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Directional recurrence and directional rigidity for infinite measure preserving actions of nilpotent lattices

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Abstract. Let Γ be a lattice in a simply connected nilpotent Lie group G. Given an infinite measure-preserving action T of Γ and a 'direction' in G (i.e. an element θ of the projective space $P(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G), some notions of recurrence and rigidity for T along θ are introduced. It is shown that the set of recurrent directions $\mathcal{R}(T)$ and the set of rigid directions for T are both G_δ. In the case where $G = \mathbb{R}^d$ and $\Gamma = \mathbb{Z}^d$, we prove that (a) for each G_δ-subset Δ of $P(\mathfrak{g})$ and a countable subset $D \subset \Delta$, there is a rank-one action T such that $D \subset \mathcal{R}(T) \subset \Delta$ and (b) $\mathcal{R}(T) = P(\mathfrak{g})$ for a generic infinite measure-preserving action T of Γ. This partly answers a question from a recent paper by Johnson and Şahin. Some applications to the directional entropy of Poisson actions are discussed. In the case where G is the Heisenberg group $H_3(\mathbb{R})$ and $\Gamma = H_3(\mathbb{Z})$, a rank-one Γ-action T is constructed for which $\mathcal{R}(T)$ is not invariant under the natural 'adjoint' G-action.

1. *Introduction*

Subdynamics is the study of the relationship between the dynamical properties of the action of a group G, and those of the action restricted to subgroups of G. In this paper, we consider measure-preserving actions defined on σ-finite standard measure spaces. In the 1980s Milnor generalized the study of subdynamics by defining a concept of *directional entropy* of a $\mathbb{Z}^d$-action in every direction, including the irrational directions for which there is no associated subgroup action [**Mi**]. To this end, he considered $\mathbb{Z}^d$ as a lattice in $\mathbb{R}^d$ and he exploited the geometry of mutual position of this lattice and the one-dimensional subspaces (i.e. directions) in $\mathbb{R}^d$. For a detailed account on the directional entropy of $\mathbb{Z}^2$-actions and some applications to topological dynamics (expansive subdynamics) we refer to [**Pa**] and references therein. In a recent paper [**JoSa**], Johnson and Şahin applied the 'directional approach' to study *recurrence properties* of infinite measure-preserving $\mathbb{Z}^2$-

actions. They were motivated by Feldman's proof of the ratio ergodic theorem [**Fel**]. In particular, they showed that, for each such an action, say, T, the set $\mathcal{R}(T)$ of all recurrent directions of T is a G_δ-subset of the circle $\mathbb{T}$. They also exhibited examples of rank-one actions T and T' with $R(T) = \emptyset$ and $\mathbb{T} \neq \mathcal{R}(T') \supset \{e^{\pi i q} \mid q \in \mathbb{Q}\}$. They raised a question: which G_δ-subsets of $\mathbb{T}$ are realizable as recurrence sets, that is, appear as $R(T)$ for some T? We answer this question in part.

- We show that *each countable* G_δ is a recurrence set.
- More generally, for each G_δ-subset Δ of the projective space $P(\mathbb{R}^d)$ and a countable subset D of Δ, there is a rank-one infinite measure-preserving free $\mathbb{Z}^d$-action T such that $D \subset \mathcal{R}(T) \subset \Delta$ (Theorem 5.2).
- We also prove that a *generic* infinite measure-preserving action T of $\mathbb{R}^d$ is recurrent in every direction: i.e. $\mathcal{R}(T) = P(\mathbb{R}^d)$ (Theorem 6.6).

In parallel to this, we introduce a concept of directional rigidity for $\mathbb{Z}^d$-actions and obtain similar results for realization of G_δ-subsets of $P(\mathbb{R}^d)$ as rigidity sets.

As a by product, we obtain some examples of Poisson $\mathbb{R}^d$-actions with the following entropy properties.

- There is a Poisson action $V = (V_g)_{g\in\mathbb{R}^d}$ of 0 entropy such that, for each non-zero $g \in \mathbb{R}^d$, the transformation V_g is Bernoulli of infinite entropy (Proposition 6.7).
- For each G_δ-subset $\Delta \subset P(\mathbb{R}^d)$ and a countable subset D of Δ, there is a Poisson action $V = (V_g)_{g\in\mathbb{R}^d}$ of zero entropy such that, for each non-zero $g \notin \bigcup_{\theta\in\Delta} \theta$, the transformation V_g is Bernoulli of infinite entropy and, for each $g \in \bigcup_{\theta\in D} \theta$, the transformation V_g is rigid and hence of zero entropy (Proposition 6.8).

In this connection we recall the main result from [**FeKa**]: there is a Gaussian action $V = (V_g)_{g\in\mathbb{Z}^2}$ of zero entropy such that every transformation V_g, $0 \neq g \in \mathbb{Z}^2$, is Bernoulli.

We extend the concepts of directional recurrence and directional rigidity to actions of lattices Γ in simply connected nilpotent Lie groups G. By a 'direction' we now mean a one-parameter subgroup in G. Thus the set of all directions is the projective space $P(\mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G. As in the Abelian case (considered originally in [**JoSa**]), we show the following.

- Given a measure-preserving action T of Γ, the set $\mathcal{R}(T)$ of all recurrent directions of T is a G_δ in $P(\mathfrak{g})$ (Theorems 3.5 and 3.6).

Since G acts on $P(\mathfrak{g})$ via the adjoint representation, we define another invariant $\mathcal{ER}(T)$ of *even recurrence* for T as the largest G-invariant subset of $\mathcal{R}(T)$.

- Some examples of rank-one actions T of the Heisenberg group $H_3(\mathbb{Z})$ are constructed for which $\mathcal{R}(T)$ is either empty (Theorem 7.1), countably infinite (Theorem 7.2) or uncountable (Theorem 7.3)†.
- An example of T is given such that $\mathcal{ER}(T) \neq \mathcal{R}(T)$ (Theorem 7.2).

Given an action T of Γ, we can define a natural analog of the 'suspension flow' corresponding to T. This is the *induced* (in the sense of Mackey) action $\widetilde{T}$ of G associated with T. Since $\mathcal{R}(T)$ coincides with the set $\mathcal{R}(\widetilde{T})$ of conservative $\mathbb{R}$-subactions of $\widetilde{T}$ in the Abelian case [**JoSa**], it is natural to conjecture that $\mathcal{ER}(T) = \mathcal{R}(\widetilde{T})$ in the general case. It remains an open problem. However, the analogous claim for the rigidity sets does not hold in the non-Abelian case (Remark 3.2).

† We consider $H_3(\mathbb{Z})$ as a lattice in the three-dimensional real Heisenberg group $H_3(\mathbb{R})$.

The outline of the paper is as follows. In §2, we introduce the main concepts and invariants related to the directional recurrence and rigidity. In §3, we discuss relationship between the directional recurrence and rigidity of an action of a lattice in a nilpotent Lie group and similar properties of the *suspension flow*, i.e. the induced action of the underlying Lie group. It is also shown here that the sets of recurrent and rigid directions are both G_δ. In §4, we recall the (C, F)-construction of rank-one actions and provide a sufficient condition for directions to be recurrent in terms of the (C, F)-parameters. This condition is used in §5 to construct rank-one actions of $\mathbb{Z}^d$ with various sets of recurrent directions. In §6, we prove that a generic $\mathbb{Z}^d$-action is recurrent in every direction. This section also contains some applications to the directional entropy of Poisson actions. In §7, we study directional recurrence of infinite measure-preserving actions of $H_3(\mathbb{Z})$. The final §8 contains a list of open problems and concluding remarks.

2. *Recurrence, even recurrence, rigidity and even rigidity along directions*

Let G be a simply connected nilpotent Lie group, $\mathfrak{g}$ the Lie algebra of G and $\exp : \mathfrak{g} \to G$ the exponential map. We note that exp is a diffeomorphism of $\mathfrak{g}$ onto G [**Mal**]. Let $P(\mathfrak{g})$ denote the set of lines (i.e. one-dimensional subspaces) in $\mathfrak{g}$. We endow $P(\mathfrak{g})$ with the usual topology of projective space. Then $P(\mathfrak{g})$ is a compact manifold. The adjoint G-action on $\mathfrak{g}$ induces a smooth G-action on $P(\mathfrak{g})$. We denote this action by the symbol '$\cdot$'. Given $v \in \mathfrak{g}\backslash\{0\}$, we let $\exp(v) := \{\exp(tv) \mid t \in \mathbb{R}\}$. Then $\exp(v)$ is a closed one-dimensional subgroup of G. We note that if $w = tv$ for some $t \in \mathbb{R}\backslash\{0\}$, then $\exp(w) = \exp(v)$. Hence, for each line $\theta \in P(\mathfrak{g})$, the notation $\exp(\theta)$ is well defined. Moreover, $g \exp(\theta)g^{-1} = \exp(g \cdot \theta)$ for each $g \in G$.

Let $R = (R_g)_{g\in G}$ be a measure-preserving action of G on a σ-finite standard measure space $(Y, \mathfrak{Y}, \nu)$.

Definition 2.1.

(i) We recall that R is called *conservative* if for each subset $B \in \mathfrak{Y}$, $\nu(B) > 0$, and for each compact set $K \subset G$, there is an element $g \in G\backslash K$, such that

$$\nu(B \cap R_g B) > 0.$$

(ii) We call R *recurrent along a line* $\theta \in P(\mathfrak{g})$ if the flow $(\exp(tv))_{t\in\mathbb{R}}$ is conservative for some (and hence for each) $v \in \theta\backslash\{0\}$.

(iii) We recall that R is called *rigid* if there is a sequence $(g_n)_{n\geq 1}$ of elements in G such that $g_n \to \infty$ and

$$\lim_{n\to\infty} \nu(B \cap R_{g_n} B) = \mu(B)$$

for each subset $B \in \mathfrak{Y}$ of finite measure.

(iv) We call R *rigid along a line* $\theta \in P(\mathfrak{g})$ if the flow $(\exp(tv))_{t\in\mathbb{R}}$ is rigid for some (and hence for each) $v \in \theta\backslash\{0\}$.

Denote by $\mathcal{R}(R)$ the set of all $\theta \in P(\mathfrak{g})$ such that R is recurrent along θ. Denote by $\mathcal{R}i(R)$ the set of all $\theta \in P(\mathfrak{g})$ such that R is rigid along θ. Of course, $\mathcal{R}i(R) \subset \mathcal{R}(R)$. It is easy to see that if a G-action R' is isomorphic to R then $\mathcal{R}(R') = \mathcal{R}(R)$ and $\mathcal{R}i(R') = \mathcal{R}i(R)$.

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PROPOSITION 2.2. *The sets $\mathcal{R}(R)$ and $\mathcal{R}i(R)$ are G-invariant.*

Proof. Let $\theta \in \mathcal{R}(R)$. Fix an element $g_0 \in G$. Take a subset $B \subset Y$ of positive measure and a compact $K \subset G$. Since R is recurrent along θ, there is $g \in \exp(\theta)$ such that $g \notin K$ such that $\nu(B \cap R_g B) > 0$. Hence

$$0 < \nu(R_{g_0} B \cap R_{g_0} R_g B) = \nu(R_{g_0} B \cap R_{g_0 g g_0^{-1}} R_{g_0} B).$$

Since $g_0 g g_0^{-1} \in \exp(g_0 \cdot \theta)$ and $g_0 g g_0^{-1} \notin g_0 K g_0^{-1}$, it follows that the flow $(R_g)_{g \in g_0 \cdot \theta}$ is conservative. Thus $\mathcal{R}(R)$ is G-invariant. In a similar way we can verify that $\mathcal{R}i(R)$ is G-invariant. □

From now on we fix a lattice Γ in G. We recall that there exists a lattice in G if and only if the structural constants of $\mathfrak{g}$ are all rational [**Mal**]. Moreover, every lattice in G is uniform [**Mal**], i.e. co-compact. We fix a right-invariant metric $\mathrm{dist}(\cdot\,,\cdot)$ on G, compatible with the topology.

