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On Goren–Oort stratification for quaternionic Shimura varieties

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ABSTRACT

Let F be a totally real field in which a prime p is unramified. We define the Goren–Oort stratification of the characteristic- p fiber of a quaternionic Shimura variety of maximal level at p . We show that each stratum is a $(\mathbb{P}^1)^r$ -bundle over other quaternionic Shimura varieties (for an appropriate integer r). As an application, we give a necessary condition for the ampleness of a modular line bundle on a quaternionic Shimura variety in characteristic p .

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1. Introduction

This paper is intended as the first in a series [TX13, TX14], in which we study the Goren–Oort stratification for quaternionic Shimura varieties. The purpose of this paper is to give a global description of the strata, saying that they are in fact $(\mathbb{P}^1)^r$ -bundles over (the special fiber of) other quaternionic Shimura varieties for a certain integer r . We fix $p > 2$ a prime number.

1.1 Motivation: the case of modular curves

Let $N \geq 5$ be an integer prime to p . Let \mathcal{X} denote the modular curve with level $\Gamma_1(N)$; it admits a smooth integral model \mathbf{X} over $\mathbb{Z}[1/N]$. We are interested in the special fiber $X := \mathbf{X} \otimes_{\mathbb{Z}[1/N]} \mathbb{F}_p$. The curve X has a natural stratification by the supersingular locus X^{ss} and the ordinary locus X^{ord} . In concrete terms, X^{ss} is defined as the zero locus of the Hasse invariant $h \in H^0(X, \omega^{\otimes(p-1)})$, where $\omega^{\otimes(p-1)}$ is the sheaf for weight $p-1$ modular forms. The following deep result of Deuring and Serre (see e.g. [Ser96]) gives an intrinsic description of X^{ss} .

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THEOREM 1.2 (Deuring and Serre). *Let \mathbb{A}^∞ denote the ring of finite adeles over \mathbb{Q} and $\mathbb{A}^{\infty,p}$ its prime-to- p part. We have a bijection of sets*

$$\{\overline{\mathbb{F}}_p\text{-points of } X^{\text{ss}}\} \longleftrightarrow B_{p,\infty}^\times \backslash B_{p,\infty}^\times(\mathbb{A}^\infty) / K_1(N) B_{p,\infty}^\times(\mathbb{Z}_p)$$

equivariant under the prime-to- p Hecke correspondences, where $B_{p,\infty}$ is the quaternion algebra over \mathbb{Q} which ramifies at exactly two places: p and ∞ , $B_{p,\infty}^\times(\mathbb{Z}_p)$ is the maximal open compact subgroup of $B_{p,\infty}^\times(\mathbb{Q}_p)$, and $K_1(N)$ is an open compact subgroup of $\text{GL}_2(\mathbb{A}^{\infty,p}) = B_{p,\infty}^\times(\mathbb{A}^{\infty,p})$ given by

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\widehat{\mathbb{Z}}^{(p)}) \mid c \equiv 0, d \equiv 1 \pmod{N} \right\}, \quad \text{where } \widehat{\mathbb{Z}}^{(p)} = \prod_{l \neq p} \mathbb{Z}_l.$$

The original proof of this theorem uses the fact that all supersingular elliptic curves over $\overline{\mathbb{F}}_p$ are isogenous and the quasi-endomorphism ring is exactly $B_{p,\infty}$. We however prefer to understand the result as: certain special cycles of the special fiber of the Shimura variety for GL_2 are just the special fiber of the Shimura variety for $B_{p,\infty}^\times$.

The aim of this paper is to generalize this theorem to the case of quaternionic Shimura varieties. For the purpose of simple presentation in this introduction, we focus on the case of Hilbert modular varieties. We will indicate how to modify the result to adapt to general cases.

1.3 Goren–Oort stratification

Let F be a totally real field and let \mathcal{O}_F denote its ring of integers. We assume that p is unramified in F . Goren and Oort [GO00] defined a stratification of the special fiber of the Hilbert modular variety X_{GL_2} . More precisely, let \mathbb{A}_F^∞ denote the ring of finite adeles of F and $\mathbb{A}_F^{\infty,p}$ its prime-to- p part. We fix an open compact subgroup $K^p \subset \text{GL}_2(\mathbb{A}_F^{\infty,p})$. Let $\mathcal{X}_{\text{GL}_2}$ denote the Hilbert modular variety (over \mathbb{Q}) with tame level K^p . Its complex points are given by

$$\mathcal{X}_{\text{GL}_2}(\mathbb{C}) = \text{GL}_2(F) \backslash (\mathfrak{h}^{\pm[F:\mathbb{Q}]} \times \text{GL}_2(\mathbb{A}_F^\infty)) / (K^p \times \text{GL}_2(\mathcal{O}_{F,p})),$$

where $\mathfrak{h}^\pm := \mathbb{C} \backslash \mathbb{R}$ and $\mathcal{O}_{F,p} := \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The Hilbert modular variety $\mathcal{X}_{\text{GL}_2}$ admits an integral model \mathbf{X}_{GL_2} over $\mathbb{Z}_{(p)}$; let X_{GL_2} denote its special fiber over $\overline{\mathbb{F}}_p$.

Since p is unramified in F , we may and will identify the p -adic embeddings of F with the homomorphisms of \mathcal{O}_F to $\overline{\mathbb{F}}_p$, i.e. $\text{Hom}(F, \overline{\mathbb{Q}}_p) \cong \text{Hom}(\mathcal{O}_F, \overline{\mathbb{F}}_p)$. Let Σ_∞ denote this set. (We shall later identify the p -adic embeddings with the real embeddings, hence the subscript ∞ .) Under the latter description, the absolute Frobenius σ acts on Σ_∞ by taking an element τ to the composite $\sigma\tau : \mathcal{O}_F \xrightarrow{\tau} \overline{\mathbb{F}}_p \xrightarrow{x \mapsto x^p} \overline{\mathbb{F}}_p$. This action decomposes Σ_∞ into a disjoint union of cycles, parametrized by all p -adic places of F .

Let \mathcal{A} denote the universal abelian variety over X_{GL_2} . The sheaf of invariant differential 1-forms $\omega_{\mathcal{A}/X_{\text{GL}_2}}$ is then locally free of rank one as a module over

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{X_{\text{GL}_2}} \cong \bigoplus_{\tau \in \Sigma_\infty} \mathcal{O}_{X_{\text{GL}_2,\tau}},$$

where $\mathcal{O}_{X_{\text{GL}_2,\tau}$ is the direct summand on which \mathcal{O}_F acts through $\tau : \mathcal{O}_F \rightarrow \overline{\mathbb{F}}_p$. We then write accordingly $\omega_{\mathcal{A}/X_{\text{GL}_2}} = \bigoplus_{\tau \in \Sigma_\infty} \omega_\tau$; each ω_τ is locally free of rank one over $\mathcal{O}_{X_{\text{GL}_2}}$.

DEFINITION 1.4. The Verschiebung map induces an \mathcal{O}_F -morphism $\omega_{A/X_{GL_2}} \rightarrow \omega_{A^{(p)}/X_{GL_2}}$, which further induces a homomorphism $h_\tau : \omega_\tau \rightarrow \omega_{\sigma^{-1}\tau}^{\otimes p}$ for each $\tau \in \Sigma_\infty$. This map then defines a global section $h_\tau \in H^0(X_{GL_2}, \omega_\tau^{\otimes -1} \otimes \omega_{\sigma^{-1}\tau}^{\otimes p})$, which we call the *partial Hasse invariant at τ* . We use X_τ to denote the zero locus of h_τ . For a subset $T \subseteq \Sigma_\infty$, we put $X_T = \bigcap_{\tau \in T} X_\tau$. These X_T give the *Goren–Oort stratification* of X_{GL_2} .

An alternative definition of X_T is given as follows: $z \in X_T(\overline{\mathbb{F}}_p)$ if and only if $\text{Hom}(\alpha_p, A_z[p])$ under the action of \mathcal{O}_F has eigenvalues given by those embeddings $\tau \in T$. We refer to [GO00] for the proof of equivalence and a more detailed discussion.

It is proved in [GO00] that each X_τ is a smooth and proper divisor and these divisors intersect transversally. Hence, X_T is smooth of codimension $\#T$ for any subset $T \subseteq \Sigma_\infty$; it is proper if $T \neq \emptyset$.

1.5 Description of the Goren–Oort strata

The goal of this paper is to give a *global* description of the Goren–Oort strata.

Prior to our paper, most works focused on the p -divisible groups of the abelian varieties, which often provides a good access to the local structure of Goren–Oort strata, e.g. dimensions and smoothness. Unfortunately, there has been little understanding of the global geometry of X_T , mostly in low dimension. We refer to the survey article [AG04] for a historical account. Recently, Helm made a break-through progress in [Hel12, Hel10] by taking advantage of the moduli problem; he was able to describe the global geometry of certain analogous strata of the special fibers of Shimura varieties of type $U(2)$.

Our proof of the main theorem of this paper is, roughly speaking, to complete Helm’s argument to cover all cases for the $U(2)$ -Shimura varieties and then transfer the results from the unitary side to the quaternionic side.

Rather than stating our main theorem in an abstract way, we prefer to give more examples to indicate the general pattern.

When $F = \mathbb{Q}$, this is discussed in § 1.1. For $F \neq \mathbb{Q}$, we fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ and hence identify $\Sigma_\infty = \text{Hom}(F, \overline{\mathbb{Q}}_p)$ with the set of real embeddings of F .

1.5.1 *F real quadratic and p inert in F.* Let ∞_1 and ∞_2 denote the two real embeddings of F and τ_1 and τ_2 the corresponding p -adic embeddings (via the fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$). Then our main Theorem 5.2 says that each X_{τ_i} is a \mathbb{P}^1 -bundle over the special fiber of the discrete Shimura variety $\overline{\text{Sh}}_{B_{\infty_1, \infty_2}^\times}$, where B_{∞_1, ∞_2} stands for the quaternion algebra over F which ramifies at both archimedean places. The intersection $X_{\tau_1} \cap X_{\tau_2}$ is isomorphic to the special fiber of the discrete Shimura variety $\overline{\text{Sh}}_{B_{\infty_1, \infty_2}^\times}(\text{Iw}_p)$, where (Iw_p) means to take Iwahori level structure at p instead. The two natural embeddings of $X_{\tau_1} \cap X_{\tau_2}$ into X_{τ_1} and X_{τ_2} induce two morphisms $\overline{\text{Sh}}_{B_{\infty_1, \infty_2}^\times}(\text{Iw}_p) \rightarrow \overline{\text{Sh}}_{B_{\infty_1, \infty_2}^\times}$; this gives (a certain variant of) the Hecke correspondence at p (see Theorem 7.16).

We remark here that it was first proved in [BG99] that the one-dimensional strata are disjoint unions of \mathbb{P}^1 and the number of such \mathbb{P}^1 is also computed in [BG99]. This computation relies on the intersection theory and does not provide a natural parametrization as we gave above. Our proof will be different from theirs. One can easily recover their counting from our natural parametrization.

1.5.2 *Quaternionic Shimura varieties.* Before proceeding, we clarify our convention on quaternionic Shimura varieties.

For \mathbf{S} an even subset of archimedean and p -adic places of F , we use $B_{\mathbf{S}}$ to denote the quaternion algebra over F which ramifies exactly at those places in \mathbf{S} . We fix an identification $B_{\mathbf{S}}^{\times}(\mathbb{A}^{\infty,p}) \cong \mathrm{GL}_2(\mathbb{A}^{\infty,p})$. We fix a maximal open compact subgroup $B_{\mathbf{S}}^{\times}(\mathcal{O}_{F,p})$ of $B_{\mathbf{S}}^{\times}(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. We use \mathbf{S}_{∞} to denote the subset of archimedean places of \mathbf{S} . The Shimura variety $\mathcal{S}h_{B_{\mathbf{S}}^{\times}}$ for the algebraic group $\mathrm{Res}_{F/\mathbb{Q}} B_{\mathbf{S}}^{\times}$ has complex points

$$\mathcal{S}h_{B_{\mathbf{S}}^{\times}}(\mathbb{C}) = B_{\mathbf{S}}^{\times}(F) \setminus (\mathfrak{h}^{\pm[F:\mathbb{Q}]-\#\mathbf{S}_{\infty}} \times B_{\mathbf{S}}^{\times}(\mathbb{A}_F^{\infty})) / (K^p \times B_{\mathbf{S}}^{\times}(\mathcal{O}_{F,p})).$$

Here and later, the tame level K^p is uniformly matched up for all quaternionic Shimura varieties.

Unfortunately, $\mathcal{S}h_{B_{\mathbf{S}}^{\times}}$ itself does not possess a moduli interpretation. We follow the construction of Carayol [Car86] to relate it with a unitary Shimura variety Y and ‘carry over’ the integral model of Y . The assumption that $p > 2$ comes from the verification of the extension property for the integral canonical model following Moonen [Moo98, Corollary 3.8].

In any case, we use $\overline{\mathrm{Sh}}_{B_{\mathbf{S}}^{\times}}$ to denote the special fiber of the Shimura variety over $\overline{\mathbb{F}}_p$. When we take the Iwahori level structure at p instead, we write $\overline{\mathrm{Sh}}_{B_{\mathbf{S}}^{\times}}(\mathrm{Iw}_p)$.

1.5.3 *F real quartic and p inert in F.* Let $\infty_1, \dots, \infty_4$ denote the four real embeddings of F , labeled so that the corresponding p -adic embeddings τ_1, \dots, τ_4 satisfy $\sigma\tau_i = \tau_{i+1}$ with the convention that $\tau_i = \tau_{i \pmod{4}}$. We list the description of the strata as follows.

Strata	Description
X_{τ_i} for each i	\mathbb{P}^1 -bundle over $\overline{\mathrm{Sh}}_{B_{\{\infty_{i-1}, \infty_i\}}}$
$X_{\{\tau_{i-1}, \tau_i\}}$ for each i	$\overline{\mathrm{Sh}}_{B_{\{\infty_{i-1}, \infty_i\}}}$
$X_{\{\tau_1, \tau_3\}}$ and $X_{\{\tau_2, \tau_4\}}$	$(\mathbb{P}^1)^2$ -bundle over $\overline{\mathrm{Sh}}_{B_{\{\infty_1, \infty_2, \infty_3, \infty_4\}}}$
$X_{\mathbf{T}}$ with $\#\mathbf{T} = 3$	\mathbb{P}^1 -bundle over $\overline{\mathrm{Sh}}_{B_{\{\infty_1, \infty_2, \infty_3, \infty_4\}}}$
$X_{\{\tau_1, \tau_2, \tau_3, \tau_4\}}$	$\overline{\mathrm{Sh}}_{B_{\{\infty_1, \infty_2, \infty_3, \infty_4\}}}(\mathrm{Iw}_p)$

In particular, we point out that for a codimension-2 stratum, its shape depends on whether the two chosen τ_i are adjacent in the cycle $\tau_1 \rightarrow \dots \rightarrow \tau_4 \rightarrow \tau_1$.

1.5.4 *F general totally real of degree g over Q and p inert in F.* As before, we label the real embeddings of F by $\infty_1, \dots, \infty_g$ such that the corresponding p -adic embeddings τ_1, \dots, τ_g satisfy $\sigma\tau_i = \tau_{i+1}$ with the convention that $\tau_i = \tau_{i \pmod{g}}$. The general statement for Goren–Oort strata takes the following form: for a subset $\mathbf{T} \subseteq \Sigma_{\infty}$, the stratum $X_{\mathbf{T}}$ is isomorphic to a $(\mathbb{P}^1)^r$ -bundle over the special fiber of another quaternion Shimura variety $\overline{\mathrm{Sh}}_{B_{\mathbf{S}(\mathbf{T})}}^{\times}$. We now explain, given \mathbf{T} , what $\mathbf{S}(\mathbf{T})$ and r are.

- When $\mathbf{T} \subsetneq \Sigma_{\infty}$, we construct $\mathbf{S}(\mathbf{T})$ as follows: if $\tau \notin \mathbf{T}$ and $\sigma^{-1}\tau, \dots, \sigma^{-m}\tau \in \mathbf{T}$, we put $\sigma^{-1}\tau, \dots, \sigma^{-2\lfloor m/2 \rfloor}\tau$ into $\mathbf{S}(\mathbf{T})$. In other words, we always have $\mathbf{T} \subseteq \mathbf{S}(\mathbf{T})$, and $\mathbf{S}(\mathbf{T})$ contains the additional element $\sigma^{-m-1}\tau$ if and only if the corresponding m is odd. The number r is the cardinality of $\mathbf{S}(\mathbf{T}) - \mathbf{T}$.
- When $\mathbf{T} = \Sigma_{\infty}$, r is always 0; for $\mathbf{S}(\mathbf{T})$, we need to distinguish the parity:
 - * if $\#\Sigma_{\infty}$ is odd, we put $\mathbf{S}(\mathbf{T}) = \Sigma \cup \{p\}$;
 - * if $\#\Sigma_{\infty}$ is even, we put $\mathbf{S}(\mathbf{T}) = \Sigma$ and we put an Iwahori level structure at p .

