $A$ is inner regular (i.e., $A$ has an inner generalized inverse) if and only if $A(\mathcal{X})$ and $A^{-1}(0)$ are complemented (in $\mathcal{X}$). The study of inner regular operators has a long and rich history, and there is a large body of information available on inner regular operators in the extant literature (see, for example, [7]). An important class of inner regular Banach space operators is that of operators $A \in B[\mathcal{X}]$ which are either left or right Fredholm. Here, we say that $A \in B[\mathcal{X}]$ is left Fredholm, $A \in \Phi_+(\mathcal{X})$ (resp, right Fredholm, $A \in \Phi_+(\mathcal{X})$) if $A \in \Phi_+(\mathcal{X})$ and $R(A)$ is complemented (resp., $A \in \Phi_-(\mathcal{X})$ and $A^{-1}(0)$ is complemented), $\Phi_+(\mathcal{X}) = \{ A \in B[\mathcal{X}] : A(\mathcal{X})$ is closed and $\dim(A^{-1}(0)) < \infty \}$ is the class of upper semi-Fredholm operators and
\( \Phi_-(\mathcal{X}) = \{ A \in \mathcal{B}[\mathcal{X}] : \dim(\mathcal{X}/A(\mathcal{X})) < \infty \} \) is the class of lower semi-Fredholm operators (see, e.g., [12]).

The problem of the perturbation of inner regular operators by compact operators is of some interest, and has been considered in the not too distant past. Thus, if an \( A \in \mathcal{B}[\mathcal{X}] \) is left Fredholm (or right Fredholm), and \( S \in \mathcal{B}[\mathcal{X}] \) is a compact operator, then \( A + S \) is left Fredholm (resp., right Fredholm) [5, 10]. This result is in a way the best possible: If \( A \in \mathcal{B}[\mathcal{X}, \mathcal{Y}] \) for Banach spaces \( \mathcal{X} \) and \( \mathcal{Y} \), \( A^{-1}(0) \) is infinite-dimensional and complemented in \( \mathcal{X} \), \( A(\mathcal{X}) \) is closed, complemented and of infinite co-dimension in \( \mathcal{Y} \), then the closure of \( (A + S)(\mathcal{X}) \) is complemented in \( \mathcal{Y} \) for every compact \( S \in \mathcal{B}[\mathcal{X}, \mathcal{Y}] \) only if \( A(\mathcal{X}) \) has a complementary subspace isomorphic to a Hilbert space [10, Theorem 3].

For an operator \( A \in \mathcal{B}[\mathcal{X}] \), let \( \mathcal{H}(\sigma(A)) \) denote the set of functions \( f \) which are holomorphic on a neighbourhood \( \mathcal{N} \) of the spectrum \( \sigma(A) \) of \( A \), and let \( \mathcal{H}_s(\sigma(A)) = \{ f \in \mathcal{H}(\sigma(A)) : f \) is non-constant on the connected components of \( \mathcal{N} \} \). Let \( \mathcal{K}[\mathcal{X}] \) denote the ideal of compact operators, and let \( \mathcal{R}[\mathcal{X}] \) denote the class of Riesz operators (i.e., operators whose non-zero translates are Fredholm). The operator \( A \) is holomorphically compact (resp., Riesz), \( A \in \text{Holo}^{-1}(\mathcal{K}[\mathcal{X}]) \) (resp., \( A \in \text{Holo}^{-1}(\mathcal{R}[\mathcal{X}]) \)), if there exists an \( f \in \mathcal{H}_s(\sigma(A)) \) such that \( f(A) \) is compact (resp., Riesz).

This paper considers perturbation of operators in \( \Phi_+(\mathcal{X}) = \Phi_+(\mathcal{X}) \cup \Phi_-(\mathcal{X}) \) by commuting operators in \( (\text{Holo}^{-1}(\mathcal{K}[\mathcal{X}]), \text{more generally}) \) \( \text{Holo}^{-1}(\mathcal{R}[\mathcal{X}]) \). It is known that if \( B \in \text{Holo}^{-1}(\mathcal{K}[\mathcal{X}]) \) (resp., \( B \in \text{Holo}^{-1}(\mathcal{R}[\mathcal{X}]) \)), then there exists a polynomial \( \pi_B(z) = \prod_{i=1}^m (z - \mu_i)^{a_i} \) for some complex numbers \( \mu_i \) and positive integers \( a_i \) (resp., \( \pi_B(z) = \prod_{i=1}^m (z_i - \mu_i) \)), which is the minimal polynomial \( \pi_B(\cdot) \) of \( B \), such that \( \pi_B(B) \) is compact (resp., Riesz).

Let \( \Phi_+(\mathcal{X}) \) denote either of \( \Phi_+(\mathcal{X}) \) and \( \Phi_-(\mathcal{X}) \). We prove (a more general version of the result) that if \( \pi_B(A) \in \Phi_+(\mathcal{X}) \), if \( [A, B] = AB - BA = 0 \) (or, more generally, \( [A, B] \) is in the “perturbation class” \( \text{Ptrb}(\Phi_+(\mathcal{X})) \) of \( \Phi_+(\mathcal{X}) \)) and \( \pi_B(B) \) is Riesz, then \( A - B \in \Phi_+(\mathcal{X}) \). The hypothesis \( B \in \text{Holo}^{-1}(\mathcal{K}[\mathcal{X}]) \) (or, \( B \in \text{Holo}^{-1}(\mathcal{R}[\mathcal{X}]) \)) enforces a decomposition \( \mathcal{X} = \bigoplus_{i=1}^m \mathcal{X}_i \) of \( \mathcal{X} \) such that \( B = \bigoplus_{i=1}^m B_i = \bigoplus_{i=1}^m B_i : \mathcal{X}_i \) with \( \bigoplus_{i=1}^m (B_i - \mu_i)^{c_i} \) compact (resp., \( \bigoplus_{i=1}^m (B_i - \mu_i) \) Riesz). Let \( B_0 = \bigoplus_{i=1}^m B_i - \mu_i \), where \( m \) and \( \mu_i \) are as above. It is proved that if \( [A, B] = 0 \) and \( B \in \text{Holo}^{-1}(\mathcal{R}[\mathcal{X}]) \), then (a) \( \pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda \mu_i) \in \Phi_+(\mathcal{X}) \) for a complex number \( \lambda \neq 0 \) if and only if \( A - \lambda(B_0 - \mu_i) \in \Phi_+(\mathcal{X}), \) and \( A \in \Phi_+(\mathcal{X}) \) if and only if \( A - B_0 \in \Phi_+(\mathcal{X}) \); (b) \( \pi_B(A, \lambda) \in \Phi_+(\mathcal{X}) \) for some \( \lambda \neq 0 \) implies \( A - \lambda B \in \Phi_+(\mathcal{X}) \). The case of operator \( A \) such \( \pi_B(A, \lambda) \) has SVEP, the single-valued extension property, or essential SVEP, at 0 is also considered.