Let $T = (T_\gamma)_{\gamma \in \Gamma}$ be a measure-preserving action of Γ on a σ-finite standard measure space $(X, \mathfrak{B}, \mu)$. Although, in general, T does not extend to a G-action on $(X, \mathfrak{B}, \mu)$, it is possible to give an analog of Definition 2.1 for T.

Definition 2.3.

(i) We call T *recurrent along a line* $\theta \in P(\mathfrak{g})$ if, for each $\epsilon > 0$ and every subset $A \in \mathfrak{B}$, $\mu(A) > 0$, there is an element $\gamma \in \Gamma \setminus \{1_\Gamma\}$ and an element $g \in \exp(\theta)$ such that $\mathrm{dist}(\gamma, g) < \epsilon$ and $\mu(A \cap T_\gamma A) > 0$.

(ii) We call T *evenly recurrent along a line* $\theta \in P(\mathfrak{g})$ if T is recurrent along every line from the G-orbit of θ.

(iii) We call T *rigid along a line* $\theta \in P(\mathfrak{g})$ if there is a sequence $(\gamma_n)_{n \geq 1}$ of elements in Γ such that $\lim_{n\to\infty} \inf_{g \in \exp(\theta)} \mathrm{dist}(\gamma_n, g) = 0$ and

$$\lim_{n\to\infty} \mu(A \cap T_{\gamma_n} A) = \mu(A)$$

for each subset $A \in \mathfrak{B}$ with $\mu(A) < \infty$†.

(iv) We call T *evenly rigid along a line* $\theta \in P(\mathfrak{g})$ if T is rigid along every line from the G-orbit of θ.

We denote by $\mathcal{R}(T)$ the set of all $\theta \in P(\mathfrak{g})$ such that T is recurrent along θ. We denote by $\mathcal{R}i(T)$ the set of all $\theta \in P(\mathfrak{g})$ such that T is rigid along θ. In a similar way, we denote by $\mathcal{E}\mathcal{R}(T)$ and $\mathcal{E}\mathcal{R}i(T)$ the set of all $\theta \in P(\mathfrak{g})$ such that T is evenly recurrent along them and evenly rigid along them, respectively.

Of course, $\mathcal{R}(T) \supset \mathcal{E}\mathcal{R}(T)$, $\mathcal{R}i(T) \supset \mathcal{E}\mathcal{R}i(T)$, $\mathcal{R}(T) \supset \mathcal{R}i(T)$ and $\mathcal{E}\mathcal{R}(T) \supset \mathcal{E}\mathcal{R}i(T)$. For G Abelian, $\mathcal{R}(T) = \mathcal{E}\mathcal{R}(T)$ and $\mathcal{R}i(T) = \mathcal{E}\mathcal{R}i(T)$. However, in general, $\mathcal{R}(T) \neq \mathcal{E}\mathcal{R}(T)$ (see Theorem 7.2 below) and $\mathcal{R}i(T) \neq \mathcal{E}\mathcal{R}i(T)$.

† This means that $T_{\gamma_n} \to \mathrm{Id}$ as $n \to \infty$ in the weak topology on the group of all μ-preserving invertible transformations of X.

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Remark 2.4.

(i) It is easy to see that if θ is 'rational', i.e. the intersection $\Gamma \cap \exp(\theta)$ is non-trivial, say, there is $\gamma_0 \neq 1_\Gamma$ such that $\Gamma \cap \exp(\theta) = \{\gamma_0^n \mid n \in \mathbb{Z}\}$, then θ is recurrent if and only if γ_0 (i.e. the action of $\mathbb{Z}$ generated by γ_0) is conservative. In a similar way, if θ is rigid if and only if γ_0 is rigid.

(ii) If $\theta \in \mathcal{R}(T)$, then we have $\{\gamma \cdot \theta \mid \gamma \in \Gamma\} \subset \mathcal{R}(T)$. In a similar way, if $\theta \in \mathcal{R}i(T)$, then we have $\{\gamma \cdot \theta \mid \gamma \in \Gamma\} \subset \mathcal{R}i(T)$. This can be shown in a similar way as in Proposition 2.2 (plus the fact that dist is right-invariant).

Given $g \in G$ and $\theta \in P(\mathfrak{g})$, we denote by $\mathrm{dist}(g, \exp(\theta))$ the distance from g to the closed subgroup $\exp(\theta)$, i.e.

$$\mathrm{dist}(g, \exp(\theta)) := \inf_{h \in \exp(\theta)} \mathrm{dist}(g, h) = \min_{h \in \exp(\theta)} \mathrm{dist}(g, h).$$

Since, in Definition 2.3(i), there is no estimate (from below) for the ratio $\mu(A \cap T_\gamma A)/\mu(A)$, the following lemma—which is equivalent to Definition 2.3(i)—is more useful for applications.

LEMMA 2.5. *Let $\theta \in \mathcal{R}(T)$. Then, for each $\epsilon > 0$, a compact $K \subset G$ and a subset $A \subset X$ of finite measure, there is a Borel subset $A_0 \subset A$ and Borel one-to-one map $R : A_0 \to A$ and a Borel map $\vartheta : A_0 \ni x \mapsto \vartheta_x \in \Gamma\backslash K$ such that $\mu(A_0) \geq 0.5\mu(A)$ and $Rx = T_{\vartheta_x} x$ and* $\mathrm{dist}(\vartheta_x, \exp(\theta)) < \epsilon$ *for all $x \in A_0$.*

Proof. We use a standard exhaustion argument. Let

$$\Gamma_\epsilon := \{\gamma \in \Gamma\backslash\{1\} \mid \mathrm{dist}(\gamma, \exp(\theta)) < \epsilon\}.$$

Enumerate the elements of Γ_ϵ: i.e. let $\Gamma_\epsilon = \{\gamma_n\}_{n \geq 1}$. We now set $A_1 := A \cap T_{\gamma_1}^{-1} A$, $B_1 := T_{\gamma_1} A_1$, $A_2 := (A\backslash(A_1 \cup B_1)) \cap T_{\gamma_2}^{-1}(A\backslash(A_1 \cup B_1))$, $B_2 := T_{\gamma_2} A_2$, and so on. Then we obtain two sequences $(A_n)_{n \geq 1}$ and $(B_n)_{n \geq 1}$ of Borel subsets of A such that $A_i \cap A_j = B_i \cap B_j = \emptyset$ whenever $i \neq j$ and $T_{\gamma_i} A_i = B_i$ for all i. We let $A_0 := \bigsqcup_{i \geq 1} A_i$ and $B_0 := \bigsqcup_{i \geq 1} B_i$. It follows, from Definition 2.3(i), that $\mu(A\backslash(A_0 \cup B_0)) = 0$. Since $\mu(A_0) = \mu(B_0)$, it follows that $\mu(A_0) \geq 0.5\mu(A)$. It remains to let $\vartheta_x := \gamma_i$ for all $x \in A_i$, $i \geq 1$. □

3. *Recurrence and rigidity along directions in terms of the induced G-actions*

Denote by $\widetilde{T} = (\widetilde{T}_g)_{g \in G}$ the action of G *induced from* T (see [**Ma**, **Zi**]). We recall that the space of $\widetilde{T}$ is the product space $(G/\Gamma \times X, \lambda \times \mu)$, where λ is the unique G-invariant probability measure on the homogeneous space G/Γ. To define $\widetilde{T}$ we first choose a Borel cross section $s : G/\Gamma \to G$ of the natural projection $G \to G/\Gamma$. Moreover, we may assume without loss of generality that $s(\Gamma) = 1_G$ and s is a homeomorphism when restricted to an open neighborhood of Γ, this neighborhood is of full measure and the measure of the boundary of the neighborhood is zero. Define a Borel map $h_s : G \times G/\Gamma \to \Gamma$ by setting

$$h_s(g, g_1\Gamma) = s(gg_1\Gamma)^{-1} g s(g_1\Gamma).$$

Then h_s satisfies the one-cocycle identity: i.e. $h_s(g_2, g_1 g\Gamma) h_s(g_1, g\Gamma) = h_s(g_2 g_1, g\Gamma)$ for all $g_1, g_2, g \in \Gamma$. We now set, for $g, g_1 \in G$ and $x \in X$,

$$\widetilde{T}_g(g_1\Gamma, x) := (gg_1\Gamma, T_{h_s(g, g_1\Gamma)} x).$$

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Then $(\widetilde{T}_g)_{g\in G}$ is a measure-preserving action of G on $(G/\Gamma \times X, \lambda \times \mu)$. We note that the isomorphism class of $\widetilde{T}$ does not depend on the choice of s.

THEOREM 3.1. *Let* $G = \mathbb{R}^d$ *and* $\Gamma = \mathbb{Z}^d$, $d \geq 1$. *Then* $\mathcal{R}(\widetilde{T}) = \mathcal{R}(T)$ *and* $\mathcal{R}i(\widetilde{T}) = \mathcal{R}i(T)$.

Proof. We consider the quotient space G/Γ as $[0, 1)^d$. Given $g = (g_1, \ldots, g_d) \in \mathbb{R}^d$, we let $[g] = (E(g_1), \ldots, E(g_d))$ and $\{g\} := (F(g_1), \ldots, F(g_d))$, where $E(.)$ and $F(.)$ denote the integer part and the fractional part of a real. If the cross section $s : [0, 1)^d \to \mathbb{R}^d$ is given by the formula $s(y) := y$, then we have $h_s(g, y) = [g + y]$ for all $g \in G$ and $y \in [0, 1)^d$.

(A) We first show that $\mathcal{R}i(T) = \mathcal{R}i(\widetilde{T})$. Let $\theta \in \mathcal{R}i(T)$. Then there are $\gamma_n \in \Gamma$ and $t_n \in \theta$ such that $\mathrm{dist}(\gamma_n, t_n) \to 0$ and $T_{\gamma_n} \to \mathrm{Id}_X$ weakly as $n \to \infty$. We claim that $\widetilde{T}_{t_n} \to \mathrm{Id}_{(G/\Gamma)\times X}$ weakly as $n \to \infty$. Indeed, let $\epsilon_n := t_n - \gamma_n$. Then

$$\widetilde{T}_{t_n}(y, x) = (\{t_n + y\}, T_{[t_n+y]}x) = (\{\epsilon_n + y\}, T_{\gamma_n} T_{[\epsilon_n+y]}x). \tag{3.1}$$

Since the Lebesgue measure of the subset $Y_n := \{y \in [0, 1)^d \mid \epsilon_n + y \in [0, 1)^d\}$ goes to one as $n \to \infty$ and $\{\epsilon_n + y\} = y$ and $[\epsilon_n + y] = 0$ for all $y \in Y_n$, it follows that $\widetilde{T}_{t_n} \to \mathrm{Id}_{(G/\Gamma)\times X}$ as $n \to \infty$. Thus we obtain that $\theta \in \mathcal{R}i(\widetilde{T})$.