1.5.5 *F* general totally real and *p* unramified in *F*. The general principle is: *different places above p work ‘independently’ in the recipe of describing the strata (e.g. which places of the quaternion algebra are ramified); so we just take the ‘product’ of all recipes for different p-adic places.*

More concretely, let $p\mathcal{O}_F = \mathfrak{p}_1 \cdots \mathfrak{p}_d$ be the prime ideal factorization. We use Σ_∞ to denote the set of all archimedean embeddings of *F*, which is identified with the set of *p*-adic embeddings. We use $\Sigma_{\infty/\mathfrak{p}_i}$ to denote those archimedean embeddings or equivalently *p*-adic embeddings that give rise to the *p*-adic place \mathfrak{p}_i . Given any subset $\mathsf{T} \in \Sigma_\infty$, we put $\mathsf{T}_{\mathfrak{p}_i} = \mathsf{T} \cap \Sigma_{\infty/\mathfrak{p}_i}$. Applying the recipe in § 1.5.4 to each $\mathsf{T}_{\mathfrak{p}_i}$ viewed as a subset of $\Sigma_{\infty/\mathfrak{p}_i}$, we get a set of places $\mathsf{S}(\mathsf{T}_{\mathfrak{p}_i})$ and a non-negative number $r_{\mathfrak{p}_i}$. We put $\mathsf{S}(\mathsf{T}) = \bigcup_{i=1}^d \mathsf{S}(\mathsf{T}_{\mathfrak{p}_i})$ and $r = \sum_{i=1}^d r_{\mathfrak{p}_i} = \sum_{i=1}^d \#(\mathsf{S}(\mathsf{T}_{\mathfrak{p}_i}) - \mathsf{T}_{\mathfrak{p}_i})$. Then X_{T} is a $(\mathbb{P}^1)^r$ -bundle over $\overline{\text{Sh}}_{B_{\mathsf{S}(\mathsf{T})}^\times}$ (with possibly some Iwahori level structure at appropriate places above *p*).

We also prove analogous results on the global description of the Goren–Oort strata on general quaternionic Shimura varieties (Theorem 5.2). We refer to the content of the paper for the statement. The modification we need to do in the general quaternionic case is that one just ‘ignores’ all ramified archimedean places and applies the above recipe formally to the set Σ_∞ after ‘depriving all ramified archimedean places’.

1.6 Method of the proof

We briefly explain the idea behind the proof. The first step is to translate the question into an analogous question on (the special fiber of) unitary Shimura varieties. We use X' to denote the special fiber of the unitary Shimura variety we start with, over which we have the universal abelian variety A' . Similar to the Hilbert case, we have naturally defined analogous Goren–Oort stratification given by divisors X'_τ . We consider $X'_\mathsf{T} = \bigcap_{\tau \in \mathsf{T}} X'_\tau$.

The idea is to prove the following sequence of isomorphisms: $X'_\mathsf{T} \xleftarrow{\cong} Y'_\mathsf{T} \xrightarrow{\cong} Z'_\mathsf{T}$, where Z'_T is the (\mathbb{P}^1) -power bundle over the special fiber of another unitary Shimura variety, which comes with a universal abelian variety B' ; Y'_T is the moduli space classifying both A' and B' together with a quasi-isogeny $A' \rightarrow B'$ of certain fixed type (with very small *p*-power degree); and the two morphisms are just simply forgetful morphisms. We defer the characterization of the quasi-isogeny to the content of the paper. To prove the two isomorphisms above, we simply check that the natural forgetful morphisms are bijective on the closed points and induce isomorphisms on the tangent spaces.

Remark 1.7. We point out that we have been deliberately working with the special fiber over the algebraic closure $\overline{\mathbb{F}}_p$. This is because the description of the stratification is *not* compatible with the action of the Frobenius. In fact, a more rigorous way to formulate the theorem is to compare the special fiber of the Shimura variety associated to $\text{GL}_2(F) \times_{F^\times} E^\times$ and that to $B_{\mathsf{S}(\mathsf{T})}^\times \times_{F^\times} E^\times$. The homomorphism from the Deligne torus into the two E^\times are in fact different, causing the incompatibility of the Frobenius action. (See Corollary 5.9 for the corresponding statement.) The result about quaternionic Shimura varieties is obtained by comparing geometric connected components of the corresponding Shimura varieties, in which we lose the Frobenius action. See Remark 5.10 for more discussion.

1.8 Ampleness of automorphic line bundles

An immediate application of the study of the global geometry of the Goren–Oort stratification is to give a necessary condition (hopefully also sufficient) for an automorphic line bundle to be ample. As before, we take *F* to be a totally real field of degree *g* in which *p* is inert for simplicity.

Let $X_{\text{GL}_2}^*$ denote the special fiber of the minimal compactification of the Hilbert modular variety. We label all p -adic embeddings as τ_1, \dots, τ_g with subindices considered modulo g , such that $\sigma\tau_i = \tau_{i+1}$. Then $\omega_{\tau_1}, \dots, \omega_{\tau_g}$ form a basis of the group of automorphic line bundles. The class $[\omega_{\tau_i}]$ in $\text{Pic}(X)_{\mathbb{Q}} := \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ of each ω_{τ_i} extends to a class in $\text{Pic}(X_{\text{GL}_2}^*)_{\mathbb{Q}}$, still denoted by $[\omega_{\tau_i}]$. For a g -tuple $\underline{k} = (k_1, \dots, k_g) \in \mathbb{Z}^g$, we put $[\omega^{\underline{k}}] = \sum_{i=1}^g k_i [\omega_{\tau_i}]$. Probably slightly contrary to the common intuition from the case of modular forms, we prove the following theorem.

THEOREM 1.9. *If the rational class of line bundle $[\omega^{\underline{k}}]$ is ample, then*

$$pk_i > k_{i-1} \quad \text{for all } i \quad (\text{and all } k_i > 0). \tag{1.9.1}$$

Here we put the second condition in parentheses because it automatically follows from the first condition. This theorem is proved in Theorem 6.5. When F is a real quadratic field, Theorem 1.9 is proved in [AG04, Theorem 8.1.1].

To see that the condition (1.9.1) is necessary, we simply restrict to each of the Goren–Oort (GO) strata X_{τ_i} , which is a \mathbb{P}^1 -bundle as we discussed before. Along each of the \mathbb{P}^1 -fibers, the line bundle $\omega^{\underline{k}}$ restricts to $\mathcal{O}(pk_i - k_{i-1})$. The condition (1.9.1) is clear.

We do expect the condition in Theorem 1.9 to be necessary (which was proved for Hilbert modular surfaces in [AG04]), but we are not able to prove it due to a combinatorics complication.

1.10 Forthcoming works in this series

We briefly advertise the other papers of this series to indicate the potential applications of the technical result in this paper. In the subsequent paper [TX13], we discuss an application to the classicality of overconvergent Hilbert modular forms, following the original proof of R. Coleman. In the third paper [TX14], we show that certain generalizations of the Goren–Oort strata realize Tate classes of the special fiber of certain Hilbert modular varieties, and hence verify the Tate conjecture under some genericity hypothesis.

1.11 Structure of the paper

In § 2, we review some basic facts about integral models of Shimura varieties, which will be used to relate the quaternionic Shimura varieties with the unitary Shimura varieties. One novelty is that we include a discussion about the ‘canonical model’ of certain discrete Shimura varieties; this can be treated uniformly together with usual Shimura varieties. In § 3, we construct the integral canonical model for quaternionic Shimura varieties, following Carayol [Car86]. However, we tailor many of the choices (e.g. the auxiliary CM field and signatures) for our later application. In § 4, we define the Goren–Oort stratification for the unitary Shimura varieties and transfer them to the quaternionic Shimura varieties; this is a straightforward generalization of the work of Goren and Oort [GO00]. In § 5, we give the global description of Goren–Oort stratification. The method is very close to that used in [Hel12]. In § 6, we give a more detailed description for Goren–Oort divisors, including a necessary condition for an automorphic line bundle to be ample, and a structure theorem relating the Goren–Oort stratification along a \mathbb{P}^1 -bundle morphism provided by Theorem 5.8. In § 7, we further study some structures of the Goren–Oort strata which will play an important role in the forthcoming paper [TX14].

1.12 Notation

1.12.1 For a scheme X over a ring R and a ring homomorphism $R \rightarrow R'$, we use $X_{R'}$ to denote the base change $X \times_{\text{Spec } R} \text{Spec } R'$.

For a field F , we use Gal_F to denote its Galois group.

For a number field F , we use \mathbb{A}_F to denote its ring of adèles and \mathbb{A}_F^∞ (respectively $\mathbb{A}_F^{\infty,p}$) to denote its finite adèles (respectively prime-to- p finite adèles). When $F = \mathbb{Q}$, we suppress the subscript F from the notation. We use the superscript cl to mean closure in certain topological groups. For example, $F^{\times,\text{cl}}$ means the closure of F^\times inside $\mathbb{A}_F^{\infty,\times}$ or $\mathbb{A}_F^{\infty,p}$ (depending on the situation). We put $\widehat{\mathbb{Z}} = \prod_l \mathbb{Z}_l$ and $\widehat{\mathbb{Z}}^{(p)} = \prod_{l \neq p} \mathbb{Z}_l$.

For each finite place \mathfrak{p} of F , let $F_{\mathfrak{p}}$ denote the completion of F at \mathfrak{p} and $\mathcal{O}_{\mathfrak{p}}$ its valuation ring, which has uniformizer $\varpi_{\mathfrak{p}}$ and residue field $k_{\mathfrak{p}}$. (When $F_{\mathfrak{p}}$ is unramified over \mathbb{Q}_p , we take $\varpi_{\mathfrak{p}}$ to be p .)

We normalize the Artin map $\text{Art}_F : F^\times \backslash \mathbb{A}_F^\times \rightarrow \text{Gal}_F^{\text{ab}}$ so that for each finite prime \mathfrak{p} , the local uniformizer at \mathfrak{p} is mapped to a geometric Frobenius at \mathfrak{p} .

1.12.2 For A an abelian scheme over a scheme S , we denote by A^\vee the dual abelian scheme, by $\text{Lie}(A)$ the Lie algebra of A , and by $\omega_{A/S}$ the module of invariant 1-differential forms of A relative to S . We sometimes omit S from the notation when the base is clear.

For a finite p -group scheme or a p -divisible group G over a perfect field k of characteristic p , we use $\mathcal{D}(G)$ to denote its *covariant* Dieudonné module. For an abelian variety A over k , we write \mathcal{D}_A for $\mathcal{D}(A[p])$ and write $\tilde{\mathcal{D}}_A$ for $\mathcal{D}(A[p^\infty])$.

1.12.3 Throughout this paper, we fix a totally real field F of degree $g > 1$ over \mathbb{Q} . Let Σ denote the set of places of F and Σ_∞ the subset of archimedean places or, equivalently, all real embeddings of F .

We fix a prime number p which is unramified in F . Let Σ_p denote the set of places of F above p . We fix an isomorphism $\iota_p : \mathbb{C} \xrightarrow{\cong} \overline{\mathbb{Q}}_p$ so that we identify Σ_∞ as the set of p -adic embeddings of F . For each $\mathfrak{p} \in \Sigma_p$, we use $\Sigma_{\infty/\mathfrak{p}}$ to denote the subset of p -adic embeddings $\tau \in \Sigma_\infty$ which induce the p -adic place \mathfrak{p} . Since p is unramified, each τ induces an embedding $\mathcal{O}_F \hookrightarrow W(\overline{\mathbb{F}}_p)$. Post-composition with the Frobenius σ on the latter induces an action of σ on the set of p -adic embeddings and makes each $\Sigma_{\infty/\mathfrak{p}}$ into one cycle. We use $\sigma\tau$ to denote this action, i.e. $\sigma\tau = \sigma \circ \tau$.

1.12.4 We will consider a CM extension E over F , in which all places above p are unramified. Let $\Sigma_{E,\infty}$ denote the set of complex embeddings of E . For $\tau \in \Sigma_\infty$, we often use $\tilde{\tau}$ to denote a/some complex embedding of E extending τ , and we write $\tilde{\tau}^c$ for its complex conjugate. Using the isomorphism ι_p above, we view $\tilde{\tau}$ and $\tilde{\tau}^c$ as p -adic embeddings of E .

Under the natural two-to-one map $\Sigma_{E,\infty} \rightarrow \Sigma_\infty$, we use $\Sigma_{E,\infty/\mathfrak{p}}$ to denote the preimage of $\Sigma_{\infty/\mathfrak{p}}$. In the case when \mathfrak{p} splits as $\mathfrak{q}\mathfrak{q}^c$ in E/F , we use $\Sigma_{E,\infty/\mathfrak{q}}$ to denote the set of complex embeddings $\tilde{\tau}$ such that $\iota_p \circ \tilde{\tau}$ induces the p -adic place \mathfrak{q} .

1.12.5 For \mathbf{S} an even subset of places of F , we denote by $B_{\mathbf{S}}$ the quaternion algebra over F ramified at \mathbf{S} . Let $\text{Nm}_{B_{\mathbf{S}}/F} : B_{\mathbf{S}} \rightarrow F$ denote the reduced norm and $\text{Tr}_{B_{\mathbf{S}}/F} : B_{\mathbf{S}} \rightarrow F$ the reduced trace.

We use the following lists of algebraic groups. Let $G_{\mathbf{S}}$ denote the algebraic group $\text{Res}_{F/\mathbb{Q}} B_{\mathbf{S}}^\times$. Let E be the CM extension of F above and put $T_{E,\tilde{\mathbf{S}}} = \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m$; see § 3.4 for the meaning of the subscript $\tilde{\mathbf{S}}$. We put $\tilde{G}_{\tilde{\mathbf{S}}} = G_{\mathbf{S}} \times T_{E,\tilde{\mathbf{S}}}$ and $G''_{\tilde{\mathbf{S}}} = G_{\mathbf{S}} \times_Z T_{E,\tilde{\mathbf{S}}}$, which is the quotient of $\tilde{G}_{\tilde{\mathbf{S}}}$ by the subgroup $Z = \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ embedded as $z \mapsto (z, z^{-1})$. Let $G'_{\tilde{\mathbf{S}}}$ denote the subgroup of $G''_{\tilde{\mathbf{S}}}$ consisting of elements (g, e) such that $\text{Nm}_{B_{\mathbf{S}}/F}(g) \cdot \text{Nm}_{E/F}(e) \in \mathbb{G}_m$.

We put $\mathbf{S}_\infty = \Sigma_\infty \cap \mathbf{S}$. For each place $\mathfrak{p} \in \Sigma_p$, we set $\mathbf{S}_{\infty/\mathfrak{p}} = \Sigma_{\infty/\mathfrak{p}} \cap \mathbf{S}$.

2. Basics of Shimura varieties

We first collect some basic facts on integral canonical models of Shimura varieties. Our main references are [Del71, Del79, Mil05, Kis10]. (Our convention follows [Mil05, Kis10].) We focus on how to transfer integral canonical models of Shimura varieties from one group to another group. This is well known to the experts. We include the discussion here for completeness. One novelty of this section, however, is that we give an appropriate definition of ‘canonical model’ for certain discrete Shimura varieties, so that the construction holds uniformly for both usual Shimura varieties and these zero-dimensional ones. This will be important for later applications to transfer the description of the Goren–Oort strata between Shimura varieties for different groups.

Notation 2.1. Fix a prime number p . Fix an isomorphism $\iota : \mathbb{C} \xrightarrow{\cong} \overline{\mathbb{Q}_p}$. We use $\overline{\mathbb{Q}}$ to denote the algebraic closure of \mathbb{Q} inside \mathbb{C} (which is then identified with the algebraic closure of \mathbb{Q} inside $\overline{\mathbb{Q}_p}$ via ι).

In this section, let G be a connected reductive group over \mathbb{Q} . We use $G(\mathbb{R})^+$ to denote the neutral connected component of $G(\mathbb{R})$. We put $G(\mathbb{Q})^+ = G(\mathbb{R})^+ \cap G(\mathbb{Q})$. We use G^{ad} to denote the adjoint group and G^{der} its derived subgroup. We use $G(\mathbb{R})_+$ to denote the preimage of $G^{\text{ad}}(\mathbb{R})^+$ under the natural homomorphism $G(\mathbb{R}) \rightarrow G^{\text{ad}}(\mathbb{R})$. We put $G(\mathbb{Q})_+ = G(\mathbb{R})_+ \cap G(\mathbb{Q})$.

For S a torus over \mathbb{Q}_p , let $S(\mathbb{Z}_p)$ denote the maximal open compact subgroup of $S(\mathbb{Q}_p)$.

2.2 Shimura varieties over \mathbb{C}

Put $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. For a real vector space V , a *Deligne homomorphism* $h : \mathbb{S} \rightarrow \text{GL}(V)$ induces a direct sum decomposition $V_{\mathbb{C}} = \bigoplus_{a,b \in \mathbb{Z}} V^{a,b}$ such that $z \in \mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times$ acts on $V^{a,b}$ via the character $z^{-a} \bar{z}^{-b}$. Let r denote the \mathbb{C} -homomorphism $\mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}}$ such that $z^{-a} \bar{z}^{-b} \circ r = (x \mapsto x^{-a})$.

A *Shimura datum* is a pair (G, X) consisting of a connected reductive group G over \mathbb{Q} and a $G(\mathbb{R})$ -conjugacy class X of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfying the following conditions:

- (SV1) for $h \in X$, only characters $z/\bar{z}, 1, \bar{z}/z$ occur in the representation of $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times$ on $\text{Lie}(G^{\text{ad}})_{\mathbb{C}}$ via $\text{Ad} \circ h$;
- (SV2) for $h \in X$, $\text{Ad}(h(i))$ is a Cartan involution on $G_{\mathbb{R}}^{\text{ad}}$; and
- (SV3) G^{ad} has no \mathbb{Q} -factor H such that $H(\mathbb{R})$ is compact.