2. Auxiliary results. Let \( \text{Inv}_i(\mathcal{X})(\text{Inv}_i(\mathcal{X})) \) denote the class of operators \( A \in \mathcal{B}[\mathcal{X}] \) which are left invertible (resp., right invertible). Let \( \mathcal{T} \) denote the Calkin homomorphism \( \mathcal{T} : \mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}]/\mathcal{K}[\mathcal{X}] \). Then, \( A \in \mathcal{K}[\mathcal{X}] \) if and only if \( \mathcal{T}(A) = 0 \), \( A \in \mathcal{R}[\mathcal{X}] \) if and only if \( \mathcal{T}(A) \) is a quasinilpotent operator, and an \( A \in \mathcal{B}[\mathcal{X}] \) is in \( \Phi_i(\mathcal{X}) \) (resp., \( \Phi_i(\mathcal{X}) \)) if and only if \( \mathcal{T}(A) \in \text{Inv}_i(\mathcal{X}) \) (resp., \( \mathcal{T}(A) \in \text{Inv}_i(\mathcal{X}) \)). Let \( B \in \text{Holo}^{-1}(\mathcal{K}[\mathcal{X}]) \). Then, there exists a function \( f \in \mathcal{H}_s(\sigma(B)) \) such that \( f(B) \in \mathcal{K}[\mathcal{X}] \), and hence such that \( \mathcal{T}(f(B)) = f(\mathcal{T}(B)) = 0 \). Since \( f(z) \) has at least a finite number of zeros, there exists a polynomial \( p(\cdot) \) such that \( f(\mathcal{T}(B)) = p(\mathcal{T}(B))g(\mathcal{T}(B)) = 0 \), where the (holomorphic on \( \sigma(B) \)) function \( g \) satisfies the property that \( g(z) \neq 0 \) on \( \sigma(B) \). But then \( p(\mathcal{T}(B)) = 0 \), and hence there exists a monic irreducible polynomial, the minimal polynomial of \( B \), which divides every other polynomial \( q(z) \) such that \( q(\mathcal{T}(B)) = 0 \). If we let \( \pi_B(z) = \prod_{i=1}^m (z - \mu_i)^{c_i} \) denote the
minimal polynomial of $B$, then $\pi_B(B) \in \mathcal{K}[\mathcal{X}]$. In the case in which $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$, so that $f(B) \in \mathcal{R}[\mathcal{X}]$ for some $f \in \mathcal{H}_c(\sigma(B))$, $f(T(B))$ is a quasinilpotent such that $f(T(B)) = p(T(B))g(T(B))$ for some polynomial $p(.)$ such that $p(T(B))$ is quasinilpotent and the function $g(.)$ is invertible. Once again there exists a minimal polynomial $\pi_B(.)$ of $B$ such that $\pi_B(B) \in \mathcal{R}[\mathcal{X}]$ We have ([11, 13, 16]):

**PROPOSITION 2.1.** The following conditions are equivalent for operators $B \in \mathcal{B}[\mathcal{X}]:$

(i) $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ (resp., $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$).

(ii) $B$ is polynomially compact (resp., polynomially Riesz).

(iii) There exists a monic irreducible polynomial $\pi_B(z) = \prod_{i=1}^{m} (z - \mu_i)$ (resp., $\pi_B(z) = \prod_{i=1}^{m} (z - \mu_i)$), the minimal polynomial of $B$, such that $\pi_B(B)$ is compact (resp., Riesz).

If $f(B) \in \mathcal{K}[\mathcal{X}] \cup \mathcal{R}[\mathcal{X}]$ is such that (the Fredholm essential spectrum) $\sigma_{e}(f(B)) \neq \emptyset$, then (it follows from the considerations above that) there exists a finite subset $\{\mu_1, \mu_2, \ldots, \mu_m\}$ of the set of complex numbers $\mathbb{C}$ such that $f(\mu_i) = 0$ for all $1 \leq i \leq m$, and there exist disjoint countable subsets $S_i = \{\mu_{i_n}\} \subset \mathbb{C}$ such that $\mu_{i_n}$ converges to $\mu_i \in S_i$ and $S_1 \cup S_2 \cup \cdots \cup S_m = \sigma(B)$. (Here, either of the sets $S_i$ may consist just of the singleton $\mu_i$, and then the quasinilpotent part $H_0(B - \mu_i) = \{x \in \mathcal{X}: \lim_{n \to \infty} \| (B - \mu_i)^n x \|^{\frac{1}{n}} = 0\}$ of $B - \mu_i$ is infinite dimensional.) Letting $P_i$ denote the spectral projection associated with the spectral set $S_i$, we then obtain spectral subspaces $X_i$ of $\mathcal{X}$ and operators $B_i = B|_{X_i}$ such that $\mathcal{X} = \bigoplus_{i=1}^{m} X_i$, $B = \bigoplus_{i=1}^{m} B_i$ and $\sigma(B_i) = \{\mu_i\}$. Furthermore, each $(B_i - \mu_i)^{\nu_i}$ is compact in the case in which $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$, and (since, for an operator $E \in \mathcal{B}[\mathcal{X}]$, $E^{\nu_i} \in \mathcal{R}[\mathcal{X}]$ if and only if $E \in \mathcal{R}[\mathcal{X}]$) each $B_i - \mu_i$ is Riesz in the case in which $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$. We have the following:

**PROPOSITION 2.2 ([8, 16]).** If $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ (resp., $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$), then there exists a finite subset $\{\mu_1, \mu_2, \ldots, \mu_m\} \subset \mathbb{C}$, a subset $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of positive integers, a decomposition $\mathcal{X} = \bigoplus_{i=1}^{m} X_i$ of $\mathcal{X}$ into closed $B$-invariant subspaces and a decomposition $B = \bigoplus_{i=1}^{m} B_i$ of $B$ such that each $(B_i - \mu_i)^{\nu_i}$ is compact (resp., each $B_i - \mu_i$ is Riesz).

### 3. Riesz perturbations.

Given operators $A, B \in \mathcal{B}[\mathcal{X}]$, let $\delta_{A,B} \in \mathcal{B}[\mathcal{B}[\mathcal{X}]]$ denote the generalized derivation $\delta_{A,B}(X) = AX - XB$, and let $\delta_{A,B}^{\nu}(X) = \delta_{A,B}^{\nu-1}(\delta_{A,B}(X))$. The operators $A, B$ are said to be quasinilpotent equivalent if

$$
\lim_{n \to \infty} \| \delta_{A,B}(I) \|^{\frac{1}{n}} = \lim_{n \to \infty} \| \delta_{B,A}(I) \|^{\frac{1}{n}} = 0.
$$

The following proposition is well known (see [14, Proposition 3.4.11], [6, Theorem 3.1]).

**PROPOSITION 3.1.** If $A, B$ are quasinilpotent equivalent operators, then $\sigma_{\alpha}(A) = \sigma_{\alpha}(B)$, where $\sigma_{\alpha}$ stands for either of the left spectrum, the right spectrum, the approximate point spectrum $\sigma_{\alpha}$, the surjectivity spectrum $\sigma_{\alpha}$ and the spectrum $\sigma$.