Conversely, let $\theta \in \mathcal{R}i(\widetilde{T})$. Then there are $t_n \in \theta$, $n \in \mathbb{N}$, such that

$$\widetilde{T}_{t_n} \to \mathrm{Id}_{(G/\Gamma)\times X} \text{ weakly as } n \to \infty. \tag{3.2}$$

It follows from (3.1) that the sequence of transformations $y \mapsto \{t_n + y\}$ of G/Γ converge to $\mathrm{Id}_{G/\Gamma}$ as $n \to \infty$. This, in turn, implies that there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ such that $\lim_{n\to\infty} \mathrm{dist}(t_n, \gamma_n) = 0$. Therefore, the Lebesgue measure of the subset $\{y \in G/\Gamma \mid [t_n + y] = \gamma_n\}$ converges to one as $n \to \infty$. Now (3.1) and (3.2) yield that $T_{\gamma_n} \to \mathrm{Id}_X$. Hence $\theta \in \mathcal{R}i(T)$.

(B) We now show that† $\mathcal{R}(T) = \mathcal{R}(\widetilde{T})$. Take $\theta \in \mathcal{R}(T)$. Given a subset $A \subset G/\Gamma \times X$ of positive measure, a compact $K \subset G$ and $\epsilon > 0$, we find two subsets $B \subset X$ and $C \subset G/\Gamma$ of finite positive measure such that

$$(\mathrm{Leb} \times \mu)(A \cap (B \times C)) > 0.99\mathrm{Leb}(B)\mu(C). \tag{3.3}$$

For $t \in G$, we set $B_t := \{y \in B \mid t + y \in B \text{ and } [t + y] = 0\}$. Then we find $\epsilon_1 > 0$ so small that $\mathrm{Leb}(B_t) > 0.5\mathrm{Leb}(B)$ for each $t \in G$ such that $\mathrm{dist}(t, 0) < \epsilon_1$. By Lemma 2.5, there are elements $\gamma_1, \ldots, \gamma_l \in \Gamma$, $t_1, \ldots, t_l \in \theta\backslash K$ and pairwise disjoint subsets $C_1, \ldots, C_l$ of C such that $\max_{1\leq j\leq l} \mathrm{dist}(\gamma_j, t_j) < \min(\epsilon, \epsilon_1)$, the sets $T_{\gamma_1}A_1, \ldots, T_{\gamma_l}C_l$ are mutually disjoint subsets of C and $\mu(\bigsqcup_{j=1}^l C_j) > 0.4\mu(C)$. We now let $A' := \bigsqcup_{j=1}^l B_{t_j} \times C_j$. Of course, A' is a subset of $B \times C$. We have

$$\widetilde{T}_{t_j}(b, c) = (\{t_j + b\}, T_{\gamma_j}c) \subset B \times C \quad \text{if } b \in B_j \text{ and } c \in C_j$$

for each $j = 1, \ldots, l$. Moreover, the sets $\widetilde{T}_{t_j}(B_j \times C_j)$, $j = 1, \ldots, l$, are pairwise disjoint and $(\mathrm{Leb} \times \mu)(\bigsqcup_{j=1}^l (B_j \times C_j)) > 0.2(\mathrm{Leb} \times \mu)(B \times C)$. It now follows

† Although this fact was originally stated in [**JoSa**], we give here an alternative proof because, in our opinion, the proof of the inclusion $\mathcal{R}(T) \subset \mathcal{R}(\widetilde{T})$ was not completed there.

from (3.3) that there is $j \in \{1, \ldots, l\}$ such that $(\mathrm{Leb} \times \mu)(\widetilde{T}_{t_j}(A \cap (B_j \times C_j) \cap A) > 0$. Hence $\theta \in \mathcal{R}(\widetilde{T})$.

Conversely, let $\theta \in \mathcal{R}(\widetilde{T})$. Given $\epsilon > 0$, let $Y = [1/2, 1/2 + \epsilon) \subset G/\Gamma$. It is easy to see that if $gY \cap Y \neq \emptyset$ for some $g \in G$, then $\mathrm{dist}(g, \Gamma) < \epsilon$ and the map $Y \ni y \mapsto [g + y] \in \mathbb{Z}^d$ is constant. Let A be a subset of X of finite positive measure. Then there is $g \in \theta$ such that $\mathrm{dist}(g, 0) > 100$ and

$$0 < (\mathrm{Leb} \times \mu)((Y \times A) \cap \widetilde{T}_g(Y \times A)) = \mathrm{Leb}(gY \cap Y)\mu(A \cap T_\gamma A),$$

where $\gamma := [g + y] \in \Gamma$ for all $y \in Y$. It follows that $\mathrm{dist}(\gamma, \theta) < \epsilon$ and $\gamma \neq 0$. Hence $\theta \in \mathcal{R}(T)$. □

Remark 3.2. We note that the equality $\mathcal{R}i(\widetilde{T}) = \mathcal{E}\mathcal{R}i(T)$ does not hold for non-Abelian nilpotent groups. Consider, for instance, the case where $G = H_3(\mathbb{R})$ and $H = H_3(\mathbb{Z})$ (see §7 for their definition). Let T be an ergodic action of $H_3(\mathbb{Z})$. We claim that $\widetilde{T}$ is not rigid and hence $\mathcal{R}i(\widetilde{T}) = \emptyset$. Indeed, if $\widetilde{T}$ were rigid, then the quotient G-action by translations on G/Γ is also rigid. However, the latter action is mixing relative to the subspace generated by all eigenfunctions [**Au-Ha**]. On the other hand, there are examples of weakly mixing $H_3(\mathbb{Z})$-actions T such that $\mathcal{R}i(T)$ contains the line passing through the center [**Da3**].

COROLLARY 3.3. *Let $G = \mathbb{R}^d$ and $\Gamma = \mathbb{Z}^d$, $d \geq 1$. If an action T of Γ is ergodic and extends to an action $\widehat{T}$ of G on the same measure space where T is defined, then $\mathcal{R}(T) = \mathcal{R}(\widehat{T})$.*

Proof. It follows, from the condition of the corollary, that the induced G-action $\widetilde{T}$ is isomorphic to the product $\widehat{T} \times D$, where D is the natural G-action by translations on G/Γ [**Zi**, Proposition 2.10]. Since D is finite measure-preserving, $\mathcal{R}(\widehat{T} \times D) = \mathcal{R}(\widehat{T})$ (see Lemma 3.4(ii) below). It remains to apply Theorem 3.1. □

We leave the proof of the following non-difficult statement to the reader as an exercise.

LEMMA 3.4. *Let $F = (F_t)_{t \in \mathbb{R}}$ be a flow-preserving a σ-finite measure and let $S = (S_t)_{t \in \mathbb{R}}$ be a probability-preserving flow.*
(i) *F is conservative if and only if the transformation F_1 is conservative.*
(ii) *F is conservative if and only if the product flow $(F_t \times S_t)_{t \in \mathbb{R}}$ is conservative*†.
(iii) *F is rigid if and only if F_1 is rigid.*

We now describe the 'topological type' of $\mathcal{R}(T)$ and $\mathcal{E}\mathcal{R}(T)$ as subspaces of $P(\mathfrak{g})$. We first consider the Abelian case and provide a short proof of [**JoSa**, Theorem 1.3] stating that $\mathcal{R}(T)$ is a G_δ.

THEOREM 3.5. *Let $G = \mathbb{R}^d$ and $\Gamma = \mathbb{Z}^d$, $d \geq 1$. The subsets $\mathcal{R}(T)$ and $\mathcal{R}i(T)$ are both G_δ in $P(\mathbb{R}^d)$.*

† A similar claim for transformations (i.e. $\mathbb{Z}$-actions) is proved in [**Aa**]. We note that (ii) follows from that claim and (i).

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Proof. Let $(\widetilde{X}, \widetilde{\mu})$ be the space of $\widetilde{T}$. Denote by $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ the group of all $\widetilde{\mu}$-preserving invertible transformations of $\widetilde{X}$. We endow it with the standard weak topology. Then $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ is a Polish group (see [**DaSi**] and references therein). Fix a norm on $\mathbb{R}^d$. Denote by $\mathcal{S}$ the unit ball in $\mathbb{R}^d$. We define a map $\mathfrak{m} : \mathcal{S} \to \mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ by setting $\mathfrak{m}(v) := \widetilde{T}_v$. It is obviously continuous. We recall that the subset $\mathfrak{R}$ of conservative infinite measure-preserving transformations of $(\widetilde{X}, \widetilde{\mu})$ is a G_δ in $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ [**DaSi**]. It follows from this fact and Lemma 3.4(i) that the set

$$\mathfrak{m}^{-1}(\mathfrak{R}) = \{v \in \mathcal{S} \mid \text{the flow } (\widetilde{T}_{tv})_{t\in\mathbb{R}} \text{ is conservative}\}$$

is a G_δ in $\mathcal{S}$: i.e. the intersection of a countable number of open subsets. Since $\mathfrak{m}^{-1}(\mathfrak{R})$ is centrally symmetric (i.e. if $v \in \mathfrak{m}^{-1}(\mathfrak{R})$, then $-v \in \mathfrak{m}^{-1}(\mathfrak{R})$), we may assume, without loss of generality, that these open sets are also centrally symmetric. The natural projection of $\mathcal{S}$ onto $P(\mathbb{R}^d)$ is just the 'gluing' of the pairs of centrally symmetric points. We note that the projection of $\mathfrak{m}^{-1}(\mathfrak{R})$ to $P(\mathbb{R}^d)$ is exactly $\mathcal{R}(\widetilde{T})$. It follows that $\mathcal{R}(\widetilde{T})$ is a G_δ in $P(\mathfrak{g})$. It remains to apply Theorem 3.1.

To show that $\mathcal{R}i(T)$ is a G_δ argue in a similar way and use the fact that the set of all rigid transformations is a G_δ in $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ [**DaSi**] and apply Lemma 3.4(iii). □

We now consider the general case (independently of Theorem 3.5).

THEOREM 3.6. *The subsets $\mathcal{R}(T)$ and $\mathcal{R}i(T)$ are both G_δ in $P(\mathfrak{g})$.*

Proof. Let $\Gamma \backslash \{1\} = \{\gamma_k \mid k \in \mathbb{N}\}$.

