ON COMMUTATIVE REDUCED FILIAL RINGS

R. R. ANDRUSZKIEWICZ and K. PRYSZCZEPKO

(Received 21 July 2009)

Abstract

A ring in which every accessible subring is an ideal is called filial. We continue the study of commutative reduced filial rings started in [R. R. Andruszkiewicz and K. Pryszczepko, ‘A classification of commutative reduced filial rings’, Comm. Algebra to appear]. In particular we describe the Noetherian commutative reduced rings and construct nontrivial examples of commutative reduced filial rings without ideals which are domains.

2000 Mathematics subject classification: primary 16D25; secondary 16D70.
Keywords and phrases: ideal, filial ring, reduced ring, p-adic numbers.

1. Introduction and preliminaries

Throughout this paper we assume that all rings are associative not necessarily with unity. We denote by \( \mathbb{Z} \) the ring of integers, and by \( \mathbb{P} \) the set of all prime integers. If \( p \in \mathbb{P} \) then we write \( \mathbb{Z}_p \) and \( \mathbb{Q}_p \) for the ring of \( p \)-adic integers and the quotient field of \( p \)-adic integers, respectively. For arbitrary \( \Pi \subseteq \mathbb{P} \) we denote \( \mathbb{Q}_\Pi = \prod_{p \in \Pi} \mathbb{Q}_p \), \( \mathbb{Z}_\Pi = \prod_{p \in \Pi} \mathbb{Z}_p \).

An associative ring \( R \) is called filial if \( A \triangleleft B \triangleleft R \) implies \( A \triangleleft R \) for all subrings \( A, B \) of \( R \). The problem of describing filial rings was raised by Szász in [12]. The problem has been studied by various authors, namely, Ehrlich [5], Filipowicz and Puczyłowski [6, 7] Sands [10] and Veldsman [13].

A ring \( R \) is strongly regular if \( a \in Ra^2 \) for every \( a \in R \). It is well known that all strongly regular rings are von Neumann regular and for commutative rings the two properties coincide. The class of all strongly regular rings \( \mathbb{S} \) forms a radical in the sense of Kurosh and Amitsur [8]. A ring is reduced if it has no nontrivial nilpotent elements.

For a torsion-free ring \( R \) let \( \Pi (R) = \{ p \in \mathbb{P} \mid pR \neq R \} \). A ring \( R \) is called a CRF-ring if \( R \) is commutative, reduced and filial. Theorem 4.4 in [2] gives the following description of the \( \mathbb{S} \)-semisimple CRF-rings with an identity.
\textsc{Theorem 1.1.} Let \( \Pi \) be an arbitrary nonempty subset of \( \mathbb{P} \). Then a ring \( R \) is an \( \mathbb{S} \)-semisimple CRF-ring with an identity, such that \( \Pi(R) = \Pi \) if and only if \( R \) is isomorphic to a subring of \( \mathbb{Q}_\Pi \) of the form \( K \cap \mathbb{Z}_\Pi \) where \( K \) is the unique strongly regular subring of \( \mathbb{Q}_\Pi \) with the same identity, such that for every \( a \in K \), \( a = (a_p)_{p \in \Pi} \), we have \( a_p \in \mathbb{Z}_p \) for almost all \( p \in \Pi \).

The above theorem is important because every CRF-ring is an extension of a commutative strongly regular ring by an \( \mathbb{S} \)-semisimple CRF-ring (see [2, Proposition 4.1]).

In the present paper we study some nontrivial consequences of Theorem 1.1. In particular, using some techniques from Boolean algebra theory we characterize Noetherian CRF-rings. We also prove a structure theorem for finitely generated CRF-rings. Finally, we describe CRF-rings without ideals which are domains, and we give some nontrivial examples of such rings.

We shall need the following result proved in [2].