We assume in the following that if an operator $B \in \mathcal{B}[\mathcal{X}]$ is such that $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ or $Holo^{-1}(\mathcal{R}[\mathcal{X}])$, then it has the minimal polynomial function of Proposition 2.1, the Banach space $\mathcal{X}$ and the operator $B$ have the decompositions $X = \bigoplus_{i=1}^{m} X_i$ and $B = \bigoplus_{i=1}^{m} B_i$ of Proposition 2.2. The operator $B_0 \in \mathcal{B}[\mathcal{X}]$ shall henceforth be
Let \( \text{Inv}_x(\mathcal{X}) \) denote operators \( A \in \mathcal{B}[\mathcal{X}] \) which are either bounded below or surjective. Given operators \( A, B \in \mathcal{B}[\mathcal{X}] \), let \([A, B]\) denote the commutator \([A, B] = AB - BA\) of \( A \) and \( B \). If \( \Phi_x(\mathcal{X}) \) denotes either of \( \Phi_+ (\mathcal{X}) \) or \( \Phi_-(\mathcal{X}) \) or \( \Phi_\pm(\mathcal{X}) = \Phi_+(\mathcal{X}) \cup \Phi_-(\mathcal{X}) \), then the perturbation class of \( \Phi_x(\mathcal{X}) \), \( \text{Ptrb}(\Phi_x(\mathcal{X})) \), is the closed two-sided ideal.

\[
\text{Ptrb}(\Phi_x(\mathcal{X})) = \{ A \in \mathcal{B}[\mathcal{X}] : A + B \in \Phi_x(\mathcal{X}) \text{ for every } B \in \Phi_x(\mathcal{X}) \}.
\]

It is seen that

\[
\text{Ptrb}(\Phi_+(\mathcal{X})) = \text{Ptrb}(\Phi_-(\mathcal{X})) = \text{Ptrb}(\Phi_+(\mathcal{X})) \cup \text{Ptrb}(\Phi_-(\mathcal{X})).
\]

Let \( \mathcal{T}_p \) denote the homomorphism

\[
\mathcal{T}_p : \mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}] / \text{Ptrb}(\Phi_x(\mathcal{X})),
\]

which is effected by the natural projection of the algebra \( \mathcal{B}[\mathcal{X}] / \text{Ptrb}(\Phi_x(\mathcal{X})) \). It is then clear that \([A, B] = AB - BA \in \text{Ptrb}(\Phi_x(\mathcal{X}))\) if and only if \( \mathcal{T}_p(AB - BA) = \mathcal{T}_p(A)\mathcal{T}_p(B) - \mathcal{T}_p(B)\mathcal{T}_p(A) = 0 \); furthermore, if the function \( f \in \text{Holo}^{-1}(\sigma(A) \cup \sigma(B)) \), in particular if \( f \) is a polynomial, then \([A, B] \in \text{Ptrb}(\Phi_x(\mathcal{X}))\) implies \( f(A)f(B) - f(B)f(A) \in \text{Ptrb}(\Phi_x(\mathcal{X})) \), and hence \( \mathcal{T}_p(f(A)f(B) - f(B)f(A)) = 0 \).

**Theorem 3.1.** Let \( A, B \in \mathcal{B}[\mathcal{X}] \) be such that \( B \in \text{Holo}^{-1}(\mathcal{R}[\mathcal{X}]) \).

(a) If \( \pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda \mu_i) \in \Phi_x(\mathcal{X}) \) for some complex number \( \lambda \) and \([A, B] \in \text{Ptrb}(\Phi_x(\mathcal{X})) \), then \( A - \lambda B \in \Phi_x(\mathcal{X}) \) if \( \lambda \neq 0 \), and \( A - B_0 \in \Phi_x(\mathcal{X}) \) whenever \( \lambda = 0 \).

(b) Suppose that \([A, B] = 0 \).

(i) If \( \lambda \neq 0 \), then \( \pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda \mu_i)^{\alpha_i} \in \Phi_x(\mathcal{X}) \) if and only if \( A - \lambda B_0 - \lambda \mu_i \in \Phi_x(\mathcal{X}) \).

(ii) (Case \( \lambda = 0 \)) \( A \in \Phi_x(\mathcal{X}) \) if and only if \( A - B_0 \in \Phi_x(\mathcal{X}) \).

(c) If \( \lambda \neq 0 \), \([A, B] = 0 \) and \( \pi_B(A, \lambda) \in \Phi_x(\mathcal{X}) \), then \( A - \lambda B \in \Phi_x(\mathcal{X}) \).

**Proof.**

(a) Define the operators \( D, E \) and \( F \) by

\[
D = E - F, \quad E = \pi_B(A, \lambda) \text{ if } \lambda \neq 0 \quad \text{and} \quad E = A^m \text{ if } \lambda = 0, \\
F = \lambda^m \pi_B(B) \text{ if } \lambda \neq 0 \quad \text{and} \quad F = B_0^m \text{ if } \lambda = 0.
\]

Then, \( F \in \mathcal{R}[\mathcal{X}] \), and the hypothesis that \([A, B] \in \text{Ptrb}(\Phi_x(\mathcal{X})) \) implies

\[
\mathcal{T}_p[E, F] = \mathcal{T}_p(E)\mathcal{T}_p(F) - \mathcal{T}_p(F)\mathcal{T}_p(E) = 0.
\]

The operator \( \mathcal{T}_p(F) \) being quasinilpotent, we have

\[
\delta^{n}_{\mathcal{T}_p(D), \mathcal{T}_p(E)}(I) = \delta^{n-1}_{\mathcal{T}_p(D), \mathcal{T}_p(E)}((-1)\mathcal{T}_p(F))
\]

\[
= \cdots = (-1)^n \mathcal{T}_p(F) = \cdots = (-1)^n \delta^n_{\mathcal{T}_p(E), \mathcal{T}_p(D)}(I),
\]

and hence \( \mathcal{T}_p(D) \) and \( \mathcal{T}_p(E) \) are quasinilpotent equivalent. Since \( E \in \Phi_x(\mathcal{X}) \),

\[
\mathcal{T}_p(E) \in \text{Inv}_x(\mathcal{X}) \iff \mathcal{T}_p(D) \in \text{Inv}_x(\mathcal{X}).
\]
Again, since
\[
T_p(D) = (T_p(A) - T_p(B))g(T_p(A), T_p(B), \lambda)
= g(T_p(A), T_p(B), \lambda)(T_p(A) - \lambda T_p(B)) \text{ if } \lambda \neq 0,
\]
and
\[
T_p(D) = T_p(A)^m - T_p(B^0)^m = (T_p(A) - T_p(B^0))g_1(T_p(A), T_p(B), \lambda)
= g_1(T_p(A), T_p(B), \lambda)(T_p(A) - T_p(B^0)) \text{ if } \lambda = 0,
\]
it follows that
\[
T_p(A) - \lambda T_p(B) \in \text{Inv}_\chi(\mathcal{X}) \text{ if } \lambda \neq 0 \text{ and }
T_p(A) - T_p(B^0) \in \text{Inv}_\chi(\mathcal{X}) \text{ if } \lambda = 0.
\]

Since
\[
A - \lambda B \text{ (resp., } A - B^0) \in \Phi_+(\mathcal{X}), \text{ if and only if }
T_p(A) - \lambda T_p(B) \text{ (resp., } T_p(A) - T_p(B^0)) \text{ is bounded below and }
A - \lambda B \text{ (resp., } A - B^0) \in \Phi_-(\mathcal{X}), \text{ if and only if }
T_p(A) - \lambda T_p(B) \text{ (resp., } T_p(A) - T_p(B^0)) \text{ is surjective,}
\]
the proof follows.