The $G(\mathbb{R})$ -conjugacy class X of h admits the structure of a complex manifold. Let X^+ denote a fixed connected component of X .

A pair (G, X) satisfying only (SV1), (SV2), and the following (SV3)' is called a *weak Shimura datum*.

- (SV3)' $G^{\text{ad}}(\mathbb{R})$ is compact (and hence connected by [Bor91, p. 277]; this forces the image of h to land in the center $Z_{\mathbb{R}}$ of $G_{\mathbb{R}}$).

For an open compact subgroup $K \subseteq G(\mathbb{A}^\infty)$, we define the *Shimura variety* for (G, X) with level K to be the quasi-projective variety $\text{Sh}_K(G, X)_{\mathbb{C}}$, whose \mathbb{C} -points are

$$\text{Sh}_K(G, X)(\mathbb{C}) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty) / K \cong G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}^\infty) / K.$$

When (G, X) is a weak Shimura datum, $\text{Sh}_K(G, X)_{\mathbb{C}}$ is just a finite set of points.

2.3 Reflex field

Let (G, X) be a (weak) Shimura datum. The *reflex field*, denoted by $E = E(G, X)$, is the field of definition of the conjugacy class of the composition $h \circ r : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$. It is a subfield of \mathbb{C} , finite over \mathbb{Q} . We refer to [Del71] for the definition of the canonical model $\text{Sh}_K(G, X)$ of $\text{Sh}_K(G, X)_{\mathbb{C}}$ over this reflex field E . We assume from now on that all (weak) Shimura varieties we consider in this section admit canonical models. (In fact, [Del71] excludes the case when (G, X) is a weak Shimura datum. We will give the meaning of the canonical model in this case in §2.8 later.)

We will always assume that K is the product $K^p K_p$ of an open compact subgroup K^p of $G(\mathbb{A}^{\infty, p})$ and an open compact subgroup K_p of $G(\mathbb{Q}_p)$. Taking the inverse limit over the open compact subgroups K^p , we have $\text{Sh}_{K_p}(G, X) := \varprojlim_{K^p} \text{Sh}_{K^p K_p}(G, X)$. This is actually a scheme locally of finite type over E carrying a natural (right) action of $G(\mathbb{A}^{\infty, p})$.

2.4 Extension property

Let \mathcal{O} be the ring of integers in a finite extension of \mathbb{Q}_p . A scheme X over \mathcal{O} is said to have the *extension property* over \mathcal{O} if for any smooth \mathcal{O} -scheme S , a map $S \otimes \text{Frac}(\mathcal{O}) \rightarrow X$ extends to S . (Such an extension is automatically unique if it exists by the normality of S .) Note that this condition is weaker than the one given in [Kis10, 2.3.7] but is enough to ensure the uniqueness.

The chosen isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_p}$ identifies E as a subfield of $\overline{\mathbb{Q}_p}$. Let E_{φ} denote the p -adic completion of E and \mathcal{O}_{φ} its valuation ring with \mathbb{F}_{φ} as the residue field. Let E_{φ}^{ur} be the maximal unramified extension of E_{φ} and $\mathcal{O}_{\varphi}^{\text{ur}}$ its valuation ring.

An *integral canonical model* $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_{\varphi}}$ of $\text{Sh}_{K_p}(G, X)$ over \mathcal{O}_{φ} is an \mathcal{O}_{φ} -scheme $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_{\varphi}}$, which is an inverse limit of smooth \mathcal{O}_{φ} -schemes $\text{Sh}_{K^p K_p}(G, X)_{\mathcal{O}_{\varphi}}$ with finite étale transition maps as K^p varies, such that:

- there is an isomorphism $\text{Sh}_K(G, X)_{\mathcal{O}_{\varphi}} \otimes_{\mathcal{O}_{\varphi}} E_{\varphi} \cong \text{Sh}_K(G, X) \otimes_E E_{\varphi}$ for each K compatible with transition maps as K varies; and
- $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_{\varphi}} = \varprojlim_{K^p} \text{Sh}_{K^p K_p}(G, X)_{\mathcal{O}_{\varphi}}$ satisfies the extension property.

Existence of integral canonical models of Shimura varieties of abelian type with hyperspecial level structure was proved by Kisin [Kis10]. Unfortunately, our application requires, in some special cases, certain non-hyperspecial level structures, as well as certain non-quasi-split groups at p . We have to establish the integral canonical models in two steps: we first prove the existence for some group G' with the same derived and adjoint groups as G (as is done in §3); we then reproduce a variant of an argument of Deligne to show that the integral canonical model for the Shimura variety for G' gives that of G . The second step is well known at least for regular Shimura varieties when K_p is hyperspecial [Kis10]. Our limited contribution here is to include some non-hyperspecial case and to cover the case of discrete Shimura varieties, in a uniform way.

HYPOTHESIS 2.5. Let (G, X) be a (weak) Shimura datum. From now on, we assume that the derived subgroup G^{der} is simply connected, which will be the case when we apply the theory later. Let Z denote the center of G . Let $\nu : G \twoheadrightarrow T$ denote the maximal abelian quotient of G . We fix an open compact subgroup K_p of $G(\mathbb{Q}_p)$ such that $\nu(K_p) = T(\mathbb{Z}_p)$ and $K_p \cap Z(\mathbb{Q}_p) = Z(\mathbb{Z}_p)$.

2.6 Geometric connected components

We put $T(\mathbb{R})^{\dagger} = \text{Im}(Z(\mathbb{R}) \rightarrow T(\mathbb{R}))$ and $T(\mathbb{Q})^{\dagger} = T(\mathbb{R})^{\dagger} \cap T(\mathbb{Q})$. Put $T(\mathbb{Q})^{?,(p)} = T(\mathbb{Q})^? \cap T(\mathbb{Z}_p)$ for $? = \emptyset$ or \dagger . Let Y denote the finite quotient $T(\mathbb{R})/T(\mathbb{R})^{\dagger}$, which is isomorphic to $T(\mathbb{Q})/T(\mathbb{Q})^{\dagger}$

because $T(\mathbb{Q})$ is dense in $T(\mathbb{R})$. The morphism $\nu : G \rightarrow T$ then induces a natural map

$$\begin{aligned} \nu : G(\mathbb{Q})_+ \backslash X^+ \times G(\mathbb{A}^\infty)/K &\xrightarrow{\nu} T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}^\infty)/\nu(K) \cong T(\mathbb{Q})^{\dagger,(p)} \backslash T(\mathbb{A}^{\infty,p})/\nu(K^p) \\ &\cong T(\mathbb{Q})^{(p)} \backslash Y \times T(\mathbb{A}^{\infty,p})/\nu(K^p). \end{aligned} \tag{2.6.1}$$

If (G, X) is a Shimura datum, this map induces an isomorphism [Mil05, Theorem 5.17] on the set of geometric connected components $\pi_0(\mathrm{Sh}_K(G, X)_{\overline{\mathbb{Q}}}) \cong T(\mathbb{Q})^{(p)} \backslash Y \times T(\mathbb{A}^{\infty,p})/\nu(K^p)$. Taking the inverse limit gives a bijection

$$\pi_0(\mathrm{Sh}_{K_p}(G, X)_{\overline{\mathbb{Q}}}) \cong T(\mathbb{Q})^{\dagger,(p),\mathrm{cl}} \backslash T(\mathbb{A}^{\infty,p}) \cong T(\mathbb{Q})^{(p),\mathrm{cl}} \backslash Y \times T(\mathbb{A}^{\infty,p}).$$

Here and later, the superscript cl means taking closure in $T(\mathbb{A}^{\infty,p})$ (or in appropriate topological groups).

2.7 Reciprocity law

Let (G, X) be a (weak) Shimura datum. The composite

$$\nu hr : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{r} \mathbb{S}_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}} \xrightarrow{\nu} T_{\mathbb{C}}$$

does not depend on the choice of h (in the conjugacy class) and is defined over the reflex field E . The Shimura reciprocity map is given by

$$\mathrm{Rec}(G, X) : \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}}(\nu hr)} T_E \xrightarrow{N_{E/\mathbb{Q}}} T.$$

We normalize the Artin reciprocity map $\mathrm{Art}_E : \mathbb{A}_E^\times / (E^\times E_{\mathbb{R}}^{\times,+})^{\mathrm{cl}} \xrightarrow{\cong} \mathrm{Gal}_E^{\mathrm{ab}}$ so that the local parameter at a finite place \mathfrak{l} is mapped to a geometric Frobenius at \mathfrak{l} , where $E_{\mathbb{R}}^{\times,+}$ is the identity connected component of $E_{\mathbb{R}}^\times$. We denote the unramified Artin map at φ by $\mathrm{Art}_\varphi : E_\varphi^\times / \mathcal{O}_\varphi^\times \rightarrow \mathrm{Gal}_{E_\varphi}^{\mathrm{ab},\mathrm{ur}}$ (again normalized so that a uniformizer is mapped to the geometric Frobenius).

The morphism $\mathrm{Rec}(G, X)$ induces a natural homomorphism

$$\mathfrak{Rec} = \mathfrak{Rec}(G, X) : \mathrm{Gal}_E^{\mathrm{ab}} \xleftarrow[\cong]{\mathrm{Art}_E} (E^\times E_{\mathbb{R}}^{\times,+})^{\mathrm{cl}} \backslash \mathbb{A}_E^\times \xrightarrow{\mathrm{Rec}(G,X)} T(\mathbb{Q})^{\mathrm{cl}} \backslash Y \times T(\mathbb{A}^\infty).$$

When (G, X) is a Shimura datum, the Shimura reciprocity law [Mil05, §13] says that the action of $\sigma \in \mathrm{Gal}_E$ on $\pi_0(\mathrm{Sh}_{K_p}(G, X)_{\overline{\mathbb{Q}}}) \cong T(\mathbb{Q})^{\mathrm{cl}} \backslash Y \times T(\mathbb{A}^\infty)/T(\mathbb{Z}_p)$ is given by multiplication by $\mathfrak{Rec}(G, X)(\sigma)$. As a corollary, $\pi_0(\mathrm{Sh}_{K_p}(G, X)_{\overline{\mathbb{Q}}}) = \pi_0(\mathrm{Sh}_{K_p}(G, X)_{E_\varphi^{\mathrm{ur}}})$, i.e. the geometric connected components are seen over an unramified extension of E_φ .

The action of $\mathrm{Gal}_{E_\varphi}^{\mathrm{ab},\mathrm{ur}} = \mathrm{Gal}_{\mathbb{F}_\varphi}$ on the geometric connected component is then given by multiplication by the image of the Galois group element under the following map:

$$\begin{aligned} \mathfrak{Rec}_\varphi = \mathfrak{Rec}_\varphi(G, X) : \mathrm{Gal}_{\mathbb{F}_\varphi} &\xrightarrow[\cong]{\mathrm{Art}_\varphi} \widehat{E_\varphi^\times / \mathcal{O}_\varphi^\times} \\ &\longrightarrow (E^\times E_{\mathbb{R}}^{\times,+})^{\mathrm{cl}} \backslash \mathbb{A}_E^\times / \mathcal{O}_\varphi^\times \xrightarrow{\mathrm{Rec}(G,X)} T(\mathbb{Q})^{(p),\mathrm{cl}} \backslash Y \times T(\mathbb{A}^{\infty,p}), \end{aligned} \tag{2.7.1}$$

where $\widehat{E_\varphi^\times / \mathcal{O}_\varphi^\times}$ denotes the profinite completion of $E_\varphi^\times / \mathcal{O}_\varphi^\times$.

2.8 Integral canonical model for weak Shimura datum

When (G, X) is a weak Shimura datum, the associated Shimura variety is, geometrically, a finite set of points. We define its canonical model by specifying the action of Gal_E . The key observation is that condition (SV3)' ensures that the morphism $\text{Rec}(G, X)$ factors as

$$\begin{array}{ccccc}
 \text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) & \xrightarrow{\text{Res}_{E/\mathbb{Q}}(hr)} & Z_E & \xrightarrow{\text{Res}_{E/\mathbb{Q}}(\nu)} & T_E \\
 & \searrow \text{Rec}_Z(G, X) & \downarrow N_{E/\mathbb{Q}} & & \downarrow N_{E/\mathbb{Q}} \\
 & & Z & \xrightarrow{\nu} & T
 \end{array}$$

We consider the natural homomorphism

$$\mathfrak{Rec}_Z : \text{Gal}_E^{\text{ab}} \xleftarrow{\cong} (E^\times E_{\mathbb{R}}^{\times,+})^{\text{cl}} \backslash \mathbb{A}_E^\times \xrightarrow{\text{Rec}_Z(G, X)} Z(\mathbb{Q})^{\text{cl}} Z(\mathbb{R}) \backslash Z(\mathbb{A}) \cong Z(\mathbb{Q})^{\text{cl}} \backslash Z(\mathbb{A}^\infty).$$

We define the *canonical model* $\text{Sh}_K(G, X)$ to be the (pro-) E -scheme whose base change to \mathbb{C} is isomorphic to $\text{Sh}_K(G, X)_{\mathbb{C}}$, such that every $\sigma \in \text{Gal}_E$ acts on its $\overline{\mathbb{Q}}$ -points by multiplication by $\mathfrak{Rec}_Z(\sigma)$. In comparison to §2.7, we have $\nu(\sigma(x)) = \mathfrak{Rec}(G, X)(\sigma) \cdot \nu(x)$ for any $x \in \text{Sh}_K(G, X)(\overline{\mathbb{Q}})$.

Since $\text{Sh}_K(G, X)$ is just a finite union of spectra of some finite extensions of E , it naturally admits an integral canonical model over \mathcal{O}_φ by taking the corresponding valuation rings. With the map \mathfrak{Rec}_φ as defined in (2.7.1), we have $\nu(\sigma(x)) = \mathfrak{Rec}_\varphi(G, X)(\sigma) \cdot \nu(x)$ for any closed point $x \in \text{Sh}_K(G, X)_{\mathcal{O}_\varphi}$ and $\sigma \in \text{Gal}_{\mathbb{F}_\varphi}$.

Notation 2.9. We put $K_p^{\text{der}} = K_p \cap G^{\text{der}}(\mathbb{Q}_p)$. Let K_p^{ad} denote the image of K_p in $G^{\text{ad}}(\mathbb{Q}_p)$. Set $G^?(\mathbb{Q})^{(p)} = G^?(\mathbb{Q}) \cap K_p^?$ for $? = \emptyset, \text{ad}$ and der ; they are the subgroups of p -integral elements. Put $G^{\text{ad}}(\mathbb{Q})^{+, (p)} = G^{\text{ad}}(\mathbb{R})^+ \cap G^{\text{ad}}(\mathbb{Q})^{(p)}$ and $G^?(\mathbb{Q})_+^{(p)} = G^?(\mathbb{R})_+ \cap G^?(\mathbb{Q})^{(p)}$ for $? = \emptyset$ or der .

2.10 A group theoretic construction

Before proceeding, we recall a pure group theoretic construction. See [Del79, §2.0.1] for more details.

Let H be a group equipped with an action r of a group Δ and $\Gamma \subset H$ a Δ -stable subgroup. Suppose that we are given a Δ -equivariant map $\varphi : \Gamma \rightarrow \Delta$, where Δ acts on itself by inner automorphisms, and suppose that for $\gamma \in \Gamma$, $\varphi(\gamma)$ acts on H as inner conjugation by γ .

Given the data above, we can first define the semi-product $H \rtimes \Delta$ using the action r . The conditions above imply that the natural map $\gamma \mapsto (\gamma, \varphi(\gamma)^{-1})$ embeds Γ as a normal subgroup of $H \rtimes \Delta$. We define the *star extension* $H *_{\Gamma} \Delta$ to be the quotient of $H \rtimes \Delta$ by this subgroup.

Two typical examples we will encounter later are

$$G^{\text{der}}(\mathbb{A}^{\infty, p}) *_{G^{\text{der}}(\mathbb{Q})^{(p)}} G(\mathbb{Q})^{(p)} \cong G^{\text{der}}(\mathbb{A}^{\infty, p}) \cdot G(\mathbb{Q})^{(p)} \quad \text{and} \quad G^{\text{der}}(\mathbb{A}^{\infty, p}) *_{G^{\text{der}}(\mathbb{Q})^{(p)}} G^{\text{ad}}(\mathbb{Q})^{(p)}.$$

2.11 The connected components of the integral model

Let (G, X) be a (weak) Shimura datum. Suppose that there exists an integral canonical model $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi}$. For K^p an open compact subgroup of $G(\mathbb{A}^{\infty, p})$, let $\text{Sh}_{K^p K_p}(G, X)_{\mathcal{O}_\varphi}^\circ$ denote the open and closed subscheme whose \mathbb{C} -points consist of the preimage of $\{1\}$ under the

ν -map in (2.6.1). When (G, X) is a Shimura datum, this gives a connected component of $\text{Sh}_{K^p K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}}$. We put

$$\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}}^\circ = \varprojlim_{K^p} \text{Sh}_{K^p K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}}^\circ \quad \text{and} \quad \text{Sh}_{K_p}(G, X)_{\overline{\mathbb{F}}_\varphi}^\circ = \text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}}^\circ \otimes_{\mathcal{O}_\varphi^{\text{ur}}} \overline{\mathbb{F}}_\varphi. \tag{2.11.1}$$

Note that the set of \mathbb{C} -points of $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}}^\circ$ is nothing but $G^{\text{der}}(\mathbb{Q})_+^{(p), \text{cl}} \backslash (X^+ \times G^{\text{der}}(\mathbb{A}^{\infty, p}))$. When (G, X) is a Shimura datum, strong approximation shows that this is a projective limit of a connected complex manifold. In any case, this implies that $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}}^\circ$ depends only on X , the groups G^{der} and G^{ad} (as opposed to the group G), and the subgroups K_p^{der} and K_p^{ad} (as opposed to K_p).