\textsc{Theorem 1.2.} If \( R \) is an \( \mathbb{S} \)-semisimple torsion-free CRF-ring without an identity, then \( R \) is isomorphic to some essential ideal of a ring \( S \), where \( S \) is a torsion-free \( \mathbb{S} \)-semisimple CRF-ring (see [2, Proposition 4.1]).

Let \( K \) be a subring of \( \mathbb{Q}_\Pi \) with the same identity. Take any \( a \in K \). Let us denote by \( \text{supp}(a) \) the set \( \{ p \in \Pi \mid a_p \neq 0 \} \). Then \( \mathcal{B}_K = \{ \text{supp}(a) \mid a \in K \} \) is a Boolean algebra.

For every \( Y \subseteq \Pi \), we define \( \chi_Y = (a_p)_{p \in \Pi} \in \mathbb{Z}_\Pi \) to be

\[
a_p = \begin{cases} 
0 & \text{if } p \notin Y \\
1 & \text{if } p \in Y.
\end{cases}
\] (1.1)

\textsc{Lemma 1.3.} Let \( \Pi \) be an arbitrary nonempty subset of \( \mathbb{P} \). Let \( K \) be a subring of \( \mathbb{Q}_\Pi \) with the same identity. Then \( K \) is a strongly regular ring if and only if for every \( a \in K \) there exists \( b \in K \) such that \( ab = \chi_{\text{supp}(a)} \). In particular, if \( K \) is a strongly regular ring, then \( \chi_Y \in K \) for every \( Y \in \mathcal{B}_K \).

\textsc{Lemma 1.4.} Let \( \Pi \) be an arbitrary nonempty subset of \( \mathbb{P} \). Let \( K \) be a strongly regular subring of \( \mathbb{Q}_\Pi \) with the same identity such that for every \( a \in K \), \( a = (a_p)_{p \in \Pi} \), we have \( a_p \in \mathbb{Z}_p \) for almost all \( p \in \Pi \). Put \( S = K \cap \mathbb{Z}_\Pi \). Then:

\begin{enumerate}
  \item every ideal \( J \) of \( K \) is of the form \( J = \{(1/n)i \mid i \in J \cap S, n \in \mathbb{N}\} \);
  \item if \( S \) is Noetherian, then \( K \) is also Noetherian;
  \item \( S \) contains a nonzero ideal which is a domain, if and only \( K \) contains a nonzero ideal which is a domain.
\end{enumerate}

\textsc{Proof.} (1) According to the proof of Theorem 4.4 of [2], \( K = \{(1/n)a \mid a \in S, n \in \mathbb{N}\} \). Let us first observe that \( J \subset K \) implies that \( J \cap S \subset S \). We claim that \( J = \{(1/n)i \mid i \in J \cap S, n \in \mathbb{N}\} \). Indeed, \((1/n)i = ((1/n) \cdot 1)i \in J \) for \( i \in J \cap S \). If \( j \in J \), there exists \( n \in \mathbb{N} \) such that \( n \cdot j \in S \). Then obviously \( j = (1/n)(nj) \).

Parts (2) and (3) are direct consequences of (1). \( \square \)
2. Finiteness conditions for \( S \)-semisimple CRF-rings

For a nonempty subset \( X \) of a ring \( R \), \( \langle X \rangle \) will denote the additive subgroup by \( X \), and \( [X] \) will denote the subring generated generated by \( X \). Let \((a, b)\) denote the greatest common divisor of given integers \( a \) and \( b \).

**Theorem 2.1.** Given a ring \( R \) with an identity element, the following conditions are equivalent.

1. \( R \) is a Noetherian \( S \)-semisimple CRF-ring.
2. \( R \cong \bigoplus_{i=1}^{n} D_i \), where \( D_i \) is a filial integral domain of characteristic 0, which is not a field for every \( i \in \{1, 2, \ldots, n\} \) and \( \Pi(D_i) \cap \Pi(D_j) = \emptyset \) for \( i \neq j \).