(b) The proof at places is similar to the one above, so we shall at points be brief. Let
\[
\mathcal{T} : \mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}]/\mathcal{K}[\mathcal{X}] \text{ denote the Calkin homomorphism. Suppose that } [A, B] = 0.
\]
Letting \( B = \oplus_{i=1}^m B_i \) with respect to the decomposition \( \mathcal{X} = \oplus_{i=1}^m \mathcal{X}_i \) of \( \mathcal{X} \), it is seen that \( A \) has a matrix representation \( A = (A_{ij})_{i,j=1}^m \) such that
\[
A_{ij}B_j = B_i A_{ij} \text{ for all } 1 \leq i, j \leq m
\]
\[\iff A_{ij}(B_j - \mu_i) = (B_i - \mu_i)A_{ij} \text{ for all } 1 \leq i, j \leq m.\]

Here, the complex numbers \( \mu_i, 1 \leq i \leq m, \) are distinct, the operators \( B_i - \mu_i \) being Riesz for all \( 1 \leq i \leq m \) and (since \( \mu_i \notin \sigma(B_j) \) for all \( 1 \leq i \neq j \leq m \)), the operator \( \mathcal{T}(B_j - \mu_i) \) is invertible for all \( 1 \leq i \neq j \leq m. \) Consequently,
\[
\mathcal{T}(A_{ij}) \mathcal{T}(B_j - \mu_i)^n = \mathcal{T}(B_i - \mu_i)^n \mathcal{T}(A_{ij})
\]
\[\iff \mathcal{T}(A_{ij}) = \mathcal{T}(B_j - \mu_i)^n \mathcal{T}(B_i - \mu_i)^n \mathcal{T}(A_{ij}).\]

We have two possibilities: Either \( \mathcal{T}(A_{ij}) \neq 0 \) or \( \mathcal{T}(A_{ij}) = 0. \) If \( \mathcal{T}(A_{ij}) \neq 0 \), then (since \( \mathcal{T}(B_j - \mu_i) \) is quasinilpotent):
\[
||\mathcal{T}(A_{ij})|| \leq ||\mathcal{T}(A_{ij})|| ||\mathcal{T}(B_j - \mu_i)^{-1}||^n ||\mathcal{T}(B_i - \mu_i)^n||
\]
\[\implies 1 \leq ||\mathcal{T}(B_j - \mu_i)^{-1}|| \lim_{n \to \infty} ||\mathcal{T}(B_i - \mu_i)^n||^{\frac{1}{n}} = 0.\]

This being a contradiction, we must have
\[
\mathcal{T}(A) = \oplus_{i=1}^m \mathcal{T}(A_{ii}), \mathcal{T}(A_{ij}) = 0 \text{ and } [A_{ii}, B_i] = 0 \text{ for all } 1 \leq i \neq j \leq m.
\]
Define the operators \( M_j, N_j \in B[\mathcal{X}_j], 1 \leq j \leq m \), by

\[
M_j = (A_{jj} - \lambda B_j) - \lambda (\mu_i - \mu_j), \quad N_j = A_{jj} - \lambda \mu_i \quad \text{for all} \quad 1 \leq i, j \leq m \quad \text{if} \quad \lambda \neq 0,
\]

and

\[
M_j = A_{jj} - B_j + \mu_j, \quad N_j = A_{jj} \quad \text{for all} \quad 1 \leq j \leq m \quad \text{if} \quad \lambda = 0.
\]

Then, the equivalences

\[
\pi_B(B) \in \mathcal{R}[\mathcal{X}] \iff \prod_{i=1}^{m} (B - \mu_i) = \prod_{i=1}^{m} (\bigoplus_{j=1}^{m} (B_j - \mu_i)) \in \mathcal{R}[\mathcal{X}]
\]

\[
\iff \prod_{i=1}^{m} (B_j - \mu_i) \in \mathcal{R}[\mathcal{X}_j] \quad \text{for all} \quad 1 \leq j \leq m
\]

\[
\iff B_j - \mu_i \in \mathcal{R}[\mathcal{X}_j] \quad \text{for all} \quad 1 \leq j \leq m
\]

and

\[
\pi_B(A, \lambda) \in \Phi_\times(\mathcal{X}) \iff \prod_{i=1}^{m} T(A - \lambda \mu_i) = \prod_{i=1}^{m} (\bigoplus_{j=1}^{m} T(A_{jj} - \lambda \mu_i)) \in \text{Inv}_\times(\mathcal{X})
\]

\[
\iff \prod_{i=1}^{m} T(A_{jj} - \lambda \mu_i) = T \{ \prod_{i=1}^{m} (A_{jj} - \lambda \mu_i) \} \in \text{Inv}_\times(\mathcal{X}_j)
\]

\[
\text{for all} \quad 1 \leq i, j \leq m
\]

\[
\iff \prod_{i=1}^{m} (A_{jj} - \lambda \mu_i) \in \Phi_\times(\mathcal{X}_j) \quad \text{for all} \quad 1 \leq i, j \leq m
\]

\[
\iff A_{jj} - \lambda \mu_i \in \Phi_\times(\mathcal{X}_j) \quad \text{for all} \quad 1 \leq i, j \leq m
\]

imply that

\[
\delta^n_{T(M_j), T(N_j)}(I_j) = (-\lambda)^{n-1} T^{n-1}(B_j - \mu_j) = \cdots = (-\lambda)^n T(B_j - \mu_j)^n = \cdots = \delta^n_{T(N_j), T(M_j)}(I_j).
\]

This implies that the operators \( T(M_j) \) and \( T(N_j) \) are quasinilpotent equivalent, and hence

\[
M_j \in \Phi_\times(\mathcal{X}_j) \iff N_j \in \Phi_\times(\mathcal{X}_j), \quad 1 \leq j \leq m.
\]
Now, if we define \( B_0 \in \mathcal{B}[\mathcal{X}] \) (as above) by \( B_0 = \oplus_{j=1}^{m}(B_j - \mu_j) \), then

\[
\mathcal{T}(A - \lambda B_0 - \lambda \mu_i) = \oplus_{j=1}^{m}\{\mathcal{T}((A_{ij} - \lambda B_j) - \lambda(\mu_i - \mu_j))\} \in \text{Inv}_x(\mathcal{X})
\]

for all \( 1 \leq i \leq m \)

\[\iff \oplus_{j=1}^{m}\mathcal{T}(A_{ij} - \lambda \mu_i) \in \text{Inv}_x(\mathcal{X}) \text{ for all } 1 \leq i \leq m\]

\[\iff \prod_{i=1}^{m}\{\oplus_{j=1}^{m}\mathcal{T}(A_{ij} - \lambda \mu_i)\} \in \text{Inv}_x(\mathcal{X})\]

\[= \prod_{i=1}^{m}\mathcal{T}(A - \lambda \mu_i) \in \text{Inv}_x(\mathcal{X})\]

\[\iff \pi_B(A, \lambda) \in \Phi_x(\mathcal{X})\]

if \( \lambda \neq 0 \), and

\[
\oplus_{j=1}^{m}\mathcal{T}(M_j) = \oplus_{j=1}^{m}\mathcal{T}(A_{ij} - B_j + \lambda_j) = \mathcal{T}(A - B_0) \in \text{Inv}_x(\mathcal{X})
\]

\[\iff \oplus_{j=1}^{m}\mathcal{T}(N_j) = \oplus_{j=1}^{m}\mathcal{T}(A_{ij}) = \mathcal{T}(\pi_B(A, 0)) \in \text{Inv}_x(\mathcal{X})\]

\[\iff \pi_B(A, 0) \in \Phi_x(\mathcal{X})\]

if \( \lambda = 0 \).