We also point out that (2.6.1) gives rise to a natural map

$$\nu: \pi_0(\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}}) = \pi_0(\text{Sh}_{K_p}(G, X)_{\overline{\mathbb{F}}_\varphi}) = \pi_0(\text{Sh}_{K_p}(G, X)_{\overline{\mathbb{Q}}}) \longrightarrow T(\mathbb{Q})^{(p), \text{cl}} \backslash Y \times T(\mathbb{A}^{\infty, p}). \tag{2.11.2}$$

By abuse of language, we call (2.11.1) the geometric connected components of the Shimura varieties, and the target of (2.11.2) the set of connected components (although this is not the case if (G, X) is a weak Shimura datum).

The Shimura varieties $\text{Sh}_{K_p}(G, X)_?$ for $? = \mathcal{O}_\varphi^{\text{ur}}$ and $\overline{\mathbb{F}}_\varphi$ admit the following actions.

(i) The natural right action of $G(\mathbb{A}^{\infty, p})$ on $\text{Sh}_{K_p}(G, X)_{\overline{\mathbb{Q}}}$ extends to a right action on $\text{Sh}_{K_p}(G, X)_?$. The subgroup $Z(\mathbb{Q})^{(p)} := Z(\mathbb{Q}) \cap G(\mathbb{Z}_p)$ acts trivially. So, the right multiplication action above factors through $G(\mathbb{A}^{\infty, p})/Z(\mathbb{Q})^{(p), \text{cl}}$. The induced action on the set of connected components is given by $\nu: G(\mathbb{A}^{\infty, p})/Z(\mathbb{Q})^{(p), \text{cl}} \rightarrow T(\mathbb{Q})^{\dagger, (p), \text{cl}} \backslash T(\mathbb{A}^{\infty, p})$.

(ii) There is a right action ρ of $G^{\text{ad}}(\mathbb{Q})^{+, (p)}$ on $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}}$ such that the induced map on \mathbb{C} -points is given by, for $g \in G^{\text{ad}}(\mathbb{Q})^{+, (p)}$,

$$\begin{aligned} \rho(g): G(\mathbb{Q})_+^{\text{cl}} \backslash X^+ \times G(\mathbb{A}^\infty)/K_p &\longrightarrow G(\mathbb{Q})_+^{\text{cl}} \backslash X^+ \times G(\mathbb{A}^\infty)/K_p \\ [x, a] &\longmapsto [g^{-1}x, \text{int}_{g^{-1}}(a)]. \end{aligned}$$

Here note that K_p is stable under the conjugation action of K_p^{ad} and hence of $G^{\text{ad}}(\mathbb{Q})^{+, (p)}$. One extends the action $\rho(g)$ to the integral model and hence to the special fiber using the extension property. Moreover, this action preserves the connected component $\text{Sh}_{K_p}(G, X)_?^\circ$.

(iii) For an element $g \in G(\mathbb{Q})_+^{(p)}$, the two actions above coincide. Putting them together, we have a right action of the group

$$\mathcal{G} := (G(\mathbb{A}^{\infty, p})/Z(\mathbb{Q})^{(p), \text{cl}}) *_{G(\mathbb{Q})_+^{(p)}/Z(\mathbb{Q})^{(p)}} G^{\text{ad}}(\mathbb{Q})^{+, (p)} \tag{2.11.3}$$

on $\text{Sh}_{K_p}(G, X)_?^\circ$. The induced action on the set of connected components is given by

$$\nu * \text{triv} : \mathcal{G} \rightarrow T(\mathbb{Q})^{\dagger, (p), \text{cl}} \backslash T(\mathbb{A}^{\infty, p}),$$

i.e. ν on the first factor and trivial on the second factor.

(iv) The Galois group $\text{Gal}(E_\varphi^{\text{ur}}/E_\varphi)$ acts on $\text{Sh}_{K_p}(G)_?$, according to (2.7.1) (and §2.8).

Let $\mathcal{E}_{G, \varphi}$ denote the subgroup of $\mathcal{G} \times \text{Gal}(E_\varphi^{\text{ur}}/E_\varphi)$ consisting of pairs (g, σ) such that $(\nu * \text{triv})(g)$ is equal to $\mathfrak{Rec}_\varphi(\sigma)^{-1}$ in $T(\mathbb{Q})^{\dagger, (p), \text{cl}} \backslash T(\mathbb{A}^{\infty, p})$. Then, by the discussion above, the group $\mathcal{E}_{G, \varphi}$ acts on the connected component $\text{Sh}_{K_p}(G, X)_?^\circ$.

Conversely, knowing $\text{Sh}_{K_p}(G, X)^\circ_\mathbb{F}_\varphi$ together with the action of $\mathcal{E}_{G,\varphi}$, we can recover the integral model $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi}$ or its special fiber $\text{Sh}_{K_p}(G, X)_{\mathbb{F}_\varphi}$ of the Shimura variety as follows. We consider the (pro-)scheme $\text{Sh}_{K_p}(G, X)^\circ_{\mathcal{O}_\varphi^{\text{ur}}} \times_{\mathcal{E}_{G,\varphi}} (\mathcal{G} \times \text{Gal}(E_\varphi^{\text{ur}}/E_\varphi))$. Since this is a projective limit of *quasi-projective* varieties, by Galois descent, it is the base change of a projective system of varieties $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi}$ from \mathcal{O}_φ to $\mathcal{O}_\varphi^{\text{ur}}$. The same argument applies to the special fiber.

In general, for a finite unramified extension \tilde{E}_φ of E_φ , we put $\mathcal{E}_{G,\tilde{E}_\varphi}$ to be the subgroup of $\mathcal{E}_{G,\varphi}$ consisting of elements whose second coordinate lives in $\text{Gal}(E_\varphi^{\text{ur}}/\tilde{E}_\varphi)$. Knowing the action of $\mathcal{E}_{G,\tilde{E}_\varphi}$ on $\text{Sh}_{K_p}(G, X)^\circ_{\mathcal{O}_\varphi^{\text{ur}}}$ or $\text{Sh}_{K_p}(G, X)^\circ_{\mathbb{F}_\varphi}$ allows one to descend the integral model to $\text{Sh}_{K_p}(G, X)_{\mathcal{O}_{\tilde{E}_\varphi}}$.

2.12 Transferring mathematical objects

One can slightly generalize the discussion above to $\mathcal{E}_{G,\tilde{E}_\varphi}$ -equivariant mathematical objects over the Shimura variety. More precisely, for $? = \mathbb{F}_\varphi, \mathcal{O}_\varphi$, by a *mathematical object* \mathcal{P} over $\text{Sh}_{K^p}(G, X)_?$, we mean, for each sufficiently small open compact subgroup K^p of $G(\mathbb{A}^{\infty,p})$, we have a (pro-)scheme or a vector bundle (with a section) \mathcal{P}_{K^p} over $\text{Sh}_{K_p K^p}(G, X)_?$, such that, for any subgroup $K_1^p \subseteq K_2^p$, $\mathcal{P}_{K_1^p}$ is the base change of $\mathcal{P}_{K_2^p}$ along the natural morphism $\text{Sh}_{K_p K_1^p}(G, X)_? \rightarrow \text{Sh}_{K_p K_2^p}(G, X)_?$. We say that \mathcal{P} is $\mathcal{G} \times \text{Gal}(E_\varphi^{\text{ur}}/\tilde{E}_\varphi)$ -equivariant if \mathcal{P} carries an action of $\mathcal{G} \times \text{Gal}(E_\varphi^{\text{ur}}/\tilde{E}_\varphi)$ that is compatible with the actions on the Shimura varieties.

Similarly, a *mathematical object* \mathcal{P}° over $\text{Sh}_{K^p}(G, X)^\circ_{\mathbb{F}_\varphi}$ is a (pro-)scheme or a vector bundle (with a section) as above, over the connected Shimura variety $\text{Sh}_{K^p}(G, X)^\circ_{\mathbb{F}_\varphi}$, viewed as a pro-scheme. It is called $\mathcal{E}_{G,\tilde{E}_\varphi}$ -equivariant if it carries an action of the group compatible with the natural group action on the base Shimura variety.

Similar to the discussion above, we have the following corollary.

COROLLARY 2.13. *There is a natural equivalence of categories between the category of $\mathcal{G} \times \text{Gal}(E_\varphi^{\text{ur}}/\tilde{E}_\varphi)$ -equivariant mathematical objects \mathcal{P} over the tower of Shimura varieties $\text{Sh}_{K_p K^p}(G, X)_?$, and the category of mathematical objects \mathcal{P}° over $\text{Sh}_{K^p}(G, X)^\circ_{\mathbb{F}_\varphi}$, equivariant for the action of $\mathcal{E}_{G,\tilde{E}_\varphi}$.*

Proof. As above, given \mathcal{P} , we can recover \mathcal{P}° by taking the inverse limit with respect to the open compact subgroup K^p and then restricting to the connected component $\text{Sh}_{K_p}(G, X)^\circ_{\mathbb{F}_\varphi}$. Conversely, we can recover \mathcal{P} from \mathcal{P}° through the isomorphism $\mathcal{P}_{\mathbb{F}_\varphi} \cong \mathcal{P}^\circ \times_{\mathcal{E}_{G,\tilde{E}_\varphi}} (\mathcal{G} \times \text{Gal}(E_\varphi^{\text{ur}}/\tilde{E}_\varphi))$ and then use Galois descent if needed. □

Remark 2.14. If one does not consider the Galois action, Theorem 2.16 below implies that

$$\text{Sh}_{K_p}(G, X)_{\mathcal{O}_\varphi^{\text{ur}}} \cong \text{Sh}_{K_p}(G, X)^\circ_{\mathcal{O}_\varphi} \times_{(G^{\text{der}}(\mathbb{A}^{\infty,p})^*_{G^{\text{der}}(\mathbb{Q})_+^{(p)}})^{G^{\text{ad}}(\mathbb{Q})_+^{(p)}}} \mathcal{G},$$

and the same applies to the mathematical objects.

LEMMA 2.15. *We have $\nu(G(\mathbb{Q})_+^{(p)}) = T(\mathbb{Q})^\dagger_{(p)}$.*

Proof. By § 2.6, we have $\nu(G(\mathbb{Q})_+) = T(\mathbb{Q})^\dagger$. The lemma follows from taking the kernels of the following morphism of exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & G^{\text{der}}(\mathbb{Q})_+ & \longrightarrow & G(\mathbb{Q})_+ & \longrightarrow & T(\mathbb{Q})^\dagger \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G^{\text{der}}(\mathbb{Q}_p)/K_p^{\text{der}} & \longrightarrow & G(\mathbb{Q}_p)/K_p & \longrightarrow & T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \longrightarrow 1 \end{array}$$

Here the left vertical arrow is surjective by the strong approximation theorem for the simply connected group $G^{\text{der}}(\mathbb{Q})$. The bottom sequence is exact because the corresponding sequences are exact both for \mathbb{Q}_p (because $H^1(\mathbb{Q}_p, G^{\text{der}}) = 0$) and for \mathbb{Z}_p (by Hypothesis 2.5). \square

The following structure theorem for $\mathcal{E}_{G,\varphi}$ is the key to transfer integral canonical models of Shimura varieties for one group to those for another group.

THEOREM 2.16. *For a finite unramified extension \tilde{E}_φ of E_φ , we have a natural short exact sequence*

$$1 \longrightarrow G^{\text{der}}(\mathbb{A}^{\infty,p}) *_{G^{\text{der}}(\mathbb{Q})_+^{(p)}} G^{\text{ad}}(\mathbb{Q})^{+,(p)} \longrightarrow \mathcal{E}_{G,\tilde{E}_\varphi} \longrightarrow \text{Gal}(E_\varphi^{\text{ur}}/\tilde{E}_\varphi) \longrightarrow 1. \tag{2.16.1}$$

Proof. By the definition of $\mathcal{E}_{G,\tilde{E}_\varphi}$, it fits into the following short exact sequence:

$$1 \longrightarrow \text{Ker}(\tilde{\mathcal{G}} \rightarrow T(\mathbb{Q})^{\dagger,(p),\text{cl}} \setminus T(\mathbb{A}^{\infty,p})) \longrightarrow \mathcal{E}_{G,\tilde{E}_\varphi} \longrightarrow \text{Gal}(E_\varphi^{\text{ur}}/\tilde{E}_\varphi) \longrightarrow 1.$$

By Lemma 2.15, the kernel above is isomorphic to

$$((G(\mathbb{Q})_+^{(p)} G^{\text{der}}(\mathbb{A}^{\infty,p}))^{\text{cl}} / Z(\mathbb{Q})^{(p),\text{cl}}) *_{G(\mathbb{Q})_+^{(p)} / Z(\mathbb{Q})^{(p)}} G^{\text{ad}}(\mathbb{Q})^{+,(p)}, \tag{2.16.2}$$

where both closures are taken inside $G(\mathbb{A}^{\infty,p})$.

We claim that we can remove the two completions. Indeed, put $Z' = Z \cap G^{\text{der}}$ and $Z'(\mathbb{Q})^{(p)} = Z'(\mathbb{Q}) \cap Z(\mathbb{Q})^{(p)}$; the latter is a finite group. Consider the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & Z'(\mathbb{Q})^{(p)} & \longrightarrow & Z(\mathbb{Q})^{(p)} \times G^{\text{der}}(\mathbb{A}^{\infty,p}) & \longrightarrow & G(\mathbb{Q})_+^{(p)} G^{\text{der}}(\mathbb{A}^{\infty,p}) & \longrightarrow & T(\mathbb{Q})^{\dagger,(p)} / \text{Im}(Z(\mathbb{Q})^{(p)}) & \longrightarrow & T(\mathbb{Q})^{(p)} & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Z'(\mathbb{Q})^{(p),\text{cl}} & \longrightarrow & Z(\mathbb{Q})^{(p),\text{cl}} \times G^{\text{der}}(\mathbb{A}^{\infty,p}) & \longrightarrow & (G(\mathbb{Q})_+^{(p)} G^{\text{der}}(\mathbb{A}^{\infty,p}))^{\text{cl}} & \longrightarrow & T(\mathbb{Q})^{\dagger,(p),\text{cl}} / \text{Im}(Z(\mathbb{Q})^{(p),\text{cl}}) & \longrightarrow & T(\mathbb{Q})^{(p),\text{cl}} & \longrightarrow & 1 \end{array}$$

By diagram chasing, it suffices to prove that the right vertical arrow is an isomorphism. Since the kernel of $Z \rightarrow T$ is finite, [Del79, § 2.0.10] implies that $\text{Im}(Z(\mathbb{Q})^{\text{cl}} \rightarrow T(\mathbb{Q})^{\text{cl}}) \cong (\text{Im}(Z(\mathbb{Q}) \rightarrow T(\mathbb{Q})))^{\text{cl}}$ and the right vertical arrow is an isomorphism.

Now the exact sequence (2.16.1) follows from a series of tautological isomorphisms

$$\begin{aligned} & (G(\mathbb{Q})_+^{(p)} \cdot G^{\text{der}}(\mathbb{A}^{\infty,p}) / Z(\mathbb{Q})^{(p)}) *_{G(\mathbb{Q})_+^{(p)} / Z(\mathbb{Q})^{(p)}} G^{\text{ad}}(\mathbb{Q})^{+,(p)} \\ & \cong \left[(G^{\text{der}}(\mathbb{A}^{\infty,p}) *_{G^{\text{der}}(\mathbb{Q})_+^{(p)}} G(\mathbb{Q})_+^{(p)}) / Z(\mathbb{Q})^{(p)} \right] *_{G(\mathbb{Q})_+^{(p)} / Z(\mathbb{Q})^{(p)}} G^{\text{ad}}(\mathbb{Q})^{+,(p)} \\ & \cong \left[G^{\text{der}}(\mathbb{A}^{\infty,p}) *_{G^{\text{der}}(\mathbb{Q})_+^{(p)}} (G(\mathbb{Q})_+^{(p)} / Z(\mathbb{Q})^{(p)}) \right] *_{G(\mathbb{Q})_+^{(p)} / Z(\mathbb{Q})^{(p)}} G^{\text{ad}}(\mathbb{Q})^{+,(p)} \\ & \cong G^{\text{der}}(\mathbb{A}^{\infty,p}) *_{G^{\text{der}}(\mathbb{Q})_+^{(p)}} G^{\text{ad}}(\mathbb{Q})^{+,(p)}. \end{aligned} \quad \square$$

COROLLARY 2.17. *Let $\varphi : G \rightarrow G'$ be a homomorphism of two reductive groups over \mathbb{Q} satisfying Hypothesis 2.5, which induces isomorphisms between the derived and adjoint groups as well as isomorphisms $G^?(\mathbb{Q})^{(p)} \cong G'^?(\mathbb{Q})^{(p)}$ for $? = \text{der}, \text{ad}$. A $G^{\text{ad}}(\mathbb{R})^+$ -conjugacy class X^+ of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ induces a $G'^{\text{ad}}(\mathbb{R})^+$ -conjugacy class X'^+ of homomorphisms $h' : \mathbb{S} \rightarrow G'_{\mathbb{R}}$. Put $X = G(\mathbb{R}) \cdot X^+$ and $X' = G'(\mathbb{R}) \cdot X'^+$. Then, for any field \tilde{E}_φ containing both E_φ and E'_φ and unramified over them, there exist a natural isomorphism of groups*

$\mathcal{E}_{G, \tilde{E}_\varphi} \xrightarrow{\cong} \mathcal{E}_{G', \tilde{E}_\varphi}$ and a natural isomorphism of geometric connected components of Shimura varieties $\mathrm{Sh}_{K_p}(G, X)^\circ_{\tilde{E}_\varphi^{\mathrm{ur}}} \cong \mathrm{Sh}_{K'_p}(G', X')^\circ_{\tilde{E}_\varphi^{\mathrm{ur}}}$, equivariant for the natural actions of the groups

$$\mathcal{E}_{G, \tilde{E}_\varphi} \xrightarrow{\cong} \mathcal{E}_{G', \tilde{E}_\varphi}.$$

As a corollary, if the Shimura variety for one of G or G' admits an integral canonical model and both E_φ and $E'_{\varphi'}$ are unramified extensions of \mathbb{Q}_p , then the other Shimura variety admits an integral canonical model.