**Proof.** Suppose a ring \( R \) with an identity satisfies (1). We first note that by Theorem 1.1 there exist a nonempty subset \( \Pi \subseteq \mathbb{P} \) and a unique strongly regular subring \( K \) of \( \mathbb{Q}_{\Pi} \) with the same identity, such that for every \( a \in K \), \( a = (a_p)_{p \in \Pi} \), we have \( a_p \in \mathbb{Z}_p \) for almost all \( p \in \Pi \) and \( R \cong K \otimes_{\mathbb{Z}_{\Pi}} \mathbb{Z}_{\Pi} \). Lemma 1.4 yields that \( K \) is Noetherian. Applying Lemma 1.3, we get that \( B_K \) is an Artinian Boolean algebra (\( B_K \) satisfies the descending chain condition).

Next, we can take pairwise disjoint atoms \( \Pi_1, \ldots, \Pi_k \in B_K \) such that \( \Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k \). This is possible thanks to some standard results in Boolean algebra theory (see [9]). A trivial verification and Lemma 1.3 show that \( \chi_{\Pi_1}, \chi_{\Pi_2}, \ldots, \chi_{\Pi_k} \in K \) are pairwise orthogonal idempotents and \( 1 = \chi_{\Pi_1} + \chi_{\Pi_2} + \cdots + \chi_{\Pi_k} \). Since \( \Pi_i \) is an atom, \( \chi_{\Pi_i} K \) is an integral domain. But \( \chi_{\Pi_i} K \) is an ideal in a strongly regular ring \( K \), hence \( \chi_{\Pi_i} K \in S \). From this we conclude that \( \chi_{\Pi_i} K \) is a field. It follows that \( K = \bigoplus_{i=1}^{k} \chi_{\Pi_i} K \) and consequently \( R \cong \bigoplus_{i=1}^{k} (\chi_{\Pi_i} K) \otimes_{\mathbb{Z}_{\Pi_i}} \mathbb{Z}_{\Pi_i} \). Moreover, [1, Theorem 8.8] gives that \( D_i = (\chi_{\Pi_i} K) \otimes_{\mathbb{Z}_{\Pi_i}} \mathbb{Z}_{\Pi_i} \) is a filial integral domain of characteristic 0 and \( \Pi(D_i) = \Pi_i \) for \( i \in \{1, 2, \ldots, k\} \).

Finally, suppose that (2) holds. Note that [4, Corollary 3] implies that \( R \) is an \( S \)-semisimple CRF-ring. From [1, Theorem 3.3] it follows that \( D_i \) is a Noetherian ring as a principal ideal domain. Obviously \( R \) is a Noetherian ring. \( \square \)

We have been working under the assumption that a ring has an identity element. This condition was essential for the above proof. We will now show how to dispense with this assumption.

**Theorem 2.2.** The following conditions on a ring \( R \) are equivalent.

1. \( R \) is a Noetherian \( S \)-semisimple CRF-ring.
2. \( R \cong \bigoplus_{i=1}^{n} m_i D_i \), where \( D_i \) is a filial integral domain of characteristic 0, which is not a field, \( m_i \in \mathbb{N} \) for every \( i \in \{1, 2, \ldots, n\} \) and \( \Pi(D_i) \cap \Pi(D_j) = \emptyset \) for \( i \neq j \).

**Proof.** Let \( R \) be a Noetherian \( S \)-semisimple CRF-ring. Let us first observe that Theorem 1.2 shows that there exists a torsion-free CRF-ring \( S \) with an identity such that \( R \) is an essential ideal in \( S \). Since \( R \) is a Noetherian ring, \( \text{End}_R(R) \) is a
Noetherian $R$-module. But $S$ is an $R$-submodule of $\text{End}_R(R)$, so $S$ is a Noetherian $R$-module. Consequently $S$ is a Noetherian ring. According to Theorem 2.1 we have $S \cong \bigoplus_{i=1}^n D_i$, where $D_i$ is a filial integral domain of characteristic 0, which is not a field for every $i \in \{1, 2, \ldots, n\}$ and $\Pi(D_i) \cap \Pi(D_j) = \emptyset$ for $i \neq j$. Since $R$ is an essential ideal of $S$ it is easy to see that $R \cong \bigoplus_{i=1}^n J_i$, where $J_i$ is an nonzero ideal of $D_i$. Applying [1, Theorem 3.3], we get $J_i \cong m_i D_i$, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$. Finally, $R \cong \bigoplus_{i=1}^n m_i D_i$. This shows that (1) implies (2).