(c) Let \( \lambda \neq 0 \). Choosing \( i = j \) in

\[\pi_B(A, \lambda) \in \Phi_x(\mathcal{X}) \iff A - \lambda(\oplus_{j=1}^{m}(B_j - \lambda_j + \mu_i)) \in \Phi_x(\mathcal{X})\]

for all \( 1 \leq i \leq m \), it then follows that

\[\pi_B(A, \lambda) \in \Phi_x(\mathcal{X}) \implies A - \lambda B \in \Phi_x(\mathcal{X}). \qedhere\]

**Remark 3.1.**

(i) Some hypothesis of the type \([A, B] \in \text{Perrb} \Phi_x(\mathcal{X})\), or \([A, B] = 0\), is essential to the validity of Theorem 3.1. To see this, consider operators \( A, B \) such that \( \pi_B(A, \lambda) \in \Phi_x(\mathcal{X}) \) and \( \pi_B(B) \) is compact. Then, since \( \mathcal{T}_p(\pi_B(B)) = 0 = \mathcal{T}(\pi_B(B)) \), \( \pi_B(A, \lambda) - \lambda^m \pi_B(B) \in \Phi_x(\mathcal{X}) \iff \pi_B(A, \lambda) \in \Phi_x(\mathcal{X}) \). This does not however imply \( A - \lambda B \) (or, \( A - B_0 \) if \( \lambda = 0 \), or \( A - \lambda B_0 - \mu_i \) if \( \lambda \neq 0 \)) \( \in \Phi_x(\mathcal{X}) \), as the following elementary example shows. Letting \( I \) denote the identity of \( \mathcal{B}[\mathcal{X}] \), define the polynomially compact operator \( B \) (with minimal polynomial \( \pi_B(z) = (z - 1)^2 \)) by \( B = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \), and let \( A = \begin{pmatrix} 2I & 0 \\ 0 & I \end{pmatrix} \). Then, with \( \lambda = 1 \), \( \pi_B(A, \lambda) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \) is invertible (hence, Fredholm). However, the operator \( A - \lambda B \) (which satisfies \( (A - \lambda B)^2 = 0 \)) is not even semi-Fredholm. Again, if we define \( A \) by \( A = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \), then \( (A - B_0)^2 = 0 \) and \( A - B_0 \) is not semi-Fredholm. Observe that neither of the hypotheses \([A, B] = 0\) or \([A, B] \in \text{Perrb} \Phi_x(\mathcal{X})\) is satisfied.

(ii) Let \( A \) and \( B \) be the operators of Theorem 3.1, parts (b) and (c). Then, \( A - \mu_i \in \Phi_x(\mathcal{X}) \) if and only if \( A_{ij} - \lambda \mu_i \in \Phi_x(\mathcal{X}) \) for all \( 1 \leq j \leq m \) and \( \mathcal{T}(A_{ij}) = 0 \) for all \( 1 \leq i \neq j \leq m \). The conclusion \( \mathcal{T}(A_{ij}) = 0 \) for all \( 1 \leq i \neq j \leq m \) implies that the operator \( A = [A_{ij}]_{1 \leq i \leq j \leq m} \) may be written as the sum \( A = A_1 + A_0 \), where
A_1 = \oplus_{j=1}^{m} A_{\beta j} and the compact (hence, Riesz) operator \( A_0 \) is defined by

\[ A_0 = [A_{\beta j}]_{1 \leq j \leq m} \text{ with } A_{ii} = 0 \text{ for all } 1 \leq i \leq m. \]

Recalling that the sum of two commuting Riesz operators is a Riesz operator, it follows that the operators \( \frac{1}{\lambda} A_0 - B_0 \) (case \( \lambda \neq 0 \)) and \( A_0 - B_0 \) (case \( \lambda = 0 \)) are Riesz operators. It is now seen that the operators

\[ A - \lambda \mu_i - \lambda B_0 = (A_1 - \lambda \mu_i) + \lambda(\frac{1}{\lambda} A_0 - B_0) \quad \text{and} \quad A_1 - \lambda \mu_i \ (\lambda \neq 0), \]

\[ A - B_0 = A_1 + (A_0 - B_0) \quad \text{and} \quad A_1 \ (\lambda = 0) \]

are quasinilpotent equivalent. Hence

\[ A_1 - \lambda \mu_i \in \Phi_{\times}(\mathcal{X}) \iff A - \lambda \mu_i - \lambda B_0 \in \Phi_{\times}(\mathcal{X}), \ \lambda \neq 0 \]

and

\[ A \in \Phi_{\times}(\mathcal{X}) \iff A - B_0 \in \Phi_{\times}(\mathcal{X}). \]

This provides an alternative to some of the argument used to prove parts (b) and (c) of Theorem 3.1.

Let \( \lambda(t) \) denote a continuous function from a connected subset \( I \) of the reals into \( \mathbb{C} \) such that \( \lambda(t_1) = 0 \) and \( \lambda(t_2) = 1 \) for some \( t_1, t_2 \in I, t_1 < t_2 \). Then, the argument of the proof of Theorem 3.1 holds with \( \lambda \) replaced by \( \lambda(t) \) and we have:

**Corollary 3.1.** Let \( A, B \in \mathcal{B}[\mathcal{X}] \) be such that \( B \in Holo^{-1}(\mathcal{R}[\mathcal{X}]) \).

(a) If \( \pi_{\mathcal{B}}(A, \lambda) = \prod_{i=1}^{m} (A - \lambda(t) \mu_i) \in \Phi_{\times}(\mathcal{X}) \) and \( [A, B] \in \text{Ptrb}(\Phi_{\times}(\mathcal{X})) \), then \( A - \lambda(t) B \in \Phi_{\times}(\mathcal{X}) \) for all \( t \in [t_1, t_2] \).

(b) If \( A, B \) commute, then

(i) \( \pi_{\mathcal{B}}(A, \lambda(t)) = \prod_{i=1}^{m} (A - \lambda(t) \mu_i) \in \Phi_{\times}(\mathcal{X}) \) if and only if \( A - \lambda(t) (B_0 + \mu_i) \in \Phi_{\times}(\mathcal{X}), 1 \leq i \leq m, \text{ for all } t \in [t_1, t_2] \);

(ii) \( \pi_{\mathcal{B}}(A, \lambda(t)) \in \Phi_{\times}(\mathcal{X}) \) if and only if \( A - B_0 \in \Phi_{\times}(\mathcal{X}) \);

(iii) \( \pi_{\mathcal{B}}(A, \lambda(t)) \in \Phi_{\times}(\mathcal{X}) \) implies \( A - \lambda(t) B \in \Phi_{\times}(\mathcal{X}) \) for all \( t \in [t_1, t_2] \).

Recalling the fact that “every locally constant function on a connected set is constant”, it follows from the local constancy of the index function “\( \text{ind} \)” that \( \text{ind}(A) = \text{ind}(A - B) = \text{ind}(A - \lambda(t) B) \) for all \( t \in [t_1, t_2] \). In particular, if \( A \in \Phi_{r}(\mathcal{X}) \) (resp., \( A \in \Phi_{s}(\mathcal{X}) \)), then \( (A - \lambda(t) B)(\mathcal{X}) \) (resp., \( (A - \lambda(t) B)^{-1}(0) \)) is complemented by a finite-dimensional subspace if and only if \( A(\mathcal{X}) \) (resp., \( A^{-1}(0) \)) is complemented by a finite-dimensional subspace.