Moreover, when there are canonical integral models, we have an equivalence of categories between the category of $\mathcal{G} \times \mathrm{Gal}(E_\varphi^{\mathrm{ur}}/\tilde{E}_\varphi)$ -equivariant mathematical objects \mathcal{P} over the tower of Shimura varieties $\mathrm{Sh}_{K_p K^p}(G, X)_?$ (for $? = \mathcal{O}_\varphi$ or \mathbb{F}_φ) and the category of $\mathcal{G}' \times \mathrm{Gal}(E_\varphi^{\mathrm{ur}}/\tilde{E}_\varphi)$ -equivariant mathematical objects \mathcal{P}' over the tower of Shimura varieties $\mathrm{Sh}_{K'_p K'^p}(G', X')_{?}$ (for $? = \mathcal{O}_{\varphi'}$ or $\mathbb{F}_{\varphi'}$).

Proof. The first part follows from Theorem 2.16 and the discussion in § 2.11. For the second part, the existence of an integral canonical model over \tilde{E}_φ follows from the first part and the discussion at the end of § 2.11. The extension property allows one to further descend the integral canonical model to \mathcal{O}_φ (or $\mathcal{O}_{\varphi'}$). The last part follows from Corollary 2.13. \square

3. Integral canonical models of quaternionic Shimura varieties

Classically, the integral model for a quaternionic Shimura variety is defined by passing to a unitary Shimura variety, as is done in the curve case by Carayol [Car86]. As we pointed out earlier that we will encounter some groups which are not quasi-split at p , Kisin’s general work [Kis10] unfortunately does not apply. We have to work out a generalization of Carayol’s construction for completeness. This will also be useful later when discussing the construction of the Goren–Oort stratification.

We tailor the choice of the unitary group to our application of Helm’s isogeny trick later. In particular, we will assume certain places above p to be inert in the CM extension.

3.1 Quaternionic Shimura varieties

Recall the notation from § 1.12.3. Let \mathbf{S} be an even subset of places of F . Put $\mathbf{S}_\infty = \mathbf{S} \cap \Sigma_\infty$ and $\mathbf{S}_p = \mathbf{S} \cap \Sigma_p$. Let $B_\mathbf{S}$ be the quaternion algebra over F ramified precisely at \mathbf{S} . Let $G_\mathbf{S}$ denote the reductive group $\mathrm{Res}_{F/\mathbb{Q}}(B_\mathbf{S}^\times)$. Then $G_{\mathbf{S}, \mathbb{R}}$ is isomorphic to

$$\prod_{\tau \in \mathbf{S}_\infty} \mathbb{H}^\times \times \prod_{\tau \in \Sigma_\infty - \mathbf{S}_\infty} \mathrm{GL}_{2, \mathbb{R}}.$$

We define the Deligne homomorphism to be $h_\mathbf{S} : \mathbb{S} \rightarrow G_{\mathbf{S}, \mathbb{R}}$, sending $z = x + \mathbf{i}y$ to $(z_{G_\mathbf{S}}^\tau)_{\tau \in \Sigma_\infty}$, where $z_{G_\mathbf{S}}^\tau = 1$ if $\tau \in \mathbf{S}_\infty$ and $z_{G_\mathbf{S}}^\tau = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ if $\tau \in \Sigma_\infty - \mathbf{S}_\infty$. Let $\mathfrak{H}_\mathbf{S}$ denote the $G_\mathbf{S}(\mathbb{R})$ -conjugacy class of the homomorphism $h_\mathbf{S}$; it is isomorphic to the product of $\#(\Sigma_\infty - \mathbf{S}_\infty)$ copies of $\mathfrak{h}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. We put $\mathfrak{H}_\mathbf{S}^+ = (\mathfrak{h}^+)_{\Sigma_\infty - \mathbf{S}_\infty}$, where \mathfrak{h}^+ denotes the upper half plane.

We will consider the following type of open compact subgroups of $G_\mathbf{S}(\mathbb{A}^\infty)$: $K = K^p K_p$, where K^p is an open compact subgroup of $B_\mathbf{S}^\times(\mathbb{A}_F^{\infty, p})$ and $K_p = \prod_{\mathfrak{p} \in \Sigma_p} K_\mathfrak{p}$ with $K_\mathfrak{p}$ an open compact subgroup of $B_\mathbf{S}^\times(F_\mathfrak{p})$.

From this point onward, we write $\mathrm{Sh}_K(G)$ instead of $\mathrm{Sh}_K(G, X)$ for Shimura varieties when the choice of X is clear. Associated to the data above, there is a Shimura variety $\mathrm{Sh}_K(G_\mathbf{S})$ whose \mathbb{C} -points are

$$\mathrm{Sh}_K(G_\mathbf{S})(\mathbb{C}) = G_\mathbf{S}(\mathbb{Q}) \backslash (\mathfrak{H}_\mathbf{S} \times G_\mathbf{S}(\mathbb{A}^\infty)) / K.$$

The reflex field F_S is a subfield of \mathbb{C} characterized as follows: an element $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ fixes F_S if and only if the subset S_∞ of Σ_∞ is preserved under the action of σ by post-composition. Following § 2.2, we put $\text{Sh}_{K_p}(G_S) = \varprojlim_{K^p} \text{Sh}_{K^p K_p}(G_S)$. (Note that the level structure at p is fixed in the inverse limit.)

Put $T_F = \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$. The reduced norm on B_S induces a homomorphism $\text{Nm} = \text{Nm}_{B_S/F} : G_S \rightarrow T_F$. This homomorphism induces a map

$$\pi_0^{\text{geom}}(\text{Sh}_K(G_S)) \longrightarrow T_F(\mathbb{Q}) \backslash (T_F(\mathbb{A}^\infty) \times \{\pm 1\}^g) / \text{Nm}(K),$$

which is an isomorphism if $S_\infty \subsetneq \Sigma_\infty$. We will make the Shimura reciprocity law (§ 2.7) explicit for $\text{Sh}_K(G_S)$ later when it is in use.

3.2 Level structure at p

We fix an isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$. For each $\mathfrak{p} \in \Sigma_p$, let $\Sigma_{\infty/\mathfrak{p}}$ denote the subset of Σ_∞ consisting of real embeddings, which, when composed with ι_p , induce the p -adic place \mathfrak{p} . We put $S_{\infty/\mathfrak{p}} = S \cap \Sigma_{\infty/\mathfrak{p}}$. Similarly, we can view the reflexive field F_S as a subfield of $\overline{\mathbb{Q}}_p$ via ι_p , which induces a p -adic place \wp of F_S . We use \mathcal{O}_\wp to denote the valuation ring and k_\wp the residue field.

In this paper, we always make the following assumption on S .

HYPOTHESIS 3.3. If B_S does not split at a p -adic place \mathfrak{p} of F , then $S_{\infty/\mathfrak{p}} = \Sigma_{\infty/\mathfrak{p}}$.

For each $\mathfrak{p} \in \Sigma_p$, we now specify the level structure $K_p \subset B_S^\times(F_p)$ of $\text{Sh}_K(G_S)$ to be considered in this paper. We distinguish four types of the prime $\mathfrak{p} \in \Sigma_p$.

- *Types α and α^\sharp :* B_S splits at \mathfrak{p} and the cardinality $\#(\Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}})$ is even. We fix an identification $B_S^\times(F_p) \simeq \text{GL}_2(F_p)$. We take K_p to be:
 - * either $\text{GL}_2(\mathcal{O}_\mathfrak{p})$; or
 - * $\text{Iw}_\mathfrak{p} = \begin{pmatrix} \mathcal{O}_\mathfrak{p}^\times & \mathcal{O}_\mathfrak{p} \\ \mathfrak{p}\mathcal{O}_\mathfrak{p} & \mathcal{O}_\mathfrak{p}^\times \end{pmatrix}$, which we allow only when $\Sigma_{\infty/\mathfrak{p}} = S_{\infty/\mathfrak{p}}$.

We name the former case as type α and the latter as type α^\sharp . (Under our definition, when $\Sigma_{\infty/\mathfrak{p}} = S_{\infty/\mathfrak{p}}$, the type of \mathfrak{p} depends on the choice of the level structure.)

- *Type β :* B_S splits at \mathfrak{p} and the cardinality $\#(\Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}})$ is odd. We fix an identification $B_S^\times(F_p) \simeq \text{GL}_2(F_p)$. We take K_p to be $\text{GL}_2(\mathcal{O}_\mathfrak{p})$.
- *Type β^\sharp :* B_S ramifies at \mathfrak{p} and $S_{\infty/\mathfrak{p}} = \Sigma_{\infty/\mathfrak{p}}$. In this case, $B_S \otimes_F F_p$ is the division quaternion algebra B_{F_p} over F_p . Let $\mathcal{O}_{B_{F_p}}$ be the maximal order of B_{F_p} . We take K_p to be $\mathcal{O}_{B_{F_p}}^\times$.

The aim of this section is to construct an integral canonical model of $\text{Sh}_K(G_S)$ over \mathcal{O}_\wp with $K_p = \prod_{\mathfrak{p}|p} K_p$ specified above. For this, we need to introduce an auxiliary CM extension and a unitary group.

3.4 Auxiliary CM extension

We choose a CM extension E over F such that:

- every place in S is inert in E/F ; and
- a place $\mathfrak{p} \in \Sigma_p$ is split in E/F if it is of type α or α^\sharp , and is inert in E/F if it is of type β or β^\sharp .

We remark that our construction slightly differs from [Car86] in that Carayol requires all places above p to split in E/F . For later convenience, we fix some totally negative element $\mathfrak{d} \in \mathcal{O}_F$ coprime to p , so that $E = F(\sqrt{\mathfrak{d}})$. (The construction will be independent of such a choice.)

Let $\Sigma_{E,\infty}$ denote the set of complex embeddings of E . We have a natural two-to-one map $\Sigma_{E,\infty} \rightarrow \Sigma_\infty$. For each $\tau \in \Sigma_\infty$, we often use $\tilde{\tau}$ to denote a complex embedding of E extending τ , whose complex conjugate is denoted by $\tilde{\tau}^c$.

We fix a choice of a subset $\tilde{S}_\infty \subseteq \Sigma_{E,\infty}$ which consists of, for each $\tau \in S_\infty$, a choice of exactly one lift $\tilde{\tau} \in \Sigma_{E,\infty}$. This choice is equivalent to a collection of the numbers $s_{\tilde{\tau}} \in \{0, 1, 2\}$ for all $\tilde{\tau} \in \Sigma_{E,\infty}$ such that:

- if $\tau \in \Sigma_\infty - S_\infty$, we have $s_{\tilde{\tau}} = 1$ for all lifts $\tilde{\tau}$ of τ ;
- if $\tau \in S_\infty$ and $\tilde{\tau}$ is the lift in \tilde{S}_∞ , we have $s_{\tilde{\tau}} = 0$ and $s_{\tilde{\tau}^c} = 2$.

We put $\tilde{S} = (S, \tilde{S}_\infty)$. Consider the torus $T_{E,\tilde{S}} = \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ together with the following choice of the Deligne homomorphism:

$$h_{E,\tilde{S}}: \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow T_{E,\tilde{S}}(\mathbb{R}) = \bigoplus_{\tau \in \Sigma_\infty} (E \otimes_{F,\tau} \mathbb{R})^\times \simeq \bigoplus_{\tau \in \Sigma_\infty} \mathbb{C}^\times$$

$$z \longmapsto (\bar{z}_{E,\tau})_\tau.$$

Here $\bar{z}_{E,\tau} = 1$ if $\tau \in \Sigma_\infty - S_\infty$ and $\bar{z}_{E,\tau} = \bar{z}$ otherwise, where, in the latter case, the isomorphism $(E \otimes_{F,\tau} \mathbb{R})^\times \simeq \mathbb{C}^\times$ is given by the lift $\tilde{\tau} \in \tilde{S}_\infty$. The reflex field $E_{\tilde{S}}$ is the subfield of \mathbb{C} corresponding to the subgroup of $\text{Aut}(\mathbb{C}/\mathbb{Q})$ which stabilizes the set $\tilde{S}_\infty \subset \Sigma_{E,\infty}$. It contains $F_{\tilde{S}}$ as a subfield. The isomorphism $\iota_p: \mathbb{C} \simeq \overline{\mathbb{Q}}_p$ determines a p -adic place $\tilde{\varphi}$ of $E_{\tilde{S}}$. We use $\mathcal{O}_{\tilde{\varphi}}$ to denote the valuation ring of the completion of $E_{\tilde{S}}$ at $\tilde{\varphi}$ and $k_{\tilde{\varphi}}$ its residue field. Note that $[k_{\tilde{\varphi}}: \mathbb{F}_p]$ is always even whenever there is a place $\mathfrak{p} \in \Sigma_p$ of type β .

We take the level structure K_E to be $K_E^p K_{E,p}$, where $K_{E,p} = (\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ and K_E^p is an open compact subgroup of $\mathbb{A}_E^{\infty,p,\times}$. This gives rise to a Shimura variety $\text{Sh}_{K_E}(T_{E,\tilde{S}})$ and its limit $\text{Sh}_{K_{E,p}}(T_{E,\tilde{S}}) = \varprojlim_{K_E^p} \text{Sh}_{K_{E,p} K_E^p}(T_{E,\tilde{S}})$. They have integral canonical models $\mathbf{Sh}_{K_E}(T_{E,\tilde{S}})$ and $\mathbf{Sh}_{K_{E,p}}(T_{E,\tilde{S}})$ over $\mathcal{O}_{\tilde{\varphi}}$, as specified in § 2.8.

We also consider the product group $G_S \times T_{E,\tilde{S}}$ with the product Deligne homomorphism

$$\tilde{h}_{\tilde{S}} = h_S \times h_{E,\tilde{S}}: \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow (G_S \times T_{E,\tilde{S}})(\mathbb{R}).$$

This gives rise to the product Shimura varieties:

$$\text{Sh}_{K \times K_E}(G_S \times T_{E,\tilde{S}}) = \text{Sh}_K(G_S) \times_{F_{\tilde{S},\varphi}} \text{Sh}_{K_E}(T_{E,\tilde{S}}),$$

$$\text{Sh}_{K_p \times K_{E,p}}(G_S \times T_{E,\tilde{S}}) = \text{Sh}_{K_p}(G_S) \times_{F_{\tilde{S},\varphi}} \text{Sh}_{K_{E,p}}(T_{E,\tilde{S}}).$$

Let $Z = \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ denote the center of G_S . Put $G''_{\tilde{S}} = G_S \times_Z T_{E,\tilde{S}}$, which is the quotient of $G_S \times T_{E,\tilde{S}}$ by Z embedded anti-diagonally as $z \mapsto (z, z^{-1})$. The corresponding Deligne homomorphism $h''_{\tilde{S}}: \mathbb{S}(\mathbb{R}) \rightarrow G''_{\tilde{S}}(\mathbb{R})$ is the one induced by $\tilde{h}_{\tilde{S}}$. We will consider open compact subgroups $K'' \subseteq G''_{\tilde{S}}(\mathbb{A}^\infty)$ of the form $K''^p K''_p$, where K''^p is an open compact subgroup of $G''_{\tilde{S}}(\mathbb{A}^{\infty,p})$ and K''_p is an open compact subgroup of $G''_{\tilde{S}}(\mathbb{Q}_p)$. Finally, the $G''_{\tilde{S}}(\mathbb{R})$ -conjugacy class of $h''_{\tilde{S}}$ can be canonically identified with \mathfrak{H}_S . We then get the Shimura variety $\text{Sh}_{K''}(G''_{\tilde{S}})$ and its limit $\text{Sh}_{K''_p}(G''_{\tilde{S}})$ over the reflex field $E_{\tilde{S}}$. The set of \mathbb{C} -points of $\text{Sh}_{K''}(G''_{\tilde{S}})$ is

$$\text{Sh}_{K''}(G''_{\tilde{S}})(\mathbb{C}) = G''_{\tilde{S}}(\mathbb{Q}) \backslash (\mathfrak{H}_S \times G''_{\tilde{S}}(\mathbb{A}^\infty)) / K''.$$

3.5 Unitary Shimura varieties

We now introduce the unitary group. Consider the morphism

$$\nu = \text{Nm}_{B/F} \times \text{Nm}_{E/F}: \quad G''_{\tilde{S}} = G_S \times_Z T_E \longrightarrow T = \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$$

$$(g, z) \longmapsto \text{Nm}(g)z\bar{z}.$$

Viewing \mathbb{G}_m naturally as a subgroup of $T = \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$, we define G'_S to be the reductive group $\nu^{-1}(\mathbb{G}_m)$. This will be our auxiliary unitary group, whose associated Shimura variety will

provide $\text{Sh}_K(G_S)$ with an integral canonical model. We will occasionally use the algebraic group $G'_{S,1} = \text{Ker } \nu$, but we view it as a reductive group over F .