Suppose that (2) holds. From [1, Theorem 3.3] we get that $D_i$ is a Noetherian ring. By filiality of $D_i$ it follows that $m_i D_i$ is a Noetherian ring for every $i \in \{1, 2, \ldots, n\}$. Consequently, $R$ is a Noetherian ring. Moreover, from [4, Corollary 3] it may be concluded that $R$ is an $S$-semisimple CRF-ring. □

Our next goal is to determine the structure of Noetherian CRF-rings. Suppose now that $R$ is a Noetherian CRF-ring such that $S(R) \neq 0$. It is easy to verify that $S(R)$ is a Noetherian ring with an identity. So $S(R)$ is a direct summand of $R$. Let $R = S(R) \oplus T$. Since $T$ satisfies conditions of Theorem 2.2 so we need only consider $S(R)$. But the standard computation shows that every strongly regular, Noetherian CRF-ring is a finite direct sum of fields (see, for instance, [11]).

Applying the above observation and Theorem 2.2, one can immediately obtain the following structure theorem.

**Theorem 2.3.** The following conditions on a ring $R$ are equivalent.

1. $R$ is a Noetherian CRF-ring.
2. $R \cong (\bigoplus_{j=1}^k F_j) \oplus (\bigoplus_{i=1}^n m_i D_i)$, where $D_i$ is a filial integral domain of characteristic 0, which is not a field, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$, $\Pi(D_i) \cap \Pi(D_t) = \emptyset$ for $i \neq t$ and $F_j$ is a field for every $j \in \{1, 2, \ldots, k\}$.

As a final result in this section, we prove an analogue of Theorem 2.3 for finitely generated $S$-semisimple CRF-rings.

**Theorem 2.4.** The following conditions on a ring $R$ are equivalent.

1. $R$ is a finitely generated CRF-ring.
2. $R \cong (\bigoplus_{j=1}^k F_j) \oplus (\bigoplus_{i=1}^n m_i D_i)$ where $D_i$ is a finitely generated subring of $\mathbb{Q}$ with identity, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$, $\Pi(D_i) \cap \Pi(D_t) = \emptyset$ for $i \neq t$ and $F_j$ is a field for every $j \in \{1, 2, \ldots, k\}$.

**Proof.** Suppose that $R$ satisfies condition (1). It is clear that $R$ is Noetherian, so by Theorem 2.3 we obtain that $R \cong (\bigoplus_{j=1}^k F_j) \oplus (\bigoplus_{i=1}^n m_i D_i)$, where $D_i$ is a filial integral domain of characteristic 0, which is not a field, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$, $\Pi(D_i) \cap \Pi(D_t) = \emptyset$ for $i \neq t$ and $F_j$ is a field for every $j \in \{1, 2, \ldots, k\}$. Moreover, every $m_i D_i$ is a homomorphic image of the ring $R$. So $m_i D_i$ is finitely generated, but by filiality of $D_i$ we have $D_i = m_i D_i + \mathbb{Z} \cdot 1$, so consequently $D_i$ is finitely generated. Applying [1, Theorem 5.1], we see at once...
that $D_i$ is a finitely generated subring of $\mathbb{Q}$. Every $F_j$ is also a homomorphic image of $R$. Hence every $F_j$ is finitely generated. But every finitely generated field is finite.