(A) We first prove that $\mathcal{R}(T)$ is a G_δ. For each $g \in G$, the map

$$P(\mathfrak{g}) \ni \theta \mapsto \mathrm{dist}(g, \exp(\theta)) := \inf_{h\in\exp(\theta)} \mathrm{dist}(g, h) \in \mathbb{R} \tag{3.4}$$

is continuous. Now for a subset $A \subset X$ with $0 < \mu(A) < \infty$ and $\epsilon > 0$, we construct a sequence $A_1, A_2, \ldots$ of subsets in A as given by (cf. with the proof of Lemma 2.5):

$$A_1 := \begin{cases} A \cap T_{\gamma_1}^{-1} A & \text{if } \mathrm{dist}(\gamma_1, \exp(\theta)) < \epsilon, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$A_2 := \begin{cases} (A \backslash (A_1 \cup T_{\gamma_1} A_1)) \cap T_{\gamma_2}^{-1}(A \backslash (A_1 \cup T_{\gamma_1} A_1)) & \text{if } \mathrm{dist}(\gamma_2, \exp(\theta)) < \epsilon, \\ \emptyset & \text{otherwise,} \end{cases}$$

and so on. Then (as in Lemma 2.5) $A_i \cap A_j = \emptyset$, $T_{\gamma_i} A_i \subset A$ and $T_{\gamma_i} A_i \cap T_{\gamma_j} A_j = \emptyset$ if $i \neq j$. For each $m \in \mathbb{N}$, we set

$$\Theta_{\epsilon,A,m} := \left\{\theta \in P(\mathfrak{g}) \;\middle|\; \sum_{j\leq m} \mu(A_j) > 0.4\mu(A)\right\}.$$

We note that for each $j > 0$, the map $P(\mathfrak{g}) \ni \theta \mapsto \mu(A_j) \in \mathbb{R}$ is lower semicontinuous. Indeed, this map is (up to a multiplicative constant) the indicator function of the subset $\{\theta \mid \mathrm{dist}(\gamma_j, \exp(\theta)) < \epsilon\}$ which is open because (3.4) is continuous. It follows that $\Theta_{\epsilon,A,m}$ is an open subset in $P(\mathfrak{g})$. Fix a countable family $\mathfrak{D}$ of subsets of finite positive measure in X such that $\mathfrak{D}$ is dense in $\mathfrak{B}$. We claim that

$$\mathcal{R}(T) = \bigcap_{D\in\mathfrak{D}} \bigcap_{l=1}^{\infty} \bigcup_{m=1}^{\infty} \Theta_{1/l,D,m}. \tag{3.5}$$

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Indeed, if T is recurrent along a line $\theta \in P(\mathfrak{g})$, then, for each $\epsilon > 0$ and each subset A of positive measure, $\mu(\bigsqcup_j A_j) \geq 0.5\mu(A)$ (as in Lemma 2.5). We then obtain that there exists $m > 0$ with $\mu(\bigsqcup_{j=1}^m A_i) > 0.4\mu(A)$. Hence $\theta \in \Theta_{\epsilon,A,m}$. Now let A run $\mathfrak{D}$ and let ϵ run $\{1/l \mid l \in \mathbb{N}\}$. Then θ belongs to the right-hand side of (3.5).

Conversely, take θ from the right-hand side of (3.5). Let A be a subset of X of positive measure. Then there is $D \in \mathfrak{D}$ such that $\mu(A \cap D) > 0.999\mu(D)$. Take $l \in \mathbb{N}$. Select $m > 0$ such that $\theta \in \Theta_{1/l,D,m}$. Then

$$\mu\bigg(\bigsqcup_{j\leq m} D_j\bigg) > 0.4\mu(D) \quad \text{and hence } \mu\bigg(\bigsqcup_{j\leq m} T_{\gamma_j} D_j\bigg) > 0.4\mu(D).$$

Therefore there is $j < d$ with $\mu(T_{\gamma_j}A \cap A) > 0$ and (because $\theta \in \Theta_{1/l,D,m}$) $\operatorname{dist}(\gamma_j, \exp(\theta)) < 1/m$.

(B) To show that $\mathcal{R}i(T)$ is G_δ we first denote by τ a metric on $\operatorname{Aut}(X, \mu)$ compatible with the weak topology. Now it suffices to note that

$$\mathcal{R}i(T) = \bigcap_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{\{n>N \mid \tau(T_{\gamma_n},\mathrm{Id})<1/k\}} \{\theta \in P(\mathfrak{g}) \mid \operatorname{dist}(\gamma_n, \exp(\theta)) < 1/k\}$$

and use (3.4). $\square$

4. (C, F)-*construction and directional recurrence of rank-one actions*

We first recall a (C, F)-construction of group actions (see [**Da1**] for a detailed exposition and various applications).

Let $(C_n)_{n>0}$ and $(F_n)_{n\geq 0}$ be two sequences of finite subsets in Γ such that the following conditions hold:

(I) $F_0 = \{1\}$, $1 \in C_n$ and $\#C_n > 1$ for all n;

(II) $F_n C_{n+1} \subset F_{n+1}$ for all n;

(III) $F_n c \cap F_n c' = \emptyset$ for all $c \neq c' \in C_{n+1}$ and n; and

(IV) $\gamma F_n C_{n+1} C_{n+2} \cdots C_m \subset F_{m+1}$ eventually in m for each $\gamma \in \Gamma$ and every n.

Then the infinite product space $X_n := F_n \times C_{n+1} \times C_{n+1} \times \cdots$ is a (compact) Cantor set. It follows from (II) and (III) that the map

$$X_n \ni (f_n, c_{n+1}, c_{n+2}, c_{n+3}, \ldots) \mapsto (f_n c_{n+1}, c_{n+2}, c_{n+3}, \ldots) \in X_{n+1}$$

is a continuous embedding. Denote by X the (topological) inductive limit of the sequence $X_1 \subset X_2 \subset \cdots$. Then X is a locally compact Cantor set. For a subset $A \subset F_n$, we let $[A]_n := \{x = (f_n, c_{n+1}, \ldots) \in X_n \mid f_n \in A\}$. Then $[A]_n$ is a compact-open subset of X. We call it an *n-cylinder*. The family of all cylinders (i.e. the family of all compact-open subsets of X) is a base of the topology in X. Given $\gamma \in \Gamma$ and $x \in X$, in view of (II) and (IV), there is n such that $x = (f_n, c_{n+1}, \ldots) \in X_n$ and $\gamma f_n \in F_n$. Then we let $T_\gamma x := (\gamma f_n, c_{n+1}, \ldots) \in X_n \subset X$. It is standard to verify that T_γ is a well defined homeomorphism of X. Moreover, $T_\gamma T_{\gamma'} = T_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$: i.e. $T := (T_\gamma)_{\gamma\in\Gamma}$ is a continuous action of Γ on X. It is called the *(C, F)-action of Γ associated with* $(C_n, F_{n-1})_{n>0}$ (see [**Da1, Da3**]). This action is free and minimal. There is a unique (up to scaling) T-invariant σ-finite Borel measure μ on X. It is easy to compute that

$$\mu([A]_n) = \frac{\#A}{\#C_1 \cdots \#C_n}$$

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for all subsets $A \subset F_n$, $n > 0$, provided that $\mu(X_0) = 1$. We note that $\mu(X) = \infty$ if and only if

$$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} = \infty. \tag{4.1}$$

Of course, (X, μ, T) is an ergodic conservative dynamical system. It is of funny rank one (see [**Da1**] and [**Da3**] for the definition). Conversely, every funny rank-one free system appears this way: i.e. it is isomorphic to a (C, F)-system for an appropriately chosen sequence $(C_n, F_{n-1})_{n\geq 1}$. We state, without proof, a lemma from [**Da3**].

LEMMA 4.1. *Let A be a finite subset F_n and let $g \in G$. Then $[A]_n \cap T_g[A]_n \neq \emptyset$ if and only if $g \in \bigcup_{m>n} AC_{n+1}\cdots C_m C_m^{-1}\cdots C_{n+1}^{-1}A^{-1}$. Furthermore, if we let*

$$\mathcal{N}_m^{g,A} := \{(a, c_{n+1}, \ldots, c_m) \in A \times C_{n+1} \times \cdots \times C_m \mid gac_{n+1}\cdots c_m \in AC_{n+1}\cdots C_m\},$$

then $\mu([A]_n \cap T_g[A]_n) = \lim_{m\to\infty} (\#\mathcal{N}_m^{g,A}/\#C_1\cdots\#C_m)$.

To state the next assertion we need more notation. Denote the natural projection by $\pi : \mathfrak{g}\setminus\{0\} \to P(\mathfrak{g})$. Let κ be a metric on $P(\mathfrak{g})$ compatible with the topology. Given two sequences $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ of finite subsets in G, we write $A_n \gg B_n$ as $n \to \infty$ if

$$\lim_{n\to\infty} \max_{a\in A_n, b\in B_n} \kappa(\pi(\log(ab), \pi(\log(a)) = 0.$$

PROPOSITION 4.2. *Let $T = (T_\gamma)_{\gamma\in\Gamma}$ be a (C, F)-action of Γ associated with a sequence $(C_n, F_{n-1})_{n=1}^\infty$ satisfying* (I)–(IV). *Then*

(i) $\mathcal{R}(T) \subset \bigcap_{\gamma\in\Gamma} \gamma \cdot (\bigcap_{n=1}^\infty \overline{\bigcup_{m\geq n} \pi(\log(C_n\cdots C_m C_m^{-1}\cdots C_n^{-1}\setminus\{1\}))})$.

(ii) *If, moreover, the group generated by all C_j, $j > 0$, is commutative and $C_j\setminus\{1\} \gg C_1\cdots C_{j-1}$ as $j \to \infty$, then*

$$\mathcal{R}(T) \subset \bigcap_{\gamma\in\Gamma} \gamma \cdot \bigg(\bigcap_{n=1}^\infty \overline{\bigcup_{m\geq n} \pi(\log(C_m C_m^{-1}\setminus\{1\}))}\bigg).$$

(iii) *If, in addition, there is $c_j \in \Gamma$ such that $C_j = \{1, c_j\}$ for each $j > 0$, then*

$$\mathcal{R}(T) \subset \bigcap_{\gamma\in\Gamma} \gamma \cdot \bigg(\bigcap_{n=1}^\infty \overline{\{\pi(\log c_m) \mid m \geq n\}}\bigg).$$

Proof. (i) Let $\theta \in \mathcal{R}(T)$. Then for each $n > 0$, there is a sequence $(\gamma_m)_{m=1}^\infty$ of elements of Γ such that $\gamma_m \neq 1$ and $\mu(T_{\gamma_m}[1]_n \cap [1]_n) > 0$ for each m and $\mathrm{dist}(\gamma_m, \exp(\theta)) \to 0$ as $m \to \infty$. Hence we deduce from Lemma 4.1 that

$$\inf\bigg\{\mathrm{dist}(\gamma, \exp(\theta)) \mid \gamma \in \bigcup_{m>n} C_{n+1}\cdots C_m C_m^{-1}\cdots C_{n+1}^{-1}\setminus\{1\}\bigg\} = 0.$$

This yields that $\theta \in \overline{\pi(\log(\bigcup_{m>n} C_{n+1}\cdots C_m C_m^{-1}\cdots C_{n+1}^{-1}\setminus\{1\}))}$. Therefore

$$\mathcal{R}(T) \subset \bigcap_{n\geq 1} \overline{\bigcup_{m>n} \pi(\log(C_{n+1}\cdots C_m C_m^{-1}\cdots C_{n+1}^{-1}\setminus\{1\}))}.$$

Since $\mathcal{R}(T)$ is invariant under Γ, in view of Remark 2.4(ii), the claim (i) follows.

(ii) Denote by A the smallest closed Lie subgroup of G containing all C_j, $j > 0$. Since A is Abelian, the restriction of log to A is a group homomorphism. Hence the condition $C_j \backslash \{1\} \gg C_1 \cdots C_{j-1}$ as $j \to \infty$ implies $C_j C_j^{-1} \backslash \{1\} \gg C_1 C_1^{-1} \cdots C_{j-1} C_{j-1}^{-1}$ as $j \to \infty$. Now (ii) easily follows from (i).