Note that the Deligne homomorphism $h''_S : \mathbb{S}(\mathbb{R}) \rightarrow G''_S(\mathbb{R})$ factors through a homomorphism $h'_S : \mathbb{S}(\mathbb{R}) \rightarrow G'_S(\mathbb{R})$. The $G'_S(\mathbb{R})$ -conjugacy class of h'_S can be canonically identified with \mathfrak{H}_S .

We will consider open compact subgroups of $G'_S(\mathbb{A}^\infty)$ of the form $K' = K'_p K'^p$, where K'_p is an open compact subgroup of $G'_S(\mathbb{Q}_p)$ (to be specified later in § 3.9) and K'^p is an open compact subgroup of $G'_S(\mathbb{A}^{\infty,p})$. We will always take K'^p to be sufficiently small so that K' is *neat* and hence the moduli problem we encounter later would be representable by a fine moduli space. Given the data above, we have a Shimura variety $\text{Sh}_{K'}(G'_S)$ whose \mathbb{C} -points are given by

$$\text{Sh}_{K'}(G'_S)(\mathbb{C}) = G'_S(\mathbb{Q}) \backslash (\mathfrak{H}_S \times G'_S(\mathbb{A}^\infty)) / K'.$$

The Shimura variety $\text{Sh}_{K'}(G'_S)$ is defined over the reflex field E_S . We put $\text{Sh}_{K'_p}(G'_S) = \varprojlim_{K'^p} \text{Sh}_{K'_p K'^p}(G'_S)$.

The upshot is the following lemma, which verifies the conditions listed in Corollary 2.17. This allows us to bring the integral canonical models of the unitary Shimura varieties to those of the quaternionic Shimura varieties.

LEMMA 3.6. *The natural diagram of morphisms of groups*

$$G_S \leftarrow G_S \times T_{E,\bar{s}} \rightarrow G''_S = G_S \times_Z T_{E,\bar{s}} \leftarrow G'_S \tag{3.6.1}$$

- (1) *is compatible with the Deligne homomorphisms; and*
- (2) *induces isomorphisms on their associated derived and adjoint groups.*

Proof. This is straightforward. □

3.7 PEL Shimura data

We put $D_S = B_S \otimes_F E$. It is isomorphic to $M_2(E)$ under Hypothesis 3.3. This is a quaternion algebra over E equipped with an involution $l \rightarrow \bar{l}$ given by the tensor product of the natural involution on B_S and the complex conjugation on E . Let D_S^{sym} denote the subsets of *symmetric* elements, i.e. those elements $\delta \in D_S$ such that $\delta = \bar{\delta}$. For any element $\delta \in (D_S^{\text{sym}})^\times$, we can define a new involution on D_S given by $l \mapsto l^* = \delta^{-1} \bar{l} \delta$. In the following Lemma 3.8, we will specify a convenient choice of such a δ .

Let V be the underlying \mathbb{Q} -vector space of D_S with the natural left D_S -module structure. Define a pairing $\psi_E : V \times V \rightarrow E$ on V by

$$\psi_E(v, w) = \text{Tr}_{D_S/E}(\sqrt{\delta} \cdot v \delta w^*), \quad v, w \in V. \tag{3.7.1}$$

It is easy to check that ψ_E is skew-hermitian over E for $*$, i.e. $\overline{\psi_E(v, w)} = -\psi_E(w, v)$ and $\psi_E(lv, w) = \psi_E(v, l^*w)$ for $l \in D_S$ and $v, w \in V$. We define the bilinear form

$$\psi = \text{Tr}_{E/\mathbb{Q}} \circ \psi_E : V \times V \rightarrow \mathbb{Q},$$

which is skew-symmetric and hermitian for $*$. One checks easily that the subgroup consisting of elements $l \in D_S^\times$ satisfying $\psi(vl, wl) = c(l)\psi(v, w)$ for some $c(l) \in \mathbb{Q}^\times$ is exactly the subgroup $G'_S \subset D_S^\times$. We make the above right action of G'_S on V into a left action by taking the transpose action. This is different from the convention used in [Car86], which used the inverse action, because taking the transpose action is naturally compatible with the setup of Hilbert modular varieties (see § 3.24), and is also compatible with our earlier choice of Deligne homomorphism.

The group G'_S is identified with the D_S -linear unitary group of V with similitudes in \mathbb{Q}^\times , i.e. for each \mathbb{Q} -algebra R , we have

$$G'_S(R) = \{g \in \text{End}_{D_S \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) = D_S^{\text{op}} \mid \psi(v {}^t g, w {}^t g) = c(g)\psi(v, w) \text{ with } c(g) \in R^\times\}. \quad (3.7.2)$$

We describe $D_{S,p} = D_S \otimes_F F_p$ by distinguishing three cases according to the types of $\mathfrak{p} \in \Sigma_p$ in § 3.2.

- *Types α or α^\sharp* : In this case, the place \mathfrak{p} splits into two primes \mathfrak{q} and $\bar{\mathfrak{q}}$ in E . We have natural isomorphisms $F_p \cong E_{\mathfrak{q}} \cong E_{\bar{\mathfrak{q}}}$. We fix an isomorphism $B_S \otimes_F F_p \simeq M_2(F_p)$ as above; then $D_{S,p} \simeq M_2(E_{\mathfrak{q}}) \oplus M_2(E_{\bar{\mathfrak{q}}})$. Under these identifications, we put $\mathcal{O}_{B_{S,p}} = M_2(\mathcal{O}_p)$ and $\mathcal{O}_{D_{S,p}} = M_2(\mathcal{O}_{\mathfrak{q}}) \oplus M_2(\mathcal{O}_{\bar{\mathfrak{q}}})$.
- *Type β* : In this case, the place \mathfrak{p} is inert in E/F and we let \mathfrak{q} denote the unique place in E above \mathfrak{p} . Using the fixed isomorphism $B_S \otimes_F F_p \simeq M_2(F_p)$, we have $D_{S,p} \simeq M_2(E_{\mathfrak{q}})$. We put $\mathcal{O}_{B_{S,p}} = M_2(\mathcal{O}_p)$ and $\mathcal{O}_{D_{S,p}} = M_2(\mathcal{O}_{\mathfrak{q}})$.
- *Type β^\sharp* : Let \mathfrak{q} be the unique place in E above \mathfrak{p} . The division quaternion algebra $B_{F_p} = B_S \otimes_F F_p$ over F_p is generated by an element $\varpi_{B_{F_p}}$ over $E_{\mathfrak{q}}$, with the relations $\varpi_{B_{F_p}}^2 = p$ and $\varpi_{B_{F_p}} a = \bar{a} \varpi_{B_{F_p}}$ for $a \in E_{\mathfrak{q}}$. We identify $B_{F_p} \otimes_{F_p} E_{\mathfrak{q}}$ with $M_2(E_{\mathfrak{q}})$ via the map

$$(a + b\varpi_{B_{F_p}}) \otimes c \longmapsto \begin{pmatrix} ac & bc \\ p\bar{b}c & \bar{a}c \end{pmatrix}. \quad (3.7.3)$$

This also identifies $D_{S,p}$ with $M_2(E_{\mathfrak{q}})$. We put $\mathcal{O}_{B_{S,p}} = \mathcal{O}_{B_{F_p}}$ and take $\mathcal{O}_{D_{S,p}}$ to be the preimage of $M_2(\mathcal{O}_{\mathfrak{q}})$ in $D_S \otimes_F F_p$.

We put $\mathcal{O}_{D_{S,p}} = \prod_{\mathfrak{p} \in \Sigma_p} \mathcal{O}_{D_{S,p}}$.

LEMMA 3.8. (1) We can choose the symmetric element $\delta \in (D_S^{\text{sym}})^\times$ above such that:

- (a) $\delta \in \mathcal{O}_{D_{S,p}}^\times$ for each $\mathfrak{p} \in \Sigma_p$ not of type β^\sharp , and $\delta \in \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathcal{O}_{D_{S,p}}^\times$ for each $\mathfrak{p} \in \Sigma_p$ of type β^\sharp ; and
- (b) the following (symmetric) bilinear form on $V_{\mathbb{R}}$ is positive definite:

$$(v, w) \mapsto \psi(v, w \cdot h'_S(\mathbf{i})).$$

- (2) Through the homomorphism $h'_S : \mathbb{S}(\mathbb{R}) \rightarrow G'_S(\mathbb{R})$, $h'_S(\mathbf{i})$ acts on the vector space $V_{\mathbb{R}}$ and gives it a Hodge structure of type $\{(-1, 0), (0, -1)\}$. For $l \in D_S$, we have

$$\text{tr}(l; V_{\mathbb{C}}/F^0 V_{\mathbb{C}}) = \left(\sum_{\tilde{\tau} \in \Sigma_{E,\infty}} s_{\tilde{\tau}} \tilde{\tau} \right) (\text{Tr}_{D_S/E}(l)).$$

The reflex field $E_{\bar{S}}$ is the subfield of \mathbb{C} generated by these traces for all $l \in D_S$.

- (3) With the choice of δ in (1), the group $G'_{S,1}$ is unramified at $\mathfrak{p} \in \Sigma_p$ not of type β^\sharp , and it is non-quasi-split at $\mathfrak{p} \in \Sigma_p$ of type β^\sharp . Moreover, $\mathcal{O}_{D_{S,p}}$ is a maximal $*$ -invariant lattice of $D_S(\mathbb{Q}_p)$ (up to scaling).

Proof. (1) Since F is dense in $F \otimes_{\mathbb{Q}} \mathbb{Q}_p \oplus F \otimes_{\mathbb{Q}} \mathbb{R}$, the symmetric elements of V are dense in the symmetric elements of $V \otimes_{\mathbb{Q}} \mathbb{Q}_p \oplus V \otimes_{\mathbb{Q}} \mathbb{R}$. The conditions at places above p are clearly open and non-empty; so are the conditions at archimedean places, which follows from the same arguments in [Car86, 2.2.4].

- (2) This follows from the same calculation as in [Car86, 2.3.2].

(3) We first remark that $G'_{\mathfrak{S},1,F_p}$ does not depend on the particular choice of δ and hence we may use a convenient δ to ease the computation. We discuss each of the types separately.

If \mathfrak{p} is of type α or α^\sharp , $G'_{\mathfrak{S},1}(F_p)$ is isomorphic to the kernel of $\mathrm{GL}_2(F_p) \times_{F_p^\times} (E_q^\times \times E_q^\times) \rightarrow F_p^\times$ given by $(l, x, y) \mapsto \mathrm{Nm}(l)xy$. Hence, $l \mapsto (l, \mathrm{Nm}(l)^{-1}, 1)$ induces an isomorphism $\mathrm{GL}_2(F_p) \rightarrow G'_{\mathfrak{S},1}(F_p)$. They are of course unramified.

If \mathfrak{p} is of type β , when we identify $D_{\mathfrak{S},p}$ with $M_2(E_q)$, the convolution $l \mapsto \bar{l}$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}$ for $a, b, c, d \in E_q$. We take the element δ to be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The hermitian form on $M_2(E_q)$ is then given by

$$\langle v, w \rangle = \mathrm{tr}_{M_2(E_q)/E_q}(v\bar{w}\delta) = -a\bar{b}' + b\bar{a}' + c\bar{d}' - d\bar{c}', \quad v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } w = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

One checks easily that $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is invariant under the $*$ -involution. So, $D_{\mathfrak{S},p}$ is isomorphic to $(\epsilon D_{\mathfrak{S},p})^{\oplus 2}$ as a $*$ -hermitian space and $G'_{\mathfrak{S},1,F_p}$ is the unitary group for $\epsilon D_{\mathfrak{S},p}$. It is clear from the expression above that $\epsilon D_{\mathfrak{S},p}$ is a hyperbolic plane [Min11, Example 3.2]. Hence, $G'_{\mathfrak{S},1,F_p}$, being the unitary group of such hermitian space, is unramified.

If \mathfrak{p} is of type β^\sharp , the identification of $D_{\mathfrak{S},p}$ with $M_2(E_q)$ using (3.7.3) implies that the convolution $l \mapsto \bar{l}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & -\bar{c}/p \\ -p\bar{b} & \bar{d} \end{pmatrix} \quad \text{for } a, b, c, d \in E_q.$$

We take the element δ to be $\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. The hermitian form on $M_2(E_q)$ is then given by

$$\langle v, w \rangle = \mathrm{Tr}_{M_2(E_q)/E_q}(v\bar{w}\delta) = a\bar{a}'/p - b\bar{b}' - c\bar{c}'/p + d\bar{d}', \quad v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } w = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \tag{3.8.1}$$

Similar to above, $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is invariant under $*$ -involution and $D_{\mathfrak{S},p}$ is isomorphic to $(\epsilon D_{\mathfrak{S},p})^{\oplus 2}$ as $*$ -hermitian spaces. The unitary group $G'_{\mathfrak{S},1,F_p}$ is just the usual unitary group of $\epsilon D_{\mathfrak{S},p}$. But the hermitian form there takes the form of $a\bar{a}'/p - b\bar{b}'$, which is a typical example of an anisotropic plane [Min11, Example 3.2]. So, $G'_{\mathfrak{S},1,F_p}$ is a non-quasi-split unitary group.

To see that $\mathcal{O}_{D_{\mathfrak{S},p}}$ is a maximal $*$ -stable lattice, it suffices to prove it for $\mathcal{O}_{D_{\mathfrak{S},p}}$ for each $\mathfrak{p} \in \Sigma_p$. When \mathfrak{p} is of type α , α^\sharp , or β , this is immediate. When \mathfrak{p} is of type β^\sharp , we write δ as $\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}u$ for $u \in \mathcal{O}_{D_{\mathfrak{S},p}}^\times$. The involution $*$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto u^{-1} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{a} & -\bar{c}/p \\ -p\bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} u = u^{-1} \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix} u \quad \text{for } a, b, c, d \in E_q.$$

It is then clear that $\mathcal{O}_{D_{\mathfrak{S},p}}$ is a maximal $*$ -stable lattice. □

3.9 Level structures at \mathfrak{p} in the unitary case

We specify our choice for K'_p corresponding to the level structure $K_p = \prod_{\mathfrak{p}|p} K_{\mathfrak{p}} \subset \prod_{\mathfrak{p}|p} (B_{\mathfrak{S}} \otimes_F F_{\mathfrak{p}})^\times$ considered in §3.2.

By (3.7.2), giving an element $g_p \in G'_p(\mathbb{Q}_p)$ is equivalent to giving tuples $(g_{\mathfrak{p}})_{\mathfrak{p} \in \Sigma_p}$ with $g_{\mathfrak{p}} \in \mathrm{End}_{D_{\mathfrak{S}} \otimes_F F_{\mathfrak{p}}}(V \otimes_F F_{\mathfrak{p}})$ such that there exists $\nu(g_p) \in \mathbb{Q}_p^\times$ independent of \mathfrak{p} satisfying

$$\psi_{E,\mathfrak{p}}(g_{\mathfrak{p}}v, g_{\mathfrak{p}}w) = \nu(g_p)\psi_{E,\mathfrak{p}}(v, w), \quad \forall v, w \in V \otimes_F F_{\mathfrak{p}},$$

where $\psi_{E,p}$ is the base change of ψ_E to $V \otimes_F F_p = D_{S,p}$. In the following, we will give a chain of lattices $\Lambda_p^{(1)} \subseteq \Lambda_p^{(2)}$ in $D_{S,p}$ for each p , and define $K'_p \subseteq G'_S(\mathbb{Q}_p)$ to be the subgroup consisting of the elements $(g_p)_{p \in \Sigma_p}$ with g_p belonging to the stabilizer of $\Lambda_p^{(1)} \subseteq \Lambda_p^{(2)}$ and with $\nu(g_p) \in \mathbb{Z}_p^\times$ independent of p .

- When p is of type α , we take $\Lambda_p^{(1)} = \Lambda_p^{(2)}$ to be $\mathcal{O}_{D_{S,p}}$.
- When p is of type α^\sharp , we take

$$\Lambda_p^{(1)} = \begin{pmatrix} \mathcal{O}_q & \mathfrak{q} \\ \mathcal{O}_q & \mathfrak{q} \end{pmatrix} \oplus \begin{pmatrix} \mathcal{O}_{\bar{q}} & \mathcal{O}_{\bar{q}} \\ \mathcal{O}_{\bar{q}} & \mathcal{O}_{\bar{q}} \end{pmatrix} \quad \text{and} \quad \Lambda_p^{(2)} = \begin{pmatrix} \mathcal{O}_q & \mathcal{O}_q \\ \mathcal{O}_q & \mathcal{O}_q \end{pmatrix} \oplus \begin{pmatrix} \bar{q}^{-1} & \mathcal{O}_{\bar{q}} \\ \bar{q}^{-1} & \mathcal{O}_{\bar{q}} \end{pmatrix}.$$

- When p is of type β , we take $\Lambda_p^{(1)} = \Lambda_p^{(2)} = \mathcal{O}_{D_{S,p}}$.
- When p is of type β^\sharp , we take

$$\Lambda_p^{(1)} = \begin{pmatrix} \mathfrak{q} & \mathcal{O}_q \\ \mathfrak{q} & \mathcal{O}_q \end{pmatrix} \subseteq \Lambda_p^{(2)} = \begin{pmatrix} \mathcal{O}_q & \mathcal{O}_q \\ \mathcal{O}_q & \mathcal{O}_q \end{pmatrix}.$$

Note that these two lattices are dual to each other under the hermitian form (3.8.1).