4. Operators with SVEP. \( A \in \mathcal{B}[\mathcal{X}] \) has the single-valued extension property at \( \lambda_0 \in \mathbb{C} \), SVEP at \( \lambda_0 \) for short, if for every open disc \( \mathcal{D}_{\lambda_0} \) centred at \( \lambda_0 \) the only holomorphic function \( f : \mathcal{D}_{\lambda_0} \to \mathcal{X} \) which satisfies

\[ (T - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathcal{D}_{\lambda_0} \]

is the function \( f \equiv 0 \). \( T \) has SVEP if it has SVEP at every \( \lambda \in \mathbb{C} \). Operators with countable spectrum have SVEP: If \( A \in \mathcal{R}[\mathcal{X}] \), then both \( A \) and (the conjugate operator) \( A^* \) have SVEP. It is known that \( f(A), A \in \mathcal{B}[\mathcal{X}] \) and \( f \in H_c(\sigma(A)) \), has SVEP at a point.
\( \lambda \) if and only if \( A \) has SVEP at every \( \mu \) such that \( f(\mu) = \lambda \) (see [1, Theorem 2.39] and [14]). If an \( A \in \mathcal{B}[X] \) has SVEP at a point \( \lambda \), then SVEP for \( B \in \mathcal{B}[X] \) at \( \lambda \) does not transfer to \( A + B \), even if \( A \) and \( B \) commute. However:

**Proposition 4.1** ([2, Theorem 0.3]). If \( A \) and \( B \) commute, and if \( B \in \mathcal{R}[X] \), then SVEP at \( \lambda \) for \( A \) implies SVEP for \( A - B \) at \( \lambda \).

Recall that the ascent (resp., descent) of \( A \in \mathcal{B}[X] \), \( \text{asc}(A) \) (resp., \( \text{dsc}(A) \)), is the least non-negative integer \( n \) such that \( A^{-n}(0) = A^{-(n+1)}(0) \) (resp., \( A^{n}(X) = A^{n+1}(X) \)); if no such integer exists, then \( \text{asc}(A) = \infty \) (resp., \( \text{dsc}(A) = \infty \)). Finite ascent (resp., descent) at a point \( \lambda \) for \( A \) implies \( \text{ind}(A - \lambda) \leq 0 \) and \( A \) has SVEP at \( \lambda \) (resp., \( \text{ind}(A - \lambda) \geq 0 \) and \( A^* \) has SVEP at \( \lambda \)); conversely, if \( A - \lambda \in \Phi_X(X) \) (resp., \( A^* - \lambda \in \Phi_X(X) \)) has SVEP at 0, then \( \text{asc}(A - \lambda) < \infty \) and 0 \( \in \text{iso}_\sigma(A) \) (resp., \( \text{dsc}(A - \lambda) < \infty \) and 0 \( \in \text{iso}_\sigma(A) \)) [1, Theorems 3.16, 3.17, 3.23, 3.27]. The operator \( A \) is upper Browder (resp., lower Browder, left Browder, right Browder, \( \text{simply Browder} \)) if it is upper Fredholm with \( \text{asc}(A) = \infty \) (resp., lower Browder with \( \text{dsc}(A) = \infty \), left Browder with \( \text{asc}(A) = \infty \), right Browder with \( \text{dsc}(A) = \infty \), Fredholm with \( \text{asc}(A) = \text{dsc}(A) = \infty \)). Let \( A \in \times - \text{Browder} \) denote that \( A \) is one of upper Browder, lower Browder, left Browder, right Brower or \( \text{simply Browder} \). It is well known, see [9, Theorem 7.92.] or [6, Proposition 2.2], that if \( A, B \in \mathcal{B}[X] \) are commuting operators, then \( AB \in \times - \text{Browder} \) if and only if \( A, B \in \times - \text{Browder} \). If an operator \( A \in \{ \Phi_+(X) \cup \Phi_-(X) \} \) (resp., \( A \in \{ \Phi_-(X) \cup \Phi_+(X) \} \)) has SVEP at 0, then \( A \) is upper or left (resp., lower or right) Browder [1, Theorem 3.52]. As before, the operator \( B_0 \in \mathcal{B}[X] \) is defined by \( B_0 = \bigoplus_{i=1}^m (B_i - \mu_i) \).

The following theorem generalizes [6, Theorem 4.1].

**Theorem 4.1.** Let \( A, B \in \mathcal{B}[X] \) be such that \( [A, B] = 0 \), \( \pi_B(B) = \prod_{i=1}^m (B - \mu_i) \in \mathcal{R}[X] \) and \( \pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda, \mu_i) \in \Phi_X(X) \) for some complex number \( \lambda \). Then

(a) \( A \in \times - \text{Browder} \) if and only if \( A - B_0 \in \times - \text{Browder} \);

(b) (i) \( \pi_B(A, \lambda) \in \times - \text{Browder} \) implies \( A - \lambda B \in \times - \text{Browder} \), and (ii) \( \pi_B(A, \lambda) \in \times - \text{Browder} \) if and only if \( A - \lambda B_0 - \lambda, \mu_i \in \times - \text{Browder} \) for all \( 1 \leq i \leq m \);

(c) if \( A \in \{ \Phi_+(X) \cup \Phi_-(X) \} \) has SVEP at 0 (resp., \( A \in \{ \Phi_-(X) \cup \Phi_+(X) \} \) and \( A^* \) has SVEP at 0), then \( A - \lambda B \) is upper or, respectively, left (resp., lower or, respectively, right) Browder.

**Proof.** We consider the case \( \times - \text{Browder} = \text{upper Browder or left Browder only} \); the proof for the other cases is similar.

(a) The operator \( B_0 = \bigoplus_{i=1}^m (B_i - \mu_i) \) being the direct sum of Riesz operators is a Riesz operator. Since \( A \) commutes with \( B_0 \), \( A - B_0 \) has SVEP at 0 if and only if \( A \) has SVEP at 0. Again, by Theorem 2.1(b.ii), \( A - B_0 \notin \Phi_X(X) \) if and only if \( A \notin \Phi_X(X) \). Hence, since an operator \( T \) is \( \times - \text{Browder} \) if and only if \( T \in \Phi_X(X) \) and \( T \) has SVEP at 0 [1, Theorem 3.52], \( A - B_0 \notin \times - \text{Browder} \) if and only if \( A \notin \times - \text{Browder} \).

(b.i) The hypothesis \( \pi_B(A, \lambda) \in \times - \text{Browder} \) implies \( A - \lambda, \mu_i \in \times - \text{Browder} \) if and only if \( A - \lambda, \mu_i \in \Phi_X(X) \) and \( A - \lambda, \mu_i \) has SVEP at 0. Since \( \pi_B(B) = \prod_{i=1}^m (B - \mu_i) \) is Riesz, there an integer \( i, 1 \leq i \leq m \), such that \( \lambda(B - \mu_i) \) is Riesz (and commutes with \( A - \lambda, \mu_i \)). Hence, \( A - \lambda B = (A - \lambda, \mu_i) - (B - \lambda, \mu_i) \) has SVEP at 0. Since \( A - \lambda B \in \Phi_X(X) \) by Theorem 2.1(c), \( A - \lambda B \in \times - \text{Browder} \).
(b.ii) The case $\lambda = 0$ being evident, we consider $\lambda \neq 0$. It is clear from Theorem 2.1(b.i) that

$$
\pi_B(A, \lambda) \in \Phi_\times(\mathcal{X}) \iff A - \lambda B - \lambda \mu_i \in \Phi_\times(\mathcal{X}).
$$

Since,

$$
\pi_B(A, \lambda) \in \times-Browder \iff A - \lambda \mu_i \in \times-Browder \quad \text{for all } 1 \leq i \leq m
$$

$$
\iff A - \lambda \mu_i \in \Phi_\times(\mathcal{X}), A - \lambda \mu_i \quad \text{has SVEP at } 0
$$

for all $1 \leq i \leq m$.