(iii) It suffices to note that $C_m C_m^{-1} \backslash \{1\} = \{c_m, c_m^{-1}\}$ and $\pi(\log c_m) = \pi(\log c_m^{-1})$. □

5. *Directional recurrence sets for actions of Abelian lattices*

In this section we consider the case of Abelian G in more detail. Our purpose here is to realize various G_δ-subsets of $P(\mathfrak{g})$ as $\mathcal{R}(T)$ for rank-one actions T of G. Since G is simply connected, there is $d > 0$ such that $G = \mathbb{R}^d$. Hence $\mathfrak{g} = \mathbb{R}^d$ and the maps exp and log are the identities. Replacing Γ with an automorphic lattice we may assume, without loss of generality, that $\Gamma = \mathbb{Z}^d$. In the subsequent work, we assume that $d > 1$ (the case $d = 1$ is trivial). By $\mathrm{dist}(\cdot\,,\cdot)$ we denote the usual distance between a point and a closed subset of $\mathbb{R}^d$. We also note that $\mathcal{ER}(T) = \mathcal{R}(T)$ for each measure-preserving action T of Γ. We now restate Proposition 4.2 for the Abelian case.

PROPOSITION 5.1. *Let $T = (T_\gamma)_{\gamma \in \mathbb{Z}^d}$ be a (C, F)-action of $\mathbb{Z}^d$ associated with a sequence $(C_n, F_{n-1})_{n=1}^\infty$ satisfying* (I)–(IV). *Then*

(i) $\mathcal{R}(T) \subset \bigcap_{n=1}^\infty \overline{\pi(\sum_{j \ge n}(C_j - C_j)\backslash\{0\})}$.

(ii) *If, moreover, $C_j \backslash \{0\} \gg C_1 \cup \cdots \cup C_{j-1}$ as $j \to \infty$, then*

$$\mathcal{R}(T) \subset \bigcap_{n=1}^\infty \overline{\bigcup_{m \ge n} \pi((C_m - C_m)\backslash\{0\})}.$$

(iii) *In, in addition, there is $c_j \in \mathbb{Z}^d$ such that $C_j = \{0, c_j\}$ for each $j > 0$, then*

$$\mathcal{R}(T) \subset \bigcap_{n=1}^\infty \overline{\{\pi(c_m) \mid m \ge n\}}.$$

The following two theorems are the main results of this section.

THEOREM 5.2. *Let Δ be a G_δ-subset of $P(\mathbb{R}^d)$ and let D be a countable subset of Δ. Then there is a rank-one free infinite measure-preserving action T of $\mathbb{Z}^d$ such that $D \subset \mathcal{R}(T) \subset \Delta$. In particular, each countable G_δ-subset (e.g. each countable compact) of $P(\mathbb{R}^d)$ is realizable as $\mathcal{R}(T)$ for some rank-one free action T of $\mathbb{Z}^d$.*

Proof. First, suppose that $\Delta \neq \emptyset$. Then, without loss of generality, we may think that $D \neq \emptyset$. Let $(\delta_n)_{n=1}^\infty$ be a sequence such that $\delta_n \in D$, for each n, and every element of D occurs in this sequence an infinite number of times. Let $(\epsilon_n)_{n=1}^\infty$ be a decreasing sequence of positive reals with $\lim_{n\to\infty} \epsilon_n = 0$. There exists an increasing sequence $L_1 \subset L_2 \subset \cdots$ of closed subsets in $P(\mathbb{R}^d)$ such that $P(\mathbb{R}^d) \backslash \Delta = \bigcup_{j \ge 1} L_j$. Let $L_1^+ \subset L_2^+ \subset \cdots$ be a sequence of open subsets in $P(\mathbb{R}^d)$ such that $L_j^+ \supset L_j$ and $\delta_j \notin \overline{L_j^+}$, for each j, and $\bigcup_{j \ge 1} L_j^+ \neq P(\mathbb{R}^d)$. We will construct, inductively, two sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$ satisfying (I)–(IV) and (4.1). We note, in advance, that, in our construction, $\#C_n = 2$ and F_n is a symmetric cube in $\mathbb{Z}^d$: i.e. there is $a_n \in \mathbb{N}$ such that

$$F_n = \{(i_1, \ldots, i_d) \mid -a_n < i_j \le a_n,\ j = 1, \ldots, d\},$$

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for each n. Suppose that we have defined the subsets $C_1, F_1, \ldots, C_{n-1}, F_{n-1}$. Our purpose is to construct C_n and F_n. Choose $c_n \in \mathbb{Z}^d$ such that $(c_n + F_{n-1}) \cap F_{n-1} = \emptyset$, $\mathrm{dist}(c_n, \delta_n) < \epsilon_n$ and

$$\max_{f \in F_{n-1}} \mathrm{dist}(c_n, c_n + f) < \epsilon_n, \tag{5.1}$$

$$\pi(c_n) \notin L_n^+. \tag{5.2}$$

For that use the fact that $\delta_n \notin \overline{L_n^+}$. We now let $C_n := \{0, c_n\}$ and define F_n to be a huge symmetric cube in $\mathbb{Z}^d$ that contains $F_{n-1} + C_n$. Continuing this construction procedure an infinite number of times we obtain infinite sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$. It is easy to see that (I)–(IV) and (4.1) are all satisfied. Let $T = (T_\gamma)_{\gamma \in \mathbb{Z}^d}$ denote the associated (C, F)-action. It is free and of rank one. Let (X, μ) be the space of this action.

We first show that $D \subset \mathcal{R}(T)$. Take $\delta \in D$, $\epsilon > 0$ and a cylinder $B \subset X$. Then there are an infinite number of $n > 0$ such that $\delta = \delta_n$ and hence $\mathrm{dist}(c_n, \delta) < \epsilon_n < \epsilon$. If n is large enough, $B = [B_{n-1}]_{n-1}$ for some subset $B_{n-1} \subset F_{n-1}$. Since $[B_{n-1}]_n \subset [B_{n-1}]_{n-1}$ and $T_{c_n}[B_{n-1}]_n = [c_n + B_{n-1}]_n \subset [B_{n-1}]_{n-1}$ with $\mu([B_{n-1}]_n) = 0.5\mu([B_{n-1}]_{n-1})$,

$$\mu(T_{c_n} B \cap B) \geq \mu(T_{c_n}[B_{n-1}]_n \cap [B]_{n-1}) = \mu([B_{n-1} + c_n]_n) = 0.5\mu(B).$$

Since each subset of finite measure in X can be approximated with a cylinder up to an arbitrary positive real, we deduce that $\delta \in \mathcal{R}(T)$.

We now show that $\mathcal{R}(T) \subset \Delta$. It follows from (5.1) that $\{c_n\} \gg F_{n-1}$ as $n \to \infty$. Hence, by Proposition 5.1(iii), $\mathcal{R}(T) \subset \bigcap_{n=1}^\infty \overline{\{\pi(c_m) \mid m \geq n\}}$. Applying (5.2), we obtain that $\pi(c_m) \notin L_m^+ \supset L_n^+ \supset L_n$, for each $m \geq n$. Hence $\mathcal{R}(T) \cap L_n = \emptyset$ for each n, which yields $\mathcal{R}(T) \subset \Delta$.

It remains to consider the case where $\Delta = \emptyset$. Fix $\theta \in P(\mathbb{R}^d)$. Suppose that we have defined the subsets $C_1, F_1, \ldots, C_{n-1}, F_{n-1}$. Choose $c_n \in \mathbb{Z}^d$ such that $(c_n + F_{n-1}) \cap F_{n-1} = \emptyset$, (5.1) is satisfied,

$$\pi(c_n) \text{ is up to } \epsilon_n \text{ close to } \theta \text{ (in the metric on } P(\mathbb{R}^d)) \text{ and} \tag{5.3}$$

$$\min_{f \in F_{n-1} - F_{n-1}} \mathrm{dist}(c_n + f, \theta) > 10. \tag{5.4}$$

We now let $C_n := \{0, c_n\}$ and define F_n to be a huge symmetric cube in $\mathbb{Z}^d$ that contains $F_{n-1} + C_n$. Continuing an infinite number of times, we obtain infinite sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$. It is easy to see that (I)–(IV) and (4.1) are all satisfied. Let $T = (T_\gamma)_{\gamma \in \mathbb{Z}^d}$ denote the associated (C, F)-action. It follows from Proposition 5.1(iii), (5.1) and (5.3) that $\mathcal{R}(T) \subset \{\theta\}$. If T were recurrent along θ, then there is $\gamma \in \mathbb{Z}^d$ such that $\gamma \neq 0$, $\mathrm{dist}(\gamma, \theta) < 0.1$ and $\mu([0]_n \cap T_\gamma[0]_n) > 0$. It follows, from Lemma 4.1, that there is $l > n$ such that $\gamma \in F_{l-1} - F_{l-1} + c_l$. This contradicts (5.4). Thus we obtain that $\mathcal{R}(T) = \emptyset$. □

THEOREM 5.3. *There is a rank-one free infinite measure-preserving action T of $\mathbb{Z}^d$ such that $\mathcal{R}(T) = P(\mathbb{R}^d)$.*

Proof. Given $t \in \mathbb{N}$ and $N > 0$, we let

$$\mathcal{K}_{t,N} := \{(i_1, \ldots, i_d) \in \mathbb{Z}^d \mid |i_j| < N \text{ and } t \text{ divides } i_j, j = 1, \ldots, d\}.$$

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Then, for each $\epsilon > 0$ and each integer $t > 0$, there is $N > 0$ such that

$$\sup_{\delta \in P(\mathbb{R}^m)} \min_{0 \neq \gamma \in \mathcal{K}_{t,N}} \mathrm{dist}(\gamma, \delta) < \epsilon. \tag{5.5}$$

Fix a sequence of positive reals ϵ_n, $n \in \mathbb{N}$, decreasing to zero. We will construct, inductively, the sequences $(F_{n-1})_{n>0}$ and $(C_n)_{n>0}$ that satisfy (I)–(IV) and (4.1). As usual, $F_0 = \{0\}$. Suppose we have defined $(F_j, C_j)_{j=1}^n$. Suppose that F_n is a symmetric cube. Denote by t_n the length of an edge of this cube. We now construct C_{n+1} and F_{n+1}. By (5.5), there is N_n such that $\min_{0 \neq \gamma \in \mathcal{K}_{3t_n,N_n}} \mathrm{dist}(\gamma, \delta) < \epsilon_n$ for each $\delta \in P(\mathbb{R}^d)$. Let $C_{n+1} := \mathcal{K}_{3t_n,M_n}$, where M_n is an integer large enough so that

$$\#\{\gamma \in \mathcal{K}_{3t_n,M_n} \mid \gamma + \mathcal{K}_{3t_n,N_n} \subset \mathcal{K}_{3t_n,M_n}\} > 0.5\#\mathcal{K}_{3t_n,M_n}. \tag{5.6}$$

Now let F_{n+1} be a huge symmetric cube in $\mathbb{Z}^d$ such that $F_{n+1} \supset F_n + C_{n+1}$. Continuing this construction process an infinite number of times, we define the infinite sequences $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$, as desired. Let T be the (C, F)-action of $\mathbb{Z}^d$ associated with these sequences. It is free and of rank one. Denote by (X, μ) the space of this action. We claim that $\mathcal{R}(T) = P(\mathbb{R}^d)$. Indeed, take $\epsilon > 0$, $\delta \in P(\mathbb{R}^d)$ and a cylinder $B \subset X$. Then there is $n > 0$ and a subset $B_n \subset F_n$ such that $B = [B_n]_n$ and $\epsilon_n < \epsilon$. There is $\gamma \in \mathcal{K}_{3t_n,N_n} \setminus \{0\}$ such that $\mathrm{dist}(\gamma, \delta) < \epsilon_n$. By (5.6), $\#(C_{n+1} \cap (C_{n+1} - \gamma)) \geq 0.5\#C_{n+1}$. Therefore

$$\begin{aligned}
\mu(T_\gamma B \cap B) &\geq \mu(T_\gamma [B_n + (C_{n+1} \cap (C_{n+1} - \gamma))]_{n+1} \cap [B_n]_n) \\
&= \mu([B_n + (C_{n+1} \cap (C_{n+1} + \gamma))]_{n+1}) \\
&\geq 0.5\mu(B).
\end{aligned}$$

The standard approximation argument implies that T is recurrent along δ. □

Remark 5.4.