Similarly, we give the level structure at p for the Shimura variety associated to the group G''_S : take K''_p to be the image of $K_p \times K_{E,p}$ under the natural map $(G_S \times T_{E,\bar{s}})(\mathbb{Q}_p) \rightarrow G''_S(\mathbb{Q}_p)$.

LEMMA 3.10. *The Shimura data for $G_S, G_S \times T_{E,\bar{s}}, G''_S$, and G'_S satisfy Hypothesis 2.5. Moreover, the natural diagram of morphisms of groups*

$$G_S \leftarrow G_S \times T_{E,\bar{s}} \rightarrow G''_S = G_S \times_Z T_{E,\bar{s}} \leftarrow G'_S \tag{3.10.1}$$

induces isomorphisms on the p -integral points of the derived and adjoint groups.

Proof. This is straightforward from definition. In fact, both $K_p^{\text{ad}} = \prod_{p \in \Sigma_p} K_p^{\text{ad}}$ and $K_p^{\text{der}} = \prod_{p \in \Sigma_p} K_p^{\text{der}}$ are products and we give the description case by case:

- if p is of type α or β , then $K_p^{\text{der}} = \text{SL}_2(\mathcal{O}_p)$ and $K_p^{\text{ad}} = \text{PGL}_2(\mathcal{O}_p)$;
- if p is of type α^\sharp , then $K_p^{\text{der}} = \text{SL}_2(\mathcal{O}_p) \cap \begin{pmatrix} \mathcal{O}_p^\times & \mathcal{O}_p \\ \mathfrak{p} & \mathcal{O}_p^\times \end{pmatrix}$ and $K_p^{\text{ad}} = \begin{pmatrix} \mathcal{O}_p^\times & \mathcal{O}_p \\ \mathfrak{p} & \mathcal{O}_p^\times \end{pmatrix} / \mathcal{O}_p^\times$; and
- if p is of type β^\sharp , then K_p^{der} and K_p^{ad} are the maximal compact open subgroups of $(B_S^\times)^{\text{der}}(F_p)$ and $(B_S^\times)^{\text{ad}}(F_p)$, respectively. □

COROLLARY 3.11. *The natural morphisms between Shimura varieties*

$$\text{Sh}_{K_p}(G_S) \longleftarrow \text{Sh}_{K_p \times K_{E,p}}(G_S \times T_{E,\bar{s}}) \longrightarrow \text{Sh}_{K''_p}(G''_S) \longleftarrow \text{Sh}_{K'_p}(G'_S) \tag{3.11.1}$$

induce isomorphisms on the geometric connected components. Moreover, the groups $\mathcal{E}_{G,\bar{\varphi}}$ defined in § 2.11 (and made explicit below) are isomorphic for each of the groups; and (3.11.1) is equivariant for the actions of the $\mathcal{E}_{G,\bar{\varphi}}$ on the geometric connected components. Moreover, if one of the Shimura varieties admits an integral canonical model, so do the others.

Proof. This follows from Corollary 2.17, for which the conditions are verified in Lemmas 3.6 and 3.10. □

3.12 Structure groups for connected Shimura varieties

In order to apply the machinery developed in § 2, we now make explicit the structure groups \mathcal{G} in (2.11.3) and $\mathcal{E}_{G,\varphi}$ in § 2.11 in the case of our interest.

We use \mathcal{G}_S (respectively $\mathcal{G}'_S, \mathcal{G}''_S$) to denote the group defined in (2.11.3) for $G = G_S$ (respectively G'_S, G''_S). Explicitly, since the center of G_S is $\text{Res}_{F/\mathbb{Q}}\mathbb{G}_m$, we have $G_S^{\text{ad}}(\mathbb{Q}) = B_S^\times/F^\times$. Taking the positive and p -integral part as in Lemma 2.15, we have $G_S^{\text{ad}}(\mathbb{Q})^{+,(p)} = B_S^{\times,>0,(p)}/\mathcal{O}_{F,(p)}^\times$, where the superscript > 0 means to take the elements whose reduced norm is positive for all real embeddings. It follows that $\mathcal{G}_S = G_S(\mathbb{A}^{\infty,p})/\mathcal{O}_{F,(p)}^{\times,\text{cl}}$. The same argument applies to \mathcal{G}''_S whose center is $\text{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ and shows that $\mathcal{G}''_S = G''_S(\mathbb{A}^{\infty,p})/\mathcal{O}_{E,(p)}^{\times,\text{cl}}$. Determination of \mathcal{G}'_S is more subtle. By Lemmas 3.6 and 3.10, we have $(G'_S)^{\text{ad}}(\mathbb{Q})^{+,(p)} = (G''_S)^{\text{ad}}(\mathbb{Q})^{+,(p)}$. So, if we use Z'_S to denote the center of G'_S , then we have

$$\begin{aligned} \mathcal{G}'_S &= (G'_S(\mathbb{A}^{\infty,p})/Z'_S(\mathbb{Q})^{(p),\text{cl}}) *_{G'_S(\mathbb{Q})_+/Z'_S(\mathbb{Q})^{(p)}} (G'_S)^{\text{ad}}(\mathbb{Q})^{+,(p)} \\ &= (G'_S(\mathbb{A}^{\infty,p})/Z'_S(\mathbb{Q})^{(p),\text{cl}}) *_{G'_S(\mathbb{Q})_+/Z'_S(\mathbb{Q})^{(p)}} (G''_S(\mathbb{Q})_+^{(p)}/\mathcal{O}_{E,(p)}^\times) \\ &= G'_S(\mathbb{A}^{\infty,p})G''_S(\mathbb{Q})_+^{(p)}/\mathcal{O}_{E,(p)}^{\times,\text{cl}}. \end{aligned} \tag{3.12.1}$$

The subgroup $G'_S(\mathbb{A}^{\infty,p})G''_S(\mathbb{Q})_+^{(p)}$ can be characterized by the following commutative diagram of an exact sequence as the pullback of the right square.

$$\begin{array}{ccccccc} 1 & \longrightarrow & G'_{S,1}(\mathbb{A}^{\infty,p}) & \longrightarrow & G''_S(\mathbb{Q})_+^{(p)}G'_S(\mathbb{A}^{\infty,p}) & \longrightarrow & \mathcal{O}_{F,(p)}^\times(\mathbb{A}^{\infty,p})^\times \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G'_{S,1}(\mathbb{A}^{\infty,p}) & \longrightarrow & G''_S(\mathbb{A}^{\infty,p}) & \longrightarrow & (\mathbb{A}_F^{\infty,p})^\times \longrightarrow 1 \end{array} \tag{3.12.2}$$

We use $\mathcal{E}_{G,S,\varphi}$ to denote the group $\mathcal{E}_{G,\varphi}$ defined in § 2.11. As an abstract group, it is isomorphic for all groups $G_S, G'_S,$ and G''_S . But we point out that it is important (see Remark 5.10) to know how they sit as subgroups of $\mathcal{G}_S \times \text{Gal}_{k_\varphi}, \mathcal{G}'_S \times \text{Gal}_{k_\varphi},$ and $\mathcal{G}''_S \times \text{Gal}_{k_\varphi},$ respectively, according to the Shimura reciprocity map.

3.13 Integral models of unitary Shimura varieties

We choose a finite extension k_0 of k_φ that contains all residual fields $k_\mathfrak{q}$ for any p -adic place \mathfrak{q} of E . Then the ring of Witt vectors $W(k_0)$ may be viewed as a subring of $\overline{\mathbb{Q}}_p,$ containing \mathcal{O}_φ as a subring.

We fix an order \mathcal{O}_{D_S} of D_S stable under the involution $l \mapsto l^*$ such that $\mathcal{O}_{D_S} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,p} \simeq \mathcal{O}_{D_S,p}$. Recall that V is the abstract \mathbb{Q} -vector space D_S . We choose and fix an \mathcal{O}_{D_S} -lattice Λ of V such that:

- for each $\mathfrak{p} \in \Sigma_p,$ we have $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_\mathfrak{p} \cong \Lambda_\mathfrak{p}^{(1)};$ and
- if we put $\widehat{\Lambda}^{(p)} := \Lambda \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$ as a lattice of $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p},$ we have

$$\widehat{\Lambda}^{(p)} \subseteq \widehat{\Lambda}^{(p),\vee} \text{ under the bilinear form } \psi, \text{ or equivalently, } \psi(\widehat{\Lambda}^{(p)}, \widehat{\Lambda}^{(p)}) \subseteq \widehat{\mathbb{Z}}^{(p)}. \tag{3.13.1}$$

We call such Λ *admissible*.

THEOREM 3.14. *Let K'_p be the open compact subgroup of $G'_S(\mathbb{Q}_p)$ considered in § 3.9, and $K'^p \subset G'_S(\mathbb{A}^{\infty,p})$ be sufficiently small so that $K' = K'^p K'_p$ is neat. Then there exists a unique smooth*

quasi-projective scheme $\mathbf{Sh}_{K'}(G'_S)$ over $W(k_0)$ representing the functor that sends a locally noetherian $W(k_0)$ -scheme S to the set of isomorphism classes of tuples $(A, \iota, \lambda, \alpha_{K'})$, as described as follows.

- (a) A is an abelian scheme over S of dimension $4g$ equipped with an embedding $\iota : \mathcal{O}_{D_S} \rightarrow \text{End}_S(A)$ such that the characteristic polynomial of the endomorphism $\iota(b)$ on $\text{Lie}(A/S)$ for $b \in \mathcal{O}_E$ is given by

$$\prod_{\tilde{\tau} \in \Sigma_{E, \infty}} (x - \tilde{\tau}(b))^{2s_{\tilde{\tau}}}.$$

- (b) $\lambda : A \rightarrow A^\vee$ is a polarization of A such that:

- (b1) the Rosati involution associated to λ induces the involution $l \mapsto l^*$ on \mathcal{O}_{D_S} ;
- (b2) $(\text{Ker } \lambda)[p^\infty]$ is a finite flat closed subgroup scheme contained in $\prod_{\mathfrak{p} \text{ of type } \beta^\sharp} A[\mathfrak{p}]$ such that $(\text{Ker } \lambda)[p^\infty] \cap A[\mathfrak{p}]$ for each \mathfrak{p} of type β^\sharp has rank $(\#k_{\mathfrak{p}})^4$; and
- (b3) the cokernel of $\lambda_* : H_1^{\text{dR}}(A/S) \rightarrow H_1^{\text{dR}}(A^\vee/S)$ is a locally free module of rank two over

$$\bigoplus_{\mathfrak{p} \text{ of type } \beta^\sharp} \mathcal{O}_S \otimes_{\mathbb{Z}_p} (\mathcal{O}_E \otimes_{\mathcal{O}_F} k_{\mathfrak{p}}).$$

- (c) $\alpha_{K'}$ is a pair $(\alpha_{K'^p}, \alpha_p)$ defined as follows.

- (c1) For each connected component S_i of S , we choose a geometric point \bar{s}_i and let $T^{(p)}(A_{\bar{s}_i})$ be the product of l -adic Tate modules of A at \bar{s}_i for all $l \neq p$. Then $\alpha_{K'^p}$ is a collection of $\pi_1(S_i, \bar{s}_i)$ -invariant K'^p -orbits of pairs $(\alpha_i^p, \nu(\alpha_i^p))$, where α_i^p is an $\mathcal{O}_{D_S} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$ -linear isomorphism $\widehat{\Lambda}^{(p)} \xrightarrow{\sim} T^{(p)}(A_{\bar{s}_i})$ and $\nu(\alpha_i^p)$ is an isomorphism $\widehat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} \widehat{\mathbb{Z}}^{(p)}(1)$ such that the following diagram commutes.

$$\begin{array}{ccc} \widehat{\Lambda}^{(p)} \times \widehat{\Lambda}^{(p)} & \xrightarrow{\psi} & \widehat{\mathbb{Z}}^{(p)} \\ \alpha_i^p \times \alpha_i^p \downarrow & & \downarrow \nu(\alpha_i^p) \\ T^{(p)}(A_{\bar{s}_i}) \times T^{(p)}(A_{\bar{s}_i}) & \xrightarrow{\lambda\text{-Weil}} & \widehat{\mathbb{Z}}^{(p)}(1) \end{array}$$

- (c2) For each prime $\mathfrak{p} \in \Sigma_p$ of type α^\sharp , let \mathfrak{q} and $\bar{\mathfrak{q}}$ be the two primes of E above \mathfrak{p} . Then α_p is a collection of \mathcal{O}_{D_S} -stable closed finite flat subgroups $\alpha_{\mathfrak{p}} = H_{\mathfrak{q}} \oplus H_{\bar{\mathfrak{q}}} \subset A[\mathfrak{q}] \oplus A[\bar{\mathfrak{q}}]$ of order $(\#k_{\mathfrak{p}})^4$ such that $H_{\mathfrak{q}}$ and $H_{\bar{\mathfrak{q}}}$ are dual to each other under the perfect pairing

$$A[\mathfrak{q}] \times A[\bar{\mathfrak{q}}] \rightarrow \mu_p$$

induced by the polarization λ .

By Galois descent, the moduli space $\mathbf{Sh}_{K'}(G'_S)$ can be defined over $\mathcal{O}_{\bar{\mathfrak{S}}}$. Moreover, if the ramification set \mathbf{S}_∞ is non-empty, $\mathbf{Sh}_{K'}(G'_S)$ is projective.

We will postpone the proof of this theorem until after Notation 3.16. The intuition behind the proof is the following. It is well known that the corresponding moduli problem of hyperspecial level is representable by a quasi-projective smooth scheme over $W(k_0)$. In our situation, the hyperspecial level occurs merely at primes $\mathfrak{p} \in \Sigma_p$ of type α^\sharp or β^\sharp , but the condition $\Sigma_{\infty/\mathfrak{p}} = \mathbf{S}_{\infty/\mathfrak{p}}$ for such primes \mathfrak{p} implies that the extra levels at those primes are representable by finite étale maps over the hyperspecial moduli. Hence, the resulting moduli problem is still smooth.

3.15 Deformation theory

We recall briefly the crystalline deformation theory of abelian varieties due to Serre–Tate and Grothendieck–Messing. This will be used in the proof of Theorem 3.14.

We start with a general situation. Let S be a \mathbb{Z}_p -scheme on which p is locally nilpotent, and $S_0 \hookrightarrow S$ be a closed immersion whose ideal sheaf \mathcal{I} is equipped with a divided power structure compatible with that on $p\mathbb{Z}_p$, e.g. $S_0 = \text{Spec } k \hookrightarrow S = \text{Spec } k[\epsilon]/(\epsilon^2)$ with k a perfect field of characteristic p . Let $(S_0/\mathbb{Z}_p)_{\text{cris}}$ be the crystalline site of S_0 over $\text{Spec } \mathbb{Z}_p$ and $\mathcal{O}_{S_0/\mathbb{Z}_p}^{\text{cris}}$ be the structure sheaf. Let A_0 be an abelian scheme over S_0 , and $H_1^{\text{cris}}(A_0/S_0)$ be the dual of the relative crystalline cohomology $H_{\text{cris}}^1(A_0/S_0)$ (or isomorphically $H_1^{\text{cris}}(A_0/S_0) = H_{\text{cris}}^1(A_0^\vee/S_0)$). Then $H_1^{\text{cris}}(A_0/S_0)$ is a crystal of locally free $\mathcal{O}_{S_0/\mathbb{Z}_p}^{\text{cris}}$ -modules whose evaluation $H_1^{\text{cris}}(A_0/S_0)_S$ at the pd-embedding $S_0 \hookrightarrow S$ is a locally free \mathcal{O}_S -module. We have a canonical isomorphism $H_1^{\text{cris}}(A_0/S_0)_S \otimes_{\mathcal{O}_S} \mathcal{O}_{S_0} \simeq H_1^{\text{dR}}(A_0/S_0)$, which is the dual of the relative de Rham cohomology of A_0/S_0 . For each abelian scheme A over S with $A \times_S S_0 \simeq A_0$, we have a canonical Hodge filtration

$$0 \rightarrow \omega_{A^\vee/S} \rightarrow H_1^{\text{cris}}(A_0/S_0)_S \rightarrow \text{Lie}(A/S) \rightarrow 0.$$

Hence, $\omega_{A^\vee/S}$ gives rise to a local direct factor of $H_1^{\text{cris}}(A_0/S_0)_S$ that lifts the subbundle $\omega_{A_0^\vee/S_0} \subseteq H_1^{\text{dR}}(A_0/S_0)$. Conversely, the theory of deformations of abelian schemes says that knowing this lift of the subbundle is also enough to recover A from A_0 . More precisely, let AV_S be the category of abelian schemes over S , and let $\text{AV}_{S_0}^+$ denote the category of pairs (A_0, ω) , where A_0 is an abelian scheme over S_0 and ω is a subbundle of $H_1^{\text{cris}}(A_0/S_0)_S$ that lifts $\omega_{A_0^\vee/S_0} \subseteq H_1^{\text{dR}}(A_0/S_0)$. The main theorem of the crystalline deformation theory (cf. [Gro74, pp. 116–118] and [MM74, ch. II, § 1]) says that *the natural functor $\text{AV}_S \rightarrow \text{AV}_{S_0}^+$ given by $A \mapsto (A \times_S S_0, \omega_{A^\vee/S})$ is an equivalence of categories.*

Let A be a deformation of A_0 corresponding to a direct factor $\omega \subseteq H_1^{\text{cris}}(A_0/S_0)_S$ that lifts $\omega_{A_0^\vee/S_0}$. If A_0 is equipped with an action ι_0 by a certain algebra R , then ι_0 deforms to an action ι of R on A if and only if $\omega_S \subseteq H_1^{\text{cris}}(A_0/S_0)_S$ is R -stable. Let $\lambda_0 : A_0 \rightarrow A_0^\vee$ be a polarization. Then λ_0 induces a natural alternating pairing [BBM82, 5.1]

$$\langle \cdot, \cdot \rangle_{\lambda_0} : H_1^{\text{cris}}(A_0/S_0)_S \times H_1^{\text{cris}}(A_0/S_0)_S \rightarrow \mathcal{O}_S,$$

which is perfect if λ_0 is prime-to- p . Then there exists a (necessarily unique) polarization $\lambda : A \rightarrow A^\vee$ that lifts λ_0 if and only if ω_S is isotropic for $\langle \cdot, \cdot \rangle_{\lambda_0}$ by [Lan13, 2.1.6.9, 2.2.2.2, and 2.2.2.6].