Suppose that (2) holds. Since $D_i$ is a finitely generated subring of $\mathbb{Q}$ with identity, there exists $M \in \mathbb{N}$ such that $D_i = [1/M]$. Hence there exists $k \in \mathbb{N}$ such that $(k, M) = 1$ and $m_i D_i = k[1/M]$ (where $k = m_i/(m_i, M)$). We will show that $m_i D_i = [k/M]$. Clearly $[k/M] \subseteq k[1/M]$. Let $a \in [1/M]$. Then there exist $l \in \mathbb{Z}$ and $t \in \mathbb{N}$ such that $a = l/M^t$. But $(k, M) = 1$, so there are integers $u, v$ such that $k^{t-1} u + M^{t-1} v = 1$. Thus $ka = (k/M)^t l u + (k/M) l v \in [k/M]$. Consequently, $m_i D_i$ is finitely generated for every $i = 1, \ldots, n$. It is obvious that every $F_j$ is finitely generated. Hence $R$ is a finitely generated. Moreover, \( \bigoplus_{j=1}^n F_j \) is a CRF-ring by [4, Corollary 3] and \( \bigoplus_{j=1}^k F_j \) is clearly a subidempotent ring. Proposition 3 of [3] implies that $R$ is filial. \[\square\]

3. CRF-rings without ideals which are domains

**Theorem 3.1.** Let $\Pi$ be an arbitrary nonempty subset of $\mathbb{P}$. Then $R$ is an $S$-semisimple CRF-ring with an identity without ideals which are domains, such that $\Pi(R) = \Pi$ if and only if $R$ is isomorphic to a subring of $\mathbb{Q}_\Pi$ of the form $K \cap \mathbb{Z}_\Pi$ where $K$ is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p \in \Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and the Boolean algebra $B_K$ is atom-free.

**Proof.** Let $R$ be an $S$-semisimple CRF-ring with an identity without ideals which are domains, such that $\Pi(R) = \Pi$. From Theorem 1.1 we have that $R$ is isomorphic to a subring of $\mathbb{Q}_\Pi$ of the form $K \cap \mathbb{Z}_\Pi$ where $K$ is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p \in \Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$. Lemma 1.4 implies that a ring $K$ does not contain an ideal which is a domain. Take any nonempty $Y \subseteq B_K$. By Lemma 1.3, $a = \chi_Y \in K$.

Let $I = Ka$ is not a domain so there exist $c, d \in I$ such that $cd = 0$. Obviously $\emptyset \neq \text{supp}(c) \subseteq Y$ and $\emptyset \neq \text{supp}(d) \subseteq Y$. Moreover, $\text{supp}(c) \cap \text{supp}(d) = \emptyset$ because $cd = 0$. Hence $\text{supp}(c) \subseteq Y$ or $\text{supp}(d) \subseteq Y$ and $B_K$ is atom-free.

Conversely, according to Lemma 1.4 it is sufficient to prove that a ring $K$ does not contain an ideal which is a domain. Let $\emptyset \neq I \triangleleft K$. Take any nonzero $a \in I$. $B_K$ is atom-free so exists $Y \subseteq B_K$ such that $\emptyset \not\subseteq Y \subseteq \text{supp}(a)$. Lemma 1.3 implies that $\chi_Y \cdot \chi_{\text{supp}(a)} \not\subseteq K$ and $a \chi_Y \cdot a \chi_{\text{supp}(a) \setminus Y}$ are nonzero elements of $I$. Finally, $I$ is not a domain and the proof is complete. \[\square\]

From Theorems 1.2 and 3.1 we can easy obtain following structure theorem.