The operator $B_0$ being a Riesz operator which commutes with $A - \lambda \mu_i$, it follows that $A - \lambda \mu_i - \lambda B_0$ has SVEP at $0$ if and only if $A - \lambda \mu_i$ has SVEP at $0$. Hence,

$$
\pi_B(A, \lambda) \in \times-Browder \iff A - \lambda B_0 - \lambda \mu_i \in \times-Browder.
$$

(c) Recall from above that if an operator $A \in \Phi_\times(\mathcal{X})$ has SVEP at $0$, then $0 \in \text{iso}\sigma(A)$. Since $\sigma(A - \lambda \mu_i) \subset \sigma(A) - \{\lambda \mu_i\}$, it follows from our hypotheses that (at worst) $\lambda \mu_i \in \text{iso}\sigma(A)$ for all $1 \leq i \leq m$. Hence, $A - \lambda \mu_i$ has SVEP at $0$. As seen above, $A - \lambda B \in \Phi_\times(\mathcal{X})$. Hence, since the operator $B - \mu_i$ is Riesz and commutes with $A - \lambda \mu_i$, $A - \lambda B_i = (A - \lambda \mu_i) - \lambda(B_i - \mu_i)$ has SVEP at $0$. Thus, [1, Theorem 3.52] implies that $A - \lambda B \in \times-Browder$. □

**Remark 4.1.** An alternative argument proving Theorem 4.1(b.i) goes as follows. If $\times =$ upper or left, then the hypotheses imply that $\pi_B(A, \lambda)$ has SVEP at $0$ and the Riesz operator $\pi_B(B)$ commutes with $\pi_B(A, \lambda)$. Hence, $\pi_B(A, \lambda) - \lambda^m \pi_B(B)$ has SVEP at $0$. Already, we know from (the proof of) Theorem 3.1 that $\pi_B(A, \lambda) - \lambda^m \pi_B(B) \in \Phi_\times(\mathcal{X})$, where $\Phi_\times(\mathcal{X}) = \Phi_+(\mathcal{X}) \cup \Phi_\ell(\mathcal{X})$. Hence, $\pi_B(A, \lambda) - \lambda^m \pi_B(B) = (A - \lambda B)g(A, B, \lambda) = g(A, B, \lambda)(A - \lambda B)$ is upper or (resp.) left Browder. This implies $A - \lambda B$ is upper or (resp.) left Browder.

**Essential SVEP.** Let $T_q : B[\mathcal{X}] \to B[\mathcal{X}_q], \mathcal{X}_q = \ell^\infty(\mathcal{X})/m(\mathcal{X})$, denote the homomorphism effecting the “essential enlargement $A \to T_q(A) = A_q^*$” of [4] (and [15, Theorems 17.6 and 17.9]). Then, $A \in B[\mathcal{X}]$ is upper semi-Fredholm (lower semi-Fredholm), $A \in \Phi_+(\mathcal{X})$ (resp., $A \in \Phi_-(\mathcal{X})$), if and only if $A_q$ is bounded below (resp., $A_q$ is surjective); $A_q = 0$ for an operator $A$ if and only if $A$ is compact, and if $A \in \mathcal{R}[\mathcal{X}]$, then $A_q$ is a quasinilpotent. Recall from Theorem 3.1(b.ii) and (c) that if $A, B \in B[\mathcal{X}]$ are such that $[A, B] = 0$, $\pi_B(B) = \prod_{i=1}^m (B - \mu_i) \in \mathcal{R}[\mathcal{X}]$ and $\pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda \mu_i) \in \Phi_\pm(\mathcal{X})$, then $A - \lambda B \in \Phi_\pm(\mathcal{X})$ if $\lambda \neq 0$ and $A - B_0 \in \Phi_\pm(\mathcal{X})$ if $\lambda = 0$. If we now assume that $\pi_B(A, \lambda) \not\in \Phi_-(\mathcal{X})$ (resp., the conjugate operator $\pi_B(A, \lambda)^* \not\in \Phi_-^*(\mathcal{X})$), $\lambda \neq 0$, has SVEP at $0$, then $A - \lambda B \in \Phi(\mathcal{X})$ is inner regular. Again, if we assume $\lambda = 0$ and $A \in \Phi_+(\mathcal{X})$ (resp., $A^* \in \Phi_-(\mathcal{X})$) has SVEP at $0$, then $A - B_0 \in \Phi(\mathcal{X})$ is inner regular. SVEP for an operator neither implies nor is implied by SVEP for its image under the homomorphisms $T_q$ [3, Remark 2.9]: We say in the following that $A$ has *essential SVEP* at a point $\lambda$ if $A_q = T_q(A)$ has SVEP at $\lambda$. The following corollary says that a result similar to the one above on the inner regularity of $A - \lambda B$ and $A - B_0$ holds with the hypotheses on SVEP replaced by hypotheses on essential SVEP.
Corollary 4.1. Let \( A, B \in \mathcal{B}[\mathcal{X}] \) be such that \([A, B] = 0\), \( \pi_B(B) = \bigoplus_{i=1}^m (B - \mu_i) \in \mathcal{R}[\mathcal{X}] \). \( \pi_B(A, \lambda) \) has essential SVEP at 0 whenever \( \pi_B(A, \lambda) \in \Phi_+(\mathcal{X}) \) and \( \pi_B(A, \lambda)^* \) has essential SVEP at 0 whenever \( \pi_B(A, \lambda) \in \Phi_+(\mathcal{X}) \), then \( A - \lambda B \in \Phi(\mathcal{X}) \) if \( \lambda \neq 0 \) and \( A - B_0 \in \Phi(\mathcal{X}) \) if \( \lambda = 0 \).