Notation 3.16. Before going to the proof of Theorem 3.14, we introduce some notation. Recall that we have an isomorphism $\mathcal{O}_{D_S, p} \simeq M_2(\mathcal{O}_E \otimes \mathbb{Z}_p)$. We denote by $\epsilon \in \mathcal{O}_{D_S, p}$ the element corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(\mathcal{O}_E \otimes \mathbb{Z}_p)$. For S a $W(k_0)$ -scheme and M an \mathcal{O}_S -module locally free of finite rank equipped with an action of $\mathcal{O}_{D_S, p}$, we call $M^\circ := \epsilon M$ the reduced part of M . We have $M = (M^\circ)^{\oplus 2}$ by Morita equivalence. Moreover, the \mathcal{O}_E -action induces a canonical decomposition

$$M^\circ = \bigoplus_{\tilde{\tau} \in \Sigma_{E, \infty}} M_{\tilde{\tau}}^\circ,$$

where \mathcal{O}_E acts on each factor $M_{\tilde{\tau}}^\circ$ by $\tilde{\tau} : \mathcal{O}_E \rightarrow W(k_0)$.

Let A be an abelian scheme over S carrying an action of \mathcal{O}_{D_S} . The construction above gives rise to locally free \mathcal{O}_S -modules $\omega_{A/S}^\circ$, $\text{Lie}(A/S)^\circ$, and $H_1^{\text{dR}}(A/S)^\circ$, which are of rank $\frac{1}{2} \dim A$, $\frac{1}{2} \dim A$, and $\dim A$, respectively. We call them the *reduced invariant differential 1-forms*, the

reduced Lie algebra, and the reduced de Rham homology of A , respectively. For each $\tilde{\tau} \in \Sigma_{E,\infty}$, we have a reduced Hodge filtration in $\tilde{\tau}$ -component

$$0 \rightarrow \omega_{A^\vee/S,\tilde{\tau}}^\circ \rightarrow H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^\circ \rightarrow \text{Lie}(A/S)_{\tilde{\tau}}^\circ \rightarrow 0. \tag{3.16.1}$$

If the abelian scheme A comes from the moduli problem in Theorem 3.14, the dimensions of these three factors are $2 - s_{\tilde{\tau}}$, 2 , and $s_{\tilde{\tau}}$, respectively, where the number $s_{\tilde{\tau}}$ is defined in § 3.4.

Proof of Theorem 3.14. The representability of $\mathbf{Sh}_{K'}(G'_S)$ by a quasi-projective scheme over $W(k_0)$ is well known (cf. for instance [Lan13, 1.4.13, 2.3.3, and 7.2.3.10]). To show the smoothness of $\mathbf{Sh}_{K'}(G'_S)$, it suffices to prove that it is formally smooth over $W(k_0)$. Let R be a noetherian $W(k_0)$ -algebra, $I \subset R$ be an ideal with $I^2 = 0$, and $R_0 = R/I$. We need to show that every point $x_0 = (A_0, \iota_0, \lambda_0, \alpha_{K',0})$ of $\mathbf{Sh}_{K'}(G'_S)$ with values in R_0 lifts to an R -valued point x of $\mathbf{Sh}_{K'}(G'_S)$. We apply the deformation theory recalled in § 3.15. The relative crystalline homology $H_1^{\text{cris}}(A_0/R_0)$ is naturally equipped with an action of $\mathcal{O}_{D_S} \otimes \mathbb{Z}_p$. Let $H_1^{\text{cris}}(A_0/R_0)^\circ := \epsilon H_1^{\text{cris}}(A_0/R_0)$ be its reduced part and $H_1^{\text{cris}}(A_0/R_0)_{R_0}^\circ$ be its evaluation on R . This is a free $R \otimes \mathcal{O}_E$ -module of rank $4[F : \mathbb{Q}]$, and we have a canonical decomposition

$$H_1^{\text{cris}}(A_0/R_0)_{R_0}^\circ = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} H_1^{\text{cris}}(A_0/R_0)_{R_0,\tilde{\tau}}^\circ.$$

The polarization λ_0 on A_0 induces a pairing

$$H_1^{\text{cris}}(A_0/R_0)_{R_0,\tilde{\tau}}^\circ \times H_1^{\text{cris}}(A_0/R_0)_{R_0,\tilde{\tau}^c}^\circ \longrightarrow R, \tag{3.16.2}$$

which is perfect for $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$ with \mathfrak{p} not of type β^\sharp . By the deformation theory § 3.15, giving a deformation of (A_0, ι_0) to R is equivalent to giving, for each $\tilde{\tau} \in \Sigma_{E,\infty}$, a direct summand $\omega_{R,\tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(A_0/R_0)_{R_0,\tilde{\tau}}^\circ$ which lifts $\omega_{A_0^\vee/R_0,\tilde{\tau}}^\circ$. Let $\mathfrak{p} \in \Sigma_p$ with $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$. We distinguish several cases.

- If $\tilde{\tau}$ restricts to $\tau \in \mathbf{S}_\infty$, $\text{Lie}(A_0/R_0)_{\tilde{\tau}}^\circ$ has rank $s_{\tilde{\tau}} \in \{0, 2\}$ by the determinant condition (a). By duality or the Hodge filtration (3.16.1), $\omega_{A_0^\vee/R_0,\tilde{\tau}}^\circ$ has rank $2 - s_{\tilde{\tau}}$, i.e. $\omega_{A_0^\vee/R_0,\tilde{\tau}}^\circ = 0$ when $s_{\tilde{\tau}} = 2$ and $\omega_{A_0^\vee/R_0,\tilde{\tau}}^\circ \cong H_1^{\text{dR}}(A_0/R_0)_{\tilde{\tau}}^\circ$ when $s_{\tilde{\tau}} = 0$. Therefore, $\omega_{R,\tilde{\tau}}^\circ = 0$ or $\omega_{R,\tilde{\tau}}^\circ = H_1^{\text{cris}}(A_0/R_0)_{R_0,\tilde{\tau}}^\circ$ is the unique lift in these cases, respectively.
- If $\tilde{\tau}$ restricts to $\tau \in \Sigma_\infty - \mathbf{S}_\infty$, then $\omega_{A_0^\vee/R_0,\tilde{\tau}}^\circ$ and $\omega_{A_0^\vee/R_0,\tilde{\tau}^c}^\circ$ are both of rank one over R_0 , and we have $\omega_{A_0^\vee/R_0,\tilde{\tau}}^\circ = (\omega_{A_0^\vee/R_0,\tilde{\tau}^c}^\circ)^\perp$ under the perfect pairing between $H_1^{\text{dR}}(A_0/R_0)_{\tilde{\tau}}^\circ$ and $H_1^{\text{dR}}(A_0/R_0)_{\tilde{\tau}^c}^\circ$ induced by λ_0 . (Note that $\tau \in \Sigma_\infty/\mathfrak{p} - \mathbf{S}_\infty$ means that \mathfrak{p} is not of type β^\sharp and hence the Weil pairing is perfect.) Within each pair $\{\tilde{\tau}, \tilde{\tau}^c\}$, we can take an arbitrary direct summand $\omega_{R,\tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(A_0/R_0)_{R_0,\tilde{\tau}}^\circ$ which lifts $\omega_{A_0^\vee/R_0,\tilde{\tau}}^\circ$, and let $\omega_{R,\tilde{\tau}^c}^\circ$ be the orthogonal complement of $\omega_{R,\tilde{\tau}}^\circ$ under the perfect pairing (3.16.2). By the Hodge filtration (3.16.1), such choices of $(\omega_{R,\tilde{\tau}}^\circ, \omega_{R,\tilde{\tau}^c}^\circ)$ form a torsor under the group

$$\text{Hom}_{R_0}(\omega_{A_0^\vee/R_0,\tilde{\tau}}^\circ, \text{Lie}(A_0)_{\tilde{\tau}}^\circ) \otimes I \cong \text{Lie}(A_0)_{\tilde{\tau}}^\circ \otimes_{R_0} \text{Lie}(A_0)_{\tilde{\tau}^c}^\circ \otimes I,$$

where in the second isomorphism we have used the fact that $\text{Lie}(A_0^\vee)_{\tilde{\tau}}^\circ \simeq \text{Lie}(A_0)_{\tilde{\tau}^c}^\circ$.

We take liftings $\omega_{R,\tilde{\tau}}^\circ$ for each $\tilde{\tau} \in \Sigma_{E,\infty}$ as above, and let (A, ι) be the corresponding deformation to R of (A_0, ι_0) . It is clear that $\bigoplus_{\tilde{\tau} \in \Sigma_\infty} (\omega_{R,\tilde{\tau}}^\circ \oplus \omega_{R,\tilde{\tau}^c}^\circ)$ is isotropic for the pairing on $H_1^{\text{cris}}(A_0/R_0)_{R_0}^\circ$ induced by λ_0 . Hence, the polarization λ_0 lifts uniquely to a polarization $\lambda : A \rightarrow A^\vee$ satisfying condition (b1) in the statement of the theorem. By the criterion

of flatness by fibers [EGAIV, 11.3.10], $\text{Ker}(\lambda)$ is a finite flat group scheme over R , and the condition (b2) is thus satisfied. Condition (b3) follows from the fact that the morphism $\lambda_* : H_1^{\text{dR}}(A/R) \rightarrow H_1^{\text{dR}}(A/R)$ is the same as $\lambda_{0,*} : H_1^{\text{cris}}(A_0/R_0)_R \rightarrow H_1^{\text{cris}}(A_0^\vee/R_0)_R$ under the canonical isomorphism $H_1^{\text{dR}}(B/R) \simeq H_1^{\text{cris}}(B_0/R_0)_R$ for $B = A_0, A_0^\vee$.

We have to show moreover that the level structure $\alpha_{K',0} = (\alpha_0^p, \alpha_{p,0})$ extends uniquely to A . It is clear for α_0^p . For $\alpha_{p,0}$, let $H_0 = \prod_{\mathfrak{p} \text{ of type } \alpha^\sharp} \alpha_{\mathfrak{p}}$ be the product of the closed subgroups in the data of $\alpha_{p,0}$. Let $f_0 : A_0 \rightarrow B_0 = A_0/H_0$ be the canonical isogeny. It suffices to show that B_0 and f_0 deform to R . The abelian variety B_0 is equipped with an induced action of \mathcal{O}_{D_S} , a polarization λ_{B_0} satisfying conditions (a) and (b). The isogeny f_0 induces canonical isomorphisms $H_1^{\text{cris}}(A_0/R_0)_{R,\tilde{\tau}}^\circ \cong H_1^{\text{cris}}(B_0/R_0)_{R,\tilde{\tau}}^\circ$ for $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$ with \mathfrak{p} not of type α^\sharp . So, for such primes \mathfrak{p} and $\tilde{\tau} \in \Sigma_{\infty/\mathfrak{p}}$, the liftings $\omega_{R,\tilde{\tau}}^\circ$ chosen above give the liftings of $\omega_{B_0^\vee/R_0,\tilde{\tau}}^\circ \subset H_1^{\text{dR}}(B_0/R_0)_{\tilde{\tau}}^\circ$. For $\tilde{\tau} \in \Sigma_{\infty/\mathfrak{p}}$ with \mathfrak{p} of type α^\sharp , we note that at each closed point x of R_0 , $\omega_{B_0^\vee/k_x,\tilde{\tau}}^\circ$ is either trivial or isomorphic to the whole $H_1^{\text{dR}}(B_0/k_x)_{\tilde{\tau}}^\circ$ as in the case for A_0 . Hence, the same holds for R_0 in place of k_x . Therefore, $\omega_{B_0^\vee/R_0,\tilde{\tau}}^\circ$ admits a unique lift to a direct summand of $H_1^{\text{cris}}(B_0/R_0)_{R,\tilde{\tau}}^\circ$. Such choices of liftings of $\omega_{B_0^\vee/R_0,\tilde{\tau}}^\circ$ give rise to a deformation B/R of B_0/R_0 . It is clear that $f_0 : A_0 \rightarrow B_0$ also lifts to an isogeny $f : A \rightarrow B$. Then the kernel of f gives the required lift of H_0 . This concludes the proof of the smoothness of $\mathbf{Sh}_{K'}(G'_S)$.

The dimension of $\mathbf{Sh}_{K'}(G'_S)$ follows from the calculation of the tangent bundle of $\mathbf{Sh}_{K'}(G'_S)$, as the following corollary shows. When the ramification set \mathbf{S}_∞ is non-empty, it is a standard argument to use a valuative criterion to check that $\mathbf{Sh}_{K'}(G'_S)$ is proper. We will postpone the proof to Proposition 4.7, where a more general statement is proved. (One can check that there is no loophole in our argument.) □

COROLLARY 3.17. *Let $S_0 \hookrightarrow S$ be a closed immersion of locally noetherian k_0 -schemes with ideal sheaf \mathcal{I} such that $\mathcal{I}^2 = 0$. Let $x_0 = (A_0, \iota_0, \lambda_0, \bar{\alpha}_{K',0})$ be an S_0 -valued point of $\mathbf{Sh}_{K'}(G'_S)$. Then the set-valued sheaf of local deformations of x_0 to S forms a torsor under the group*

$$\bigoplus_{\tau \in \Sigma_\infty - \mathbf{S}_\infty} (\text{Lie}(A_0)_{\tilde{\tau}}^\circ \otimes \text{Lie}(A_0)_{\tilde{\tau}^c}^\circ) \otimes \mathcal{I}.$$

In particular, the tangent bundle $\mathcal{T}_{\mathbf{Sh}_{K'}(G'_S)}$ of $\mathbf{Sh}_{K'}(G'_S)$ is canonically isomorphic to

$$\bigoplus_{\tau \in \Sigma_\infty - \mathbf{S}_\infty} \text{Lie}(\mathbf{A}')_{\tilde{\tau}}^\circ \otimes \text{Lie}(\mathbf{A}')_{\tilde{\tau}^c}^\circ,$$

where $\mathbf{A}' = \mathbf{A}'_{S,K'}$ denotes the universal abelian scheme over $\mathbf{Sh}_{K'}(G'_S)$.

Proof. A deformation of x_0 is determined by the liftings $\omega_{S,\tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(A_0/S_0)_{S,\tilde{\tau}}^\circ$ of $\omega_{A_0^\vee/S_0,\tilde{\tau}}^\circ$ for $\tilde{\tau} \in \Sigma_{E,\infty}$. From the proof of Theorem 3.14, we see that the choices for $\omega_{S,\tilde{\tau}}^\circ$ are unique if $\tilde{\tau}$ restricts to $\tau \in \mathbf{S}_\infty$. For $\tau \in \Sigma_\infty - \mathbf{S}_\infty$, the possible liftings $\omega_{S,\tilde{\tau}}^\circ$ and $\omega_{S,\tilde{\tau}^c}^\circ$ determine each other and form a torsor under the group

$$\text{Hom}_{\mathcal{O}_{S_0}}(\omega_{A_0^\vee/S_0,\tilde{\tau}}^\circ, \text{Lie}(A_0)_{\tilde{\tau}}^\circ) \otimes_{\mathcal{O}_{S_0}} \mathcal{I} \simeq \text{Lie}(A_0)_{\tilde{\tau}}^\circ \otimes \text{Lie}(A_0)_{\tilde{\tau}^c}^\circ \otimes_{\mathcal{O}_{S_0}} \mathcal{I}.$$

The statement for the local lifts of x_0 to S follows immediately. Applying this to the universal case, we obtain the second part of the corollary. □

Remark 3.18. We remark that the moduli space $\mathbf{Sh}_{K'}(G'_S)$ does not depend on the choice of the admissible lattice Λ in § 3.13; but the universal abelian scheme \mathbf{A}' does in the following way. If Λ_1 and Λ_2 are two admissible lattices, we put $\widehat{\Lambda}_i^{(p)} := \Lambda_i \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$ and we use \mathbf{A}'_i to denote the corresponding universal abelian variety over $\mathbf{Sh}_{K'}(G'_S)$ and $\bar{\alpha}_{K'^p, i}^p$ to denote the universal level structure (away from p) for $i = 1, 2$.

Then there is a natural prime-to- p quasi-isogeny $\eta : \mathbf{A}'_1 \dashrightarrow \mathbf{A}'_2$ such that

$$\begin{array}{ccc} \widehat{\Lambda}