

How to take into account the relativistic effects in dynamical studies of comets

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Abstract. Comet-like orbits with low perihelion distances tend to be affected by relativistic effects. In this work we discuss the origin of the relativistic corrections, how they affect the orbital evolution and how to implement these corrections in a numerical integrator. We also propose a model that mimics the principal relativistic effects and, contrary to the original “exact” formula, has low computational cost. Our model is appropriated for numerical simulations but not for precise ephemeris computations.

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1. Introduction

The problem of computing relativistic effects in the Solar System is of increasing importance for study of low perihelia and low semimajor axis populations. It is a usual practice in textbooks to show the relativistic effect in a planet’s perihelion starting from a simplified problem that gives rise to a unique radial perturbation to the Newtonian potential conserving the angular momentum. But, a more rigorous analysis (Brumberg 1991, Shahid-Saless & Yeomans 1994) gives rise also to a transverse component so the angular moment is not more conserved and the expression for the relativistic perturbation becomes:

$$\Delta\ddot{\mathbf{r}} = \frac{\mu}{r^3 c^2} \left[\left(\frac{4\mu}{r} - \mathbf{v}^2 \right) \mathbf{r} + 4(\mathbf{v}\cdot\mathbf{r})\mathbf{v} \right] \quad (1.1)$$

where $\mu = k^2 m_{\odot}$ (Anderson *et al.* 1975, Quinn *et al.* 1991). This expression, that has no normal component, is valid when considering the relativistic effects generated only by a spherically symmetric and non rotating Sun. When the rotation of the Sun is considered, a new effect called gravitomagnetic Lense-Thirring effect appears (Soffel 1989, Iorio 2005a). Furthermore, if all bodies have relativistic contributions a more complex expression should be used as explained in, for example, Benitez & Gallardo (2008). The time in Eq. (1.1) should be considered as the *coordinate time*; in order to compare with observables it is necessary to transform, for example, to Barycentric Coordinate Time (TCB) or Terrestrial Time (TT) (Shahid-Saless & Yeomans 1994, Soffel *et al.* 2003).

In order to avoid either the computational cost or the non symplectic form of Eq. (1.1), some simpler models that only depend on r have been proposed which mimic or reproduce the correct relativistic shift in the argument of the perihelia of the bodies under the gravitational effect of a central mass (Nobili & Roxburgh 1986, Saha & Tremaine 1992). But, as pointed out by Saha & Tremaine (1994), these models correctly recover

the perihelion motion but fail in reproducing the evolution of the mean anomaly, M , point that we will discuss in this work.

These r -dependent secular models have a disadvantage when used to compute high eccentricity low perihelion orbits in constant time-steps algorithms of the type “advance-impulse-advance”, as is the usual case in symplectic algorithms. In this case, the relativistic perturbation when computed as an impulse, generates spurious results near the perihelion where the relativistic perturbation is stronger and poorly sampled. This is because the expression for the acceleration has a term that goes with $1/r^3$ and for low r values this impulse can be excessively high. For example, in the Solar System the relativistic perturbation for an object at $r < 0.1$ AU is greater than the gravitational perturbation by Jupiter. Saha & Tremaine (1994), in a symplectic method for planetary integrations, incorporate part of the relativistic correction (1.1) in the “advance” part of the algorithm but an acceleration proportional to $1/r^3$ remains as part of the “impulse”.

In this work we review the relativistic effects on orbital elements with special attention to mean anomaly (Rubincam 1977, Calura *et al.* 1997, Iorio 2007) and we propose an r -independent model that correctly reproduces the secular evolution of the argument of the perihelion even for high eccentricity orbits which is very useful for constant timestep algorithms.

2. Secular Relativistic Effects due to the Sun

By means of the Gauss planetary equations, averaged on an orbital period, one can compute the secular variations of orbital elements produced by Eq. (1.1). The only non zero variations obtained for this acceleration are those corresponding to the argument of perihelion and the mean anomaly:

$$\langle \dot{\omega} \rangle = \frac{3}{c^2(1-e^2)} \sqrt{\frac{\mu^3}{a^5}} \quad (2.1)$$

$$\langle \dot{M} \rangle = \frac{3}{c^2} \sqrt{\frac{\mu^3}{a^5}} \left(2 - \frac{5}{\sqrt{1-e^2}} \right) \quad (2.2)$$

where units are radians per day. Eq. (2.1) is the well known relativistic effect on the argument of the perihelion and, for small eccentricities, Eq. (2.2) coincides, for example, with the approximate expressions for the secular drift in M given by Iorio (2005b) and with the secular drift in M generated by the relativistic model for low eccentricities used by Vitagliano (1997). The exact formula, valid for all eccentricities, is Eq. (2.2) and can be checked numerically integrating a particle using the classic model and then the relativistic one. The effect in M is several times greater than in ω .