Proof. We consider the case in which \( \pi_B(A, \lambda) \in \Phi_+(\mathcal{X}) \) and \( \pi_B(A, \lambda)^* \) has essential SVEP at 0: The proof for the other case is similar. Arguing as in the proof of Theorem 3.1, the hypotheses \([A, B] = 0\), \( \pi_B(B) \in \mathcal{R}[\mathcal{X}] \) and \( \pi_B(A, \lambda) \in \Phi_+(\mathcal{X}) \) imply that if \( \lambda \neq 0 \), then

\[
A - \lambda \mu_i \quad \text{and} \quad A - \lambda B \in \Phi_+(\mathcal{X}) \quad \text{for all} \quad 1 \leq i \leq m
\]

and if \( \lambda = 0 \), then

\[
A \quad \text{and} \quad A - B_0 \in \Phi_+(\mathcal{X}) \quad \iff \quad T_q(A) \quad \text{and} \quad T_q(A - B_0) \quad \text{are bounded below.}
\]

Since \( T_q(A - \lambda \mu_i) \) is bounded below for all \( 1 \leq i \leq m \) implies \( \pi_B(A, \lambda) \) is bounded below, it follows from the hypothesis \( T_q(\pi_B(A, \lambda)^*) \) has SVEP that

\[
T_q(\pi_B(A, \lambda)) \quad \text{is invertible} \quad \iff \quad T_q(A - \lambda \mu_i) \quad \text{is invertible for all} \quad 1 \leq i \leq m
\]

[1, Corollary 2.24]. Letting \( A \) and \( B \) have the representations \( A = [A_{ij}]_{1 \leq i, j \leq m} \in B(\bigoplus_{j=1}^m \mathcal{X}_j) \) and \( B = \bigoplus_{j=1}^m B_j \in B(\bigoplus_{j=1}^m \mathcal{X}_j) \) (as in the proof of Theorem 3.1), this implies that \( T_q(A_{ij} - \lambda \mu_j) \) is invertible, and \( T_q(B_i - \mu_j) \) is quasinilpotent, for all \( 1 \leq j \leq m \). Since the operators \( T_q(A_{ij} - \lambda \mu_j) \) and \( T_q(B_j - \mu_j) \) commute, \( \sigma(T_q(A_{ij} - \lambda \mu_j)) = \sigma(T_q(A_{ij} - \lambda \mu_j)) - \{0\} \) and \( \sigma(A_{ij} - B_j + \mu_j) \subset \sigma(T_q(\pi_B(A_{ij} - \lambda \mu_j))) - \{0\} \) for all \( 1 \leq j \leq m \). Hence, the operators \( T_q(A_{ij} - \lambda \mu_j) \) and \( T_q(A_{ij} - B_j + \mu_j) \) are invertible for all \( 1 \leq j \leq m \). But then

\[
T_q(A - \lambda B) = T_q(\bigoplus_{j=1}^m (A_{ij} - B_j)) \quad \text{invertible} \quad \iff \quad A - \lambda B \in \Phi(\mathcal{X})
\]

and

\[
T_q(A - B_0) = T_q(\bigoplus_{j=1}^m (A_{ij} - B_j + \mu_j)) \quad \text{invertible} \quad \iff \quad A - B_0 \in \Phi(\mathcal{X}).
\]

This completes the proof. \( \square \)

5. A perturbed inner regular operator. If \( A \in \Phi_+(\mathcal{X}) \), \( \Phi_+ = \Phi_k \) or \( \Phi_r \), then \( A \) has an inner generalized inverse, which we shall denote by \( A^\dagger \) in the following. Clearly, the operator \( AA^\dagger \) is (then) a projection from \( \mathcal{X} \) onto \( A(\mathcal{X}) \), and \( I - A^\dagger A \) is a projection from \( \mathcal{X} \) onto \( A^{-1}(0) \). Let \( N \) denote a complement of \( A(\mathcal{X}) \) and let \( M \) denote a complement of \( A^{-1}(0) \). Then, \( A : M \oplus A^{-1}(0) \to A(\mathcal{X}) \oplus N \) has a matrix \( A = A_1 \oplus 0 \), where \( A_1 \in B[M, A(\mathcal{X})] \) is invertible. If \( A^\dagger \) is any generalized inverse of \( A \) such that \( A^\dagger A(\mathcal{X}) = M \) and \( (AA^\dagger)^{-1}(0) = N \), then \( A_{M,N,E}^\dagger = A^\dagger : A(\mathcal{X}) \oplus N \to M \oplus A^{-1}(0) \) has the form \( A_{M,N,E}^\dagger = A_1^{-1} \oplus E \) for some arbitrary \( E \in B[N, A^{-1}(0)] \) [7, Page 37]. Now, let \( A, B \in \mathcal{B}[\mathcal{X}] \) be such that \( B \in Holo^{-1}(\mathcal{R}[\mathcal{X}]) \) (with minimal polynomial \( \pi_B(z) \), defined as in Theorem 3.1), \( AB - BA \in \text{Prtr}(\Phi_k(\mathcal{X})) \) and \( \pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda \mu_i) \in \Phi_k(\mathcal{X}) \) for some scalar \( \lambda \). Then, the operators \( A - \lambda B \) if \( \lambda \neq 0 \) and \( A - B_0 \) if \( \lambda = 0 \) (with the operator \( B_0 \) as earlier defined) are in \( \Phi_k(\mathcal{X}) \). Letting \( S \) denote either of the operators
A − λB and A − B₀, it then follows that S has an inner generalized inverse S†. In general, A(𝓧) and S(𝓧), also A⁻¹(0) and S⁻¹(0), are quite distinct. However:

**Theorem 5.1.** If AA† = SS† and A†A = S†S, then A and S have the same range and the same null space, and S† has a representation

\[ S^\dagger = (I - \lambda A_{N,M,E}^\dagger B)^{-1}A_{N,M,F}^\dagger \text{ if } \lambda \neq 0, \text{ and} \]

\[ S^\dagger = (I - A_{N,M,E}^\dagger B_0)^{-1}A_{N,M,F}^\dagger \text{ if } \lambda = 0. \]

Here, N is a complement of A(𝓧), M is a complement of A⁻¹(0) and E, F ∈ B[N, A⁻¹(0)] are arbitrary.

**Proof.** If AA† = SS† and A†A = S†S, then

\[ S(𝓧) = SS^\dagger(𝓧) = AA^\dagger(𝓧) = A(𝓧), \text{ and} \]

\[ S^{-1}(0) = (S^\dagger S)^{-1}(0) = (A^\dagger A)^{-1}(0) = A^{-1}(0). \]

Now, choose the subspaces N, M as above. For A₁ = A|ₘ, S₁ = S|ₘ and every E ∈ B[N, A⁻¹(0)], if λ ≠ 0, then the operator

\[ I - \lambda A_{N,M,E}^\dagger B = I + A_{N,M,E}^\dagger (S - A) \]

\[ = I + \left( A_{1}^{-1} 0 \\ 0 E \right) \left( S_{1} - A_{1} 0 \\ 0 0 \right) = \left( A_{1}^{-1} S_{1} 0 1 \right) \]

from M ⊕ A⁻¹(0) into A(𝓧) ⊕ N is invertible with the inverse satisfying

\[ (I + A_{N,M,E}^\dagger (S - A))^{-1}A_{N,M,F}^\dagger = \left( S_{1}^{-1} A_{1} 0 1 \right) \left( A_{1}^{-1} 0 F \right) = \left( S_{1}^{-1} 0 F \right) \]

for every operator F ∈ B[N, A⁻¹(0)]. Again, if λ = 0, then

\[ I - \lambda A_{N,M,E}^\dagger B_0 = I + A_{N,M,E}^\dagger (S - A) = \left( A_{1}^{-1} S_{1} 0 1 \right) \]

from M ⊕ A⁻¹(0) into A(𝓧) ⊕ N is invertible with the inverse (as before) satisfying

\[ (I + A_{N,M,E}^\dagger (S - A))^{-1}A_{N,M,F}^\dagger = \left( S_{1}^{-1} A_{1} 0 1 \right) \left( A_{1}^{-1} 0 F \right) = \left( S_{1}^{-1} 0 F \right) \]

for every operator F ∈ B[N, A⁻¹(0)]. Evidently, SS†S = S, where S† = (I + A_{N,M,E}^\dagger (S - A))^{-1}A_{N,M,F}^\dagger.

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**References**


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