**Theorem 3.2.** $R$ is an $S$-semisimple CRF-ring without ideals which are domains if and only if $R$ is isomorphic to some essential ideal of a ring of the form $K \cap \mathbb{Z}_\Pi$, where $K$ is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p \in \Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and the Boolean algebra $B_K$ is atom-free.
4. Example

**Example 4.1.** Let $p$ be any prime number. Let $A_{i,k} = \{p^i t + k \mid t \in \mathbb{N}\}$ for $i \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, p^i - 1\}$. Let

$$
\mathcal{D} = \left\{ \bigcup_{j=1}^{n} X_j \mid n \in \mathbb{N}, \forall j \in \mathbb{N}_0 \exists t \in [0,1,\ldots,p^i-1] \ X_j = A_{i,k} \right\}.
$$

It is easy to see that for $i_1 \leq i_2$, $A_{i_1,k_1} \cap A_{i_2,k_2} = \begin{cases} A_{i_2,k_2} & \text{if } k_1 \equiv k_2 \mod p^{i_1} \\ \emptyset & \text{if } k_1 \not\equiv k_2 \mod p^{i_1} \end{cases}$.

So every element of $\mathcal{D}$ can be written as a disjoint sum of sets $A_{i,k}$. This means that if $X, Y \in \mathcal{D}$ then $X \cap Y \in \mathcal{D}$. Next, it is also clear that $A'_{i,k} = \mathbb{N} \setminus A_{i,k} = \bigcup_{j=0}^{p^i-1} A_{j,k}$. So $\mathcal{D}$ is a field of sets. Of course, for every $A_{i,k}$ and for every $j > i$, $A_{i,k} \supseteq A_{i,j}$.

**Example 4.2.** Let $\Pi = \{p_1, p_2, \ldots\}$ be any infinite subset of prime numbers. Let $\mathcal{D}$ be any atom-free Boolean algebra of subsets of $\Pi$. Such an algebra does exist, by Example 4.1. In $\mathbb{Q}_\Pi$ we define

$$
K = [a \chi_Y : Y \in \mathcal{D}, 0 \neq a \in \mathbb{Q}].
$$

It is easy to see that

$$
K = \langle a \chi_Y : Y \in \mathcal{D}, 0 \neq a \in \mathbb{Q} \rangle.
$$

Hence every nonzero $d \in K$ can be written in the form

$$
d = a_1 \chi_{Y_1} + a_2 \chi_{Y_2} + \cdots + a_k \chi_{Y_k}
$$

where $0 \neq a_i \in \mathbb{Q}$, $\emptyset \neq Y_i \in \mathcal{D}$ for every $i \in \{1, 2, \ldots, k\}$, $Y_i \cap Y_j = \emptyset$ for $i \neq j$ and $\text{supp}(d) = Y_1 \cup Y_2 \cup \cdots \cup Y_k$. We claim that $K$ is strongly regular. Let $d$ be as in (4.3). Put $d' = a_1^{-1} \chi_{Y_1} + a_2^{-1} \chi_{Y_2} + \cdots + a_k^{-1} \chi_{Y_k}$. Obviously $d' \in K$; moreover, $d \cdot d' = \chi_{\text{supp}(d)} \in K$. So by Lemma 1.3, $K$ is strongly regular subring of $\mathbb{Q}_\Pi$. Clearly $K \cap \mathbb{Z}_\Pi \neq \{0\}$. It is easy to see that $\mathcal{B}_K$ is atom-free, so Theorem 3.1 implies that $K \cap \mathbb{Z}_\Pi$ is a nonzero $\mathbb{S}$-semisimple CRF-ring, without an ideal which is a domain. Moreover, $\Pi(K \cap \mathbb{Z}_\Pi) = \Pi$.

**Acknowledgement**

The authors would like to thank the referee for many valuable suggestions.
References


R. R. ANDRUSZKIEWICZ, Institute of Mathematics, University of Białystok, 15-267 Białystok, Akademicka 2, Poland

e-mail: randrusz@math.uwb.edu.pl

K. PRYSZCZEPKO, Institute of Mathematics, University of Białystok, 15-267 Białystok, Akademicka 2, Poland

e-mail: karolp@math.uwb.edu.pl