3. Models that Mimic the Secular Relativistic Effects

3.1. Proposed Models

If we are interested in computing the relativistic effects due to the Sun on a small body, we could just introduce Eq. (1.1) into an integrator. The problem is that this acceleration depends on both vectors position and velocity of the particle, so the speed of the integrator may be slowed down in order to calculate accurately the vectorial products at small heliocentric distances. To overcome this difficulty, some alternative simpler models have been created in the last two decades. The relativistic precession of the argument of

perihelion is correctly reproduced defining the radial perturbation:

$$R = -\frac{6\mu^2}{c^2 r^3} \quad (3.1)$$

(Nobili & Roxburgh 1986). Inserting this R in the time averaged Gauss planetary equations, the exact secular drifts generated by Eq. (1.1) are recovered except for mean anomaly. Saha & Tremaine (1992) added one more term into (3.1) in order to account for both ω and M drifts:

$$R = -\frac{6\mu^2}{c^2 r^3} + \frac{3\mu^2}{ac^2} \left(\frac{4}{\sqrt{1-e^2}} - 1 \right) \frac{1}{r^2} \quad (3.2)$$

Hence, with this perturbation, Eq. (2.1) and Eq. (2.2) are recovered. This apparent improvement to the original model of Nobili & Roxburgh (1986) is not so at all as pointed out by Saha & Tremaine (1994). The point is that given an initial osculating a_0 the mean \bar{a} generated by the evolution under Eq. (1.1) is different from the mean \bar{a}_R generated by the evolution under Eq. (3.2). Then, model (3.2) will not reproduce the secular evolution of an object with the correct \bar{a} but with a different mean semimajor axis given by \bar{a}_R . In consequence, the model introduces a secular drift in M and no evident progress is done in comparison with Eq. (3.1). This problem can be solved taking appropriate initial conditions.

3.2. Our r -independent Model

The corrections exposed above, though computationally better than Eq. (1.1) because the \mathbf{v} -dependence is eliminated, still have the problem that near perihelion the perturbation can be high enough to introduce numerical errors in constant time-step integrators. In order to overcome this difficulty, we propose the following constant radial perturbation:

$$R = \frac{3\mu^2}{c^2 a^3 \sqrt{(1-e^2)^3}} \quad (3.3)$$

which generates the expected variation in the argument of perihelion given by Eq. (2.1). However, for mean anomaly we obtain a different expression:

$$\langle \dot{M} \rangle = -\frac{9}{c^2} \sqrt{\frac{\mu^3}{a^5 (1-e^2)^3}} \quad (3.4)$$

The discrepancy for the drift on M with respect to Eq. (2.2) is irrelevant in numerical simulations because, as we have explained, the predictions for M given by models cannot coincide with the true relativistic effect if the same set of initial conditions for all models are taken. The advantage of model (3.3) is that it maintains the precision of the numerical integration even for very small perihelion orbits while the original Eq. (1.1) and the other models need a strong reduction in the step size. For low eccentricity orbits, both Eq. (2.2) and Eq. (3.4) give the same result.

3.3. On Initial Conditions

Initial conditions should be used according to the theory they were determined. For example, the model of Saha & Tremaine (1992) can be used confidently for generating precise ephemeris only if the initial conditions were obtained adjusting observations to this model. That is the underlying idea of the very accurate method for computing ephemeris of low eccentricity orbits proposed by Vitagliano (1997). But, it is not possible to follow the exact evolution of the mean anomaly if there is no consistence between the

initial conditions (which should also be very precise) and the model used. Then, any of the models we have analyzed are valid to follow the secular evolution of an orbit and, in particular, our constant radial perturbation model is computationally more convenient.

For illustrative purposes, in order to compare the variations of the orbital elements produced by the different models given by Eq. (1.1), Eq. (3.2) and Eq. (3.3), we integrated numerically our planetary system for one million years. Results corroborate what we have already stated by means of the Gauss equations: the secular relativistic evolution of all orbital elements is very well reproduced. And even with strong differences in M , of the order of 10^2 degrees, the orbital evolution of the different models is almost undistinguishable. For planet Mercury for example, at the end of the integration, differences of the order of 10^{-5} in e , 10^{-3} degrees on ω , 10^{-4} degrees on Ω , and even lower differences for i were obtained.

4. Conclusions

At least three models are able to reproduce the secular drift in ω but for high eccentricity orbits only the model by Saha & Tremaine (1992) can follow the exact secular drift in M and only if appropriate initial conditions are taken. For secular evolution studies of numerous populations by means of numerical integrations, if no precise ephemeris are required, our constant radial r -independent model is very convenient specially if low perihelion orbits are involved. For accurate ephemeris computations of real bodies, the original formula (1.1), or even the full relativistic N-body one should be used with a control of the integrator's precision in the case of low perihelion orbits.

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