(i) If we choose M_m in the above construction large enough so that the inequality

$$\#\{\gamma \in \mathcal{K}_{3t_n,M_n} \mid \gamma + \mathcal{K}_{3t_n,N_n} \subset \mathcal{K}_{3t_n,M_n}\} > (1 - n^{-1})\#\mathcal{K}_{3t_n,M_n}$$

holds in place of (5.6), then the corresponding (C, F)-action T will possess the stronger property $\mathcal{R}i(T) = P(\mathbb{R}^d)$.

(ii) In a similar way, the statement of Theorem 5.2 remains true if we replace $\mathcal{R}(T)$ with $\mathcal{R}i(T)$.

6. *The generic $\mathbb{Z}^d$-action is recurrent in every direction*

Let (X, μ) be a σ-finite non-atomic standard measure space. We recall that the group of all μ-preserving invertible transformations of X is denoted by $\mathrm{Aut}(X, \mu)$. It is endowed with the weak topology under which it is a Polish space. For a nilpotent Lie group G, we denote by $\mathcal{A}_\mu^G$ the set of all μ-preserving actions of G on (X, μ). We consider every element $A \in \mathcal{A}_\mu^G$ as a continuous homomorphism $g \mapsto A_g$ from G to $\mathrm{Aut}(X, \mu)$. The group $\mathrm{Aut}(X, \mu)$ acts on $\mathcal{A}_\mu^G$ by conjugation: i.e. $(S \cdot A)_g := SA_gS^{-1}$ for all $g \in G$, $S \in \mathrm{Aut}(X, \mu)$ and $A \in \mathcal{A}_\mu^G$. We endow $\mathcal{A}_\mu^G$ with the compact-open topology: i.e. the topology of uniform convergence on the compact subsets of G.

The following lemma is well known. We state it without proof.

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LEMMA 6.1. *$\mathcal{A}_\mu^G$ is a Polish space. The action of* $\mathrm{Aut}(X, \mu)$ *on this space is continuous.*

Let S^1 be the unit sphere in $\mathfrak{g}$ and let $K := \exp(S^1)$.

LEMMA 6.2. *Let $\mu(X) = 1$. Then the subset*

$$\mathcal{Z} := \{A \in \mathcal{A}_\mu^G \mid h(A_g) = 0 \textit{ for each } g \in K\}$$

is an invariant G_δ in $\mathcal{A}_\mu^G$.

Proof. Denote by $\mathcal{P}$ the set of all finite partitions of X. Fix a countable subset $\mathcal{P}_0 \subset \mathcal{P}$ which is dense in $\mathcal{P}$ in the natural topology. For each $P \in \mathcal{P}_0$ and $n > 0$, the map

$$\mathcal{A}_\mu^G \times K \ni (A, g) \mapsto H\left(P \,\middle|\, \bigvee_{j=1}^{n} A_g^{-j} P\right) \in \mathbb{R}$$

is continuous. Therefore the map

$$m_{P,n} : \mathcal{A}_\mu^G \ni A \mapsto m_{P,n}(A) := \max_{g \in K} H\left(P \,\middle|\, \bigvee_{j=1}^{n} A_g^{-j} P\right) \in \mathbb{R}$$

is well defined and continuous. Hence the subset

$$\mathcal{Z}' := \bigcap_{P \in \mathcal{P}_0} \bigcap_{r=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{l > N} \{A \in \mathcal{A}_\mu^G \mid m_{P,l}(A) < 1/r\}$$

is a G_δ in $\mathcal{A}_\mu^G$. We now show that $\mathcal{Z}' = \mathcal{Z}$. It is easy to see that $\mathcal{Z}' \subset \mathcal{Z}$ because $h(A_g) = \sup_{P \in \mathcal{P}_0} H(P \mid \bigvee_{j=1}^{\infty} A_g^{-j} P)$. Conversely, let $A \in \mathcal{Z}$. Fix $P \in \mathcal{P}_0$, $r > 1$ and $N > 0$. Then for each $g \in K$, there is $l_g > N$ such that $H(P \mid \bigvee_{j=1}^{l_g} A_g^{-j} P) < 1/r$. Of course, this inequality holds in a neighborhood of g in G. Since K is compact and the map $\mathbb{N} \ni n \mapsto H(P \mid \bigvee_{j=1}^{n} A_g^{-j} P)$ decreases, there is $l > N$ such that $H(P \mid \bigvee_{j=1}^{l} A_g^{-j} P) < 1/r$ for all $g \in K$: i.e. $m_{P,l}(A) < 1/r$. This means that $A \in \mathcal{Z}'$. It is obvious that $\mathcal{Z}$ is $\mathrm{Aut}(X, \mu)$-invariant. □

Let Γ be a co-compact lattice in G. Fix a a cross section $s : G/\Gamma \to G$ of the natural projection $G \to G/\Gamma$ such that the subset $s(G/\Gamma)$ is relatively compact in G. Denote by h_s the corresponding one-cocycle. Given a Γ-action T on (X, μ), we construct (via h_s) the induced G-action $\widetilde{T}$ on the space $(G/\Gamma \times X, \lambda \times \mu)$. In the following lemma we show that the 'inducing' functor is continuous.

LEMMA 6.3. *The map $\mathcal{A}_\mu^\Gamma \ni T \mapsto \widetilde{T} \in \mathcal{A}_{\lambda_{G/\Gamma} \times \mu}^G$ is continuous.*

Idea of the proof. It is enough to note that, for each compact subset $K \subset G$, the set $F := \{h_s(g, y) \mid g \in K, y \in G/\Gamma\} \subset \Gamma$ is finite. Therefore, given two Γ-actions T and T', if the transformation T_γ is 'close' to T'_γ for each $\gamma \in F$, then the transformation $\widetilde{T}_g$ is 'close' to $\widetilde{T}'_g$ uniformly on K. □

From now on let $\mu(X) = \infty$. Denote by $(X^\bullet, \mu^\bullet)$ the Poisson suspension of (X, μ). Given $R \in \mathrm{Aut}(X, \mu)$, let $R^\bullet$ stand for the Poisson suspension of R (see [**Ro**, **Ja-Ru**]). We note that $\mathrm{Aut}(X^\bullet, \mu^\bullet)$ is a topological $\mathrm{Aut}(X, \mu)$-module.

LEMMA 6.4. *The map* $\mathrm{Aut}(X, \mu) \ni R \mapsto R^\bullet \in \mathrm{Aut}(X^\bullet, \mu^\bullet)$ *is a continuous homomorphism.*

Idea of the proof. Let U_R and $U_{R^\bullet}$ denote the Koopman unitary operators generated by R and $R^\bullet$, respectively. Then it is enough to note that $U_{R^\bullet}$ is unitarily equivalent in a canonical way to the exponent $\bigoplus_{n\geq 0} U_R^{\odot n}$ (see [**Ne**, **Ro**]) and the map $U_R \mapsto U_R^{\odot n}$ is continuous in the weak operator topology for each n. □

LEMMA 6.5. *Let a transformation* $R \in \mathrm{Aut}(X, \mu)$ *be non-conservative. If there is an ergodic countable transformation subgroup* $N \subset \mathrm{Aut}(X, \mu)$ *such that*

$$\{SR^n x \mid n \in \mathbb{Z}\} = \{R^n Sx \mid n \in \mathbb{Z}\} \quad \textit{at almost every } x \in X \textit{ for each } S \in N, \tag{6.1}$$

then $R^\bullet$ *is a Bernoulli transformation of infinite entropy.*

Proof. We consider Hopf decomposition of X: i.e. a partition of X into two R-invariant subsets X_d and X_c such that the restriction of R to X_d is totally dissipative and the restriction of R to X_d is conservative (see [**Aa**]). By the hypothesis, $\mu(X_d) > 0$. It follows, from (6.1), that X_d is invariant under N. Since N is ergodic, $\mu(X_c) = 0$: i.e. R is totally dissipative so there is a subset $W \subset X$ such that $X = \bigcup_{n\in\mathbb{Z}} R^n W$ (mod 0) and $R^n W \cap T^m W = \emptyset$ if $n \neq m$. Therefore $R^\bullet$ is Bernoulli [**Ro**]. Since $\mu \restriction W$ is not purely atomic, $h(R^\bullet) = \infty$ [**Ro**]. □

We now state the main result of this section.

THEOREM 6.6. *The subset* $\mathcal{V}$ *of* $\mathbb{Z}^d$*-actions* T *on* (X, μ) *with* $\mathcal{R}(T) = P(\mathbb{R}^d)$ *is residual in* $\mathcal{A}_\mu^{\mathbb{Z}^d}$.

Proof. Let λ denote Haar measure on the torus $\mathbb{R}^d/\mathbb{Z}^d$. It follows, from Lemmata 5.3 and 5.4, that the mapping

$$\mathcal{A}_\mu^{\mathbb{Z}^d} \ni T \mapsto \widetilde{T}^\bullet \in \mathcal{A}_{\lambda\times\mu}^{\mathbb{R}^d}$$

is continuous. Let $\mathcal{Z} := \{A \in \mathcal{A}_{(\lambda\times\mu)^\bullet}^{\mathbb{R}^d} \mid h(A_g) = 0 \text{ for each } g \in \mathbb{R}^d\}$. By Lemma 6.2, $\mathcal{Z}$ is a G_δ in $\mathcal{A}_\mu^{\mathbb{R}^d}$. Hence the subset $\mathcal{W} := \{T \in \mathcal{A}_\mu^{\mathbb{Z}^d} \mid \widetilde{T}^\bullet \in \mathcal{Z}\}$ is a G_δ in $\mathcal{A}_\mu^{\mathbb{Z}^d}$. Of course, $\mathcal{W}$ is $\mathrm{Aut}(X, \mu)$-invariant. It is well known that the subset $\mathcal{E} := \{T \in \mathcal{A}_\mu^{\mathbb{Z}^d} \mid T \text{ is ergodic}\}$ is an $\mathrm{Aut}(X, \mu)$-invariant G_δ in $\mathcal{A}_\mu^{\mathbb{Z}^d}$. Hence the intersection $\mathcal{W} \cap \mathcal{E}$ is also an $\mathrm{Aut}(X, \mu)$-invariant G_δ in $\mathcal{A}_\mu^{\mathbb{Z}^d}$. Take an action $T \in \mathcal{A}^{\mathbb{Z}^d} \cap \mathcal{E}$ and a line $\theta \in P(\mathbb{R}^d)$. If $\theta \notin \mathcal{R}(T)$, then $\theta \notin \mathcal{R}(\widetilde{T})$. Since T is ergodic, $\widetilde{T}$ is also ergodic. Hence the $\mathbb{Q}^d$-action $(\widetilde{T}_q)_{q\in\mathbb{Q}^d}$ is also ergodic. Then, by Lemma 6.5, $h(\widetilde{T}_r^\bullet) = \infty$ for each $r \in \theta$, $r \neq 0$. Therefore $T \notin \mathcal{W}$. This yields that $\mathcal{W} \cap \mathcal{E} \subset \mathcal{V}$. It remains to show that $\mathcal{W} \cap \mathcal{E}$ is dense in $\mathcal{A}_\mu^{\mathbb{Z}^d}$. Let T be an ergodic free action of $\mathbb{Z}^d$ such that $\mathcal{R}i(T) = P(\mathbb{R}^d)$ (see Remark 5.4(i) and Theorem 5.3). By Theorem 3.1, $\mathcal{R}i(\widetilde{T}) = P(\mathbb{R}^d)$. Then, in view of Lemma 6.4, for each $g \in \mathbb{R}^d$, the transformation $\widetilde{T}_g^\bullet$ is rigid. Hence $h(\widetilde{T}_g^\bullet) = 0$. Thus $T \in \mathcal{W} \cap \mathcal{E}$. It follows, from the Rokhlin lemma for the infinite measure-preserving free $\mathbb{Z}^d$-actions, that the conjugacy class of T (i.e. the $\mathrm{Aut}(X, \mu)$-orbit of T) is dense in $\mathcal{A}_\mu^{\mathbb{Z}^d}$ (see, e.g., [**DaSi**]). Of course, the conjugacy class of T is a subset of $\mathcal{W} \cap \mathcal{E}$. □

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Using some ideas from the proof of the above theorem, we can prove the following proposition.

PROPOSITION 6.7. *There is a Poisson action*† *V of $\mathbb{R}^d$ of zero entropy such that, for each $0 \neq g \in \mathbb{R}^m$, the transformation V_g is Bernoullian and of infinite entropy.*

Proof. By Theorem 5.2, there exists rank-one (by cubes) infinite measure-preserving actions T of $\mathbb{Z}^d$ such that $\mathcal{R}(T) = \emptyset$. Then $\widetilde{T}^\bullet$ is a Poisson (finite measure-preserving) action of $\mathbb{R}^d$. We note that $h(\widetilde{T}^\bullet) = h(\widetilde{T}^\bullet \restriction \mathbb{Z}^d) = h((\widetilde{T} \restriction \mathbb{Z}^d)^\bullet)$. We note $\widetilde{T} \restriction \mathbb{Z}^d = I \times T$, where I denotes the trivial action of $\mathbb{Z}^d$ on the torus $(\mathbb{R}^d/\mathbb{Z}^d, \lambda)$. It follows, from [**Ja-Ru**], that $h((I \times T)^\bullet) = h(T^\bullet)$. Since T is of rank one, $h(T^\bullet) = 0$, by [**Ja-Ru**]‡. Thus we obtain that $h(\widetilde{T}^\bullet) = 0$. On the other hand, arguing as in the proof of Theorem 6.6, we deduce, from Theorem 3.1 and Lemma 6.5, that, for each $g \in \mathbb{R}^d \setminus \{0\}$, the transformation $\widetilde{T}_g^\bullet$ is Bernoulli and of infinite entropy. □

In a similar way, using Remark 5.4(ii), we can show the following more general statement.

PROPOSITION 6.8. *Let Δ be a G_δ-subset of $P(\mathbb{R}^d)$ and let D be a countable subset of Δ. Then there is a Poisson action V of $\mathbb{R}^d$ of zero entropy such that, for each non-zero $g \notin \bigcup_{\theta\in\Delta}\theta$, the transformation V_g is Bernoulli and of infinite entropy and, for each $g \in \bigcup_{\theta\in D}\theta$, the transformation V_g is rigid (and hence of zero entropy).*

7. *Directional recurrence for actions of the Heisenberg group*

Consider now the three-dimensional real Heisenberg group $H_3(\mathbb{R})$, which is perhaps the simplest example of a non-commutative simply connected nilpotent Lie group. We recall that

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, t_1, t_2, t_3 \in \mathbb{R} \right\}.$$

We introduce the notation

$$a(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad c(t) := \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the maps $\mathbb{R} \ni t \mapsto a(t) \in H_3(\mathbb{R})$, $\mathbb{R} \ni t \mapsto b(t) \in H_3(\mathbb{R})$, $\mathbb{R} \ni t \mapsto c(t) \in H_3(\mathbb{R})$ are continuous homomorphisms, the subset $\{c(t) \mid t \in \mathbb{R}\}$ is the center of $H_3(\mathbb{R})$, $a(t_1)b(t_2) = b(t_2)a(t_1)c(t_1t_2)$ for all $t_1, t_2 \in \mathbb{R}$ and

$$\begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} = c(t_3)b(t_2)a(t_1) \quad \text{for all } t_1, t_2, t_3 \in \mathbb{R}.$$

† We recall that a probability-preserving action of a group G is called Poisson if it is isomorphic to the Poisson suspension of an infinite measure-preserving action of G.
‡ This fact was proved in [**Ja-Ru**] only for $d = 1$. However, in the general case, the proof is similar.

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We also note that the Lie algebra of $H_3(\mathbb{R})$ is

$$\mathfrak{h}_3(\mathbb{R}) := \left\{ \begin{pmatrix} 0 & t_1 & t_3 \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

The exponential map $\exp : \mathfrak{h}_3(\mathbb{R}) \to H_3(\mathbb{R})$ is given by the formula

$$\exp \begin{pmatrix} 0 & t_1 & t_3 \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t_1 & t_3 + \dfrac{t_1 t_2}{2} \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The adjoint action of $H_3(\mathbb{R})$ on $\mathfrak{h}_3(\mathbb{R})$ is given by the formula

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \gamma + x\beta - y\alpha \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}.$$

We also give an example of a right-invariant metric d on $H_3(\mathbb{R})$: that is,

$$d(c(t_3)b(t_2)a(t_1), c(t_3')b(t_2')a(t_1')) := |t_1 - t_1'| + |t_2 - t_2'| + |t_3 - t_3' + t_2'(t_1' - t_1)|.$$

Let Γ be a lattice in $H_3(\mathbb{R})$. It is well known (see, e.g., [**DaLe**]) that there is $k > 0$ such that Γ is automorphic to the lattice

$$\{c(n_3/k)b(n_2)a(n_1) \mid n_1, n_2, n_3 \in \mathbb{Z}\}.$$

From now on, we will assume that $k = 1$ and hence

$$\Gamma = H_3(\mathbb{Z}) := \{c(n_3)b(n_2)a(n_1) \mid n_1, n_2, n_3 \in \mathbb{Z}\}.$$

Let $F_n := \{c(j_3)b(j_2)a(j_1) \mid |j_1| < L_n, |j_2| < L_n, |j_3| < M_n\}$, where L_n and M_n are positive integers. It is easy to verify that if $L_n \to \infty$, $M_n \to \infty$ and $L_n/M_n \to 0$ as $n \to \infty$, then $(F_n)_{\geq 1}$ is a Følner sequence in $H_3(\mathbb{Z})$.

In the following three theorems, we construct rank-one actions of $H_3(\mathbb{Z})$ with various sets of recurrence and rigidity: empty, countable and uncountable.

THEOREM 7.1. *There is a rank-one free infinite measure-preserving action T of $H_3(\mathbb{Z})$ such that $\mathcal{R}(T) = \emptyset$.*

Proof. Let $C_n := \{1, a(t_n)\}$, where $(t_n)_{n\in\mathbb{N}}$ is a sequence of integers that grows fast, and let $(F_n)_{n\geq 0}$ be a Følner sequence in $H_3(\mathbb{R})$ such that (I)–(IV) and (4.1) are satisfied and, in addition, $C_n \backslash \{1\} \gg C_1 \cdots C_{n-1}$ as $n \to \infty$. Denote by T the (C, F)-action of $H_3(\mathbb{Z})$ associated with $(C_n, F_{n-1})_{n\in\mathbb{N}}$. Let $\theta \in P(\mathfrak{h}_3(\mathbb{R}))$ stand for the line in $\mathfrak{h}_3(\mathbb{R})$ that passes through the vector $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Since $\pi(\log a(t_n)) = \theta$, we deduce, from Proposition 4.2(iii), that

$$\mathcal{R}(T) \subset \bigcap_{\gamma\in\Gamma} \gamma \cdot \left(\bigcap_{n=1}^{\infty} \overline{\{\pi(\log a(t_m)) \mid m \geq n\}} \right) \subset \bigcap_{\gamma\in\Gamma} \{\gamma \cdot \theta\} = \emptyset. \qquad \square$$

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Given $t \in \mathbb{R}$, let $\theta_t \in P(\mathfrak{h}_3(\mathbb{R}))$ be the line in $\mathfrak{h}_3(\mathbb{R})$ that passes through the vector $\begin{pmatrix} 0 & 1 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\exp(\theta_t) \ni c(t)a(1)$. We also denote by θ_∞ the line in $\mathfrak{h}_3(\mathbb{R})$ that passes through the vector $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Of course, the set $\{\theta_l \mid l \in \mathbb{Z}\}$ is the $H_3(\mathbb{Z})$-orbit $\{\gamma \cdot \theta_0 \mid \gamma \in H_3(\mathbb{Z}\}$ of θ_0. The point θ_∞ is the only limit point of this orbit in $P(\mathfrak{h}_3(\mathbb{R}))$. In a similar way, the set $\{\theta_t \mid t \in \mathbb{R}\}$ is the $H_3(\mathbb{R})$-orbit of θ_0. The closure of this orbit is the union of this orbit with the limit point θ_∞.

THEOREM 7.2. *There is a rank-one free infinite measure-preserving action T of $H_3(\mathbb{Z})$ such that $\mathcal{R}(T) = \{\theta_l \mid l \in \mathbb{Z}\} \cup \{\theta_\infty\}$. Therefore $\mathcal{ER}(T) = \{\theta_\infty\}$ and hence $\mathcal{R}(T) \neq \mathcal{ER}(T)$.*

Proof. We let

$$F_n := \{c(j_3)b(j_2)a(j_1) \mid |j_1| < L_n, |j_2| < L_n, |j_3| < M_n\} \quad \text{and}$$
$$C_n := \{c(ik_n)a(jk_n) \mid j = 0, 1 \text{ and } |i| \leq I_n\},$$

where $(L_n)_{n\geq 1}$, $(M_n)_{n\geq 1}$, $(k_n)_{n\geq 1}$ and $(I_n)_{n\geq 1}$ are sequences of integers chosen in such a way such that:

($\bullet$) (I)–(IV) from §4 and (4.1) are satisfied;
($*$) $C_n \setminus \{1\} \gg C_1 \cdots C_{n-1}$ as $n \to \infty$;
($\diamond$) $L_n \to \infty$, $M_n \to \infty$, $L_n/M_n \to 0$; and
($\circ$) $I_n \to +\infty$, $L_{n-1}/I_n \to 0$.

Denote by T the (C, F)-action of $H_3(\mathbb{Z})$ associated with $(C_n, F_{n-1})_{n\in\mathbb{N}}$. It is well defined in view of ($\bullet$). Moreover, $(F_n)_{n\geq 1}$ is a Følner sequence in $H_3(\mathbb{Z})$ in view of ($\diamond$). It is standard to verify that

$$\overline{\bigsqcup_{m>n} \pi(\log(C_m C_m^{-1} \setminus \{1\}))} = \{\theta_l \mid l \in \mathbb{Z}\} \cup \{\theta_\infty\}$$

for each $n > 0$. Hence, by Proposition 4.2(ii), $\mathcal{R}(T) \subset \{\theta_n \mid n \in \mathbb{Z}\} \cup \{\theta_\infty\}$. In view of Remark 2.4(ii), to prove the converse inclusion it suffices to show that $\theta_1, \theta_\infty \in \mathcal{R}(T)$. For $n \geq 1$, take a subset $D \subset F_{n-1}$. It follows, from the definition of F_{n-1}, that, for each $\gamma \in D$, there is $j \in \mathbb{Z}$ such that $|j| < L_{n-1}$ and $a(k_n)\gamma a(-k_n) = \gamma c(jk_n)$. Let

$$C'_n := \{w \in C_n \mid c(jk_n)a(k_n)w \in C_n \text{ whenever } |j| < L_{n-1}\}. \tag{7.1}$$

Then $C'_n = \{c(ik_n) \mid |i| < I_n, |i \pm L_{n-1}| < I_n\}$. Hence $\#C'_n/\#C_n \to 1/2$ as $n \to \infty$, in view of ($\circ$), and hence

$$\max_{D\subset F_{n-1}} |\mu([D]_{n-1})/\mu([DC'_n]_n) - 1/2| \to 0 \tag{7.2}$$

as $n \to \infty$. On the other hand, in view of (7.1), we have

$$T_{a(k_n)}[DC'_n]_n = \bigsqcup_{\gamma\in D} T_{a(k_n)}[\gamma C'_n]_n = \bigsqcup_{\gamma\in D} [a(k_n)\gamma a(-k_n)a(k_n)C'_n]_n \subset \bigsqcup_{\gamma\in D} [\gamma C_n]_n.$$

Thus $T_{a(k_n)}[DC'_n]_n \subset [D]_{n-1}$. Since $a(k_n) \in \exp(\theta_1)$ and (7.2) holds, it follows that T is recurrent along θ_1. To prove that $\theta_\infty \in \mathcal{R}(T)$, we let

$$C''_n := \{w \in C_n \mid c(k_n)w \in C_n\}.$$

Then $\#C''_n/\#C_n \to 1$ and hence $\max_{D\subset F_{n-1}} |\mu([D]_{n-1})/\mu([DC'_n]_n) - 1| \to 0$ as $n \to \infty$. Moreover, $T_{c(k_n)}[DC''_n]_n \subset [DC_n]_n = [D]_{n-1}$. Hence T is recurrent along θ_∞. □

THEOREM 7.3. *There is a rank-one free infinite measure-preserving action T of $H_3(\mathbb{Z})$ such that $\mathcal{R}(T) = \mathcal{R}i(T) = \{\theta_t \mid t \in \mathbb{R}\} \cup \{\theta_\infty\} = \mathcal{ER}(T) = \mathcal{ER}i(T)$.*

Proof. Let

$$\begin{aligned} F_n &:= \{c(j_3)b(j_2)a(j_1) \mid |j_1| < L_n, |j_2| < L_n, |j_3| < M_n\}, \\ C_n &:= \{c(jk_n)a(ik_n) \mid |j| \le l_n J_n, |i| \le l_n I_n\}, \\ C_n^0 &:= \{c(jk_n)a(ik_n) \mid |j| \le l_n, |i| \le l_n\}, \end{aligned}$$

where $(L_n)_{n\ge1}$, $(M_n)_{n\ge1}$, $(k_n)_{n\ge1}$, $(I_n)_{n\ge1}$, $(J_n)_{n\ge1}$ and $(l_n)_{n\ge1}$ are sequences of integers such that ($\bullet$), ($*$), ($\diamond$) hold,

($\triangle$) $\sup_{t\in\mathbb{R}\cup\{\infty\}} \min_{1\ne\gamma\in C_n^0} \operatorname{dist}(\gamma, \theta_t) < 1/n$ and

($\blacktriangle$) $\#(\{w \in C_n \mid \bigcup_{d\in F_{n-1}} \bigcup_{c\in C_n^0} d^{-1}cdw \subset C_n\}) > (1 - 1/n)\#C_n$,

for each $n \in \mathbb{N}$. Denote by T the (C, F)-action of $H_3(\mathbb{Z})$ associated with $(C_n, F_{n-1})_{n\in\mathbb{N}}$. It is standard to verify that

$$\overline{\bigcup_{m>n} \pi(\log(C_m C_m^{-1}\setminus\{1\}))} = \{\theta_t \mid t \in \mathbb{R}\} \cup \{\theta_\infty\}.$$

Hence, by Proposition 4.2(ii), $\mathcal{R}(T) \subset \{\theta_t \mid t \in \mathbb{R}\} \cup \{\theta_\infty\}$. To prove the converse inclusion, we take θ_t for some $t \in \mathbb{R} \cup \{\infty\}$. By ($\triangle$), there is $\gamma \in C_n^0\setminus\{1\}$ such that $\operatorname{dist}(\gamma, \theta_t) < 1/n$. Let

$$C_n' := \left\{ w \in C_n \,\middle|\, \bigcup_{d\in F_{n-1}} d^{-1}\gamma dwC_n^0 \subset C_n \right\}.$$

Then $\#C_n'/\#C_n > 1 - 1/n$, in view of ($\blacktriangle$), and hence, for each subset $D \subset F_{n-1}$, we have $\mu([D]_{n-1}\setminus[DC_n']_n) < \mu([D]_n)/n$. On the other hand,

$$T_\gamma[DC_n']_n = \bigsqcup_{d\in D} T_\gamma[dC_n']_n = \bigsqcup_{d\in D} [dd^{-1}\gamma dC_n']_n \subset \bigsqcup_{d\in D} [dC_n]_n = [D]_{n-1}.$$

It follows that T is rigid along θ_t. Thus we have shown that $\{\theta_t \mid t \in \mathbb{R}\} \cup \{\theta_\infty\} \subset \mathcal{R}i(T)$. □

8. *Some open problems and concluding remarks*

(1) Which G_δ-subsets of $P(\mathfrak{g})$ are realizable as $\mathcal{R}(T)$ or $\mathcal{R}i(T)$ for an ergodic infinite measure-preserving action T of Γ? In particular, let $\theta \in P(\mathfrak{g})$. Is the subset $P(\mathfrak{g})\setminus\{\theta\}$ is realizable? In the case where $G = \mathbb{R}^2$ and $\Gamma = \mathbb{Z}^2$, $P(\mathfrak{g})$ is homeomorphic to the circle. Is a proper arc of this circle realizable?

(2) Suppose that a subset of $P(\mathfrak{g})$ is realizable as $\mathcal{R}(T)$ or $\mathcal{R}i(T)$. Can we choose T in the class of rank-one actions?

(3) In view of Theorem 3.1 and Remark 3.2, do we have $\mathcal{R}(\widetilde{T}) = \mathcal{ER}(T)$ in the non-Abelian case?

(4) Does Corollary 3.3 extend to the non-Abelian case: i.e. does $\mathcal{ER}(T) = \mathcal{R}(\widehat{T})$, where $\widehat{T}$ is an extension of T to a G-action on the same measure space, where T is defined?

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(5) A multiple recurrence (and even recurrence) along directions can be defined in the following way. Let T be a measure-preserving action of Γ on a σ-finite measure space (X, μ) and let $p \in \mathbb{N}$. We call T *p-recurrent along a line* $\theta \in P(\mathfrak{g})$ if, for each $\epsilon > 0$ and every subset $A \subset X$ of positive measure, there is an element $\gamma \in \Gamma \setminus \{1_\Gamma\}$ and an element $g \in \exp(\theta)$ such that $\mathrm{dist}(\gamma, g) < \epsilon$ and $\mu(A \cap T_\gamma A \cap \cdots \cap T_\gamma^p A) > 0$. Denote by $\mathcal{R}_p(T)$ the set of all $\theta \in P(\mathfrak{g})$ such that T is p-recurrent along θ. Then $\mathcal{R}(T) = \mathcal{R}_1(T) \supset \mathcal{R}_2(T) \supset \cdots$ and $\bigcap_{p \geq 1} \mathcal{R}_p(T) \supset \mathcal{R}i(T)$. We note that all these inclusions are strict and every set $\mathcal{R}_p(T)$ is a G_δ. The results obtained in this work for $\mathcal{R}(T)$ extend to $\mathcal{R}_p(T)$ with similar proofs for each p.

(6) Let T be a (C, F)-action of Γ associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (I)–(IV) and (4.1) from §4. Given $d > 0$, we denote by $C_n^{\otimes d}$ and $F_n^{\otimes d}$ the dth Cartesian power of C_n and F_n, respectively. Then the sequence $(C_n^{\otimes d}, F_{n-1}^{\otimes d})_{n \geq 1}$ of subsets in Γ^d satisfies (I)–(IV) and (4.1) from §4. It is easy to see that the (C, F)-action $T^{\otimes d}$ of Γ^d is canonically isomorphic to the dth tensor product of T: i.e. $T^{\otimes d}_{(\gamma_1, \ldots, \gamma_d)} = T_{\gamma_1} \times \cdots \times T_{\gamma_d}$ for all $\gamma_1, \ldots, \gamma_d \in \Gamma$. The Lie algebra $\mathfrak{g}^d$ of G^d is $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (d times). There is a natural shiftwise action of the permutation group Σ_d on $\mathfrak{g}^d$. This action pushes down to the projective space $P(\mathfrak{g}^d)$. It is easy to see that the sets $\mathcal{R}(T^{\otimes d})$ and $\mathcal{R}i(T^{\otimes d})$ are invariant under Σ_d. In the case where $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, Theorem 5.2 can be refined in the following way: given a Σ_d-invariant subset $\Delta \subset P(\mathbb{R}^d)$ and a countable Σ_d-invariant subset D of Δ, there is a rank-one free infinite measure-preserving action T of $\mathbb{Z}$ such that $D \subset \mathcal{R}(T^{\otimes d}) \subset \Delta$. In particular, each countable Σ_d-invariant G_δ-subset D of $P(\mathbb{R}^d)$ is realizable as $\mathcal{R}(T^{\otimes d})$ for some rank-one free action T of $\mathbb{Z}$. This generalizes and refines partly† one of the main results from the recent paper by Adams and Silva [**AdSi**]: for each Σ_2-invariant subset D of *rational* directions, there is a rank-one action T of $\mathbb{Z}$ such that D is the intersection of $\mathcal{R}(T^{\otimes 2})$ with the set of all rational directions in $\mathbb{R}^2$. We also note that the $\mathbb{Z}^d$-action T, constructed in Theorem 5.3, has the form $T = S^{\otimes d}$ for a (C, F)-action S of $\mathbb{Z}$.

(7) The theory of directional recurrence can be generalized in a natural way from infinite measure-preserving Γ-actions to non-singular Γ-actions.

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† This refinement is partial because we consider only the recurrence set while Adams and Silva studied, simultaneously, the set of rational ergodic directions for $T^{\otimes 2}$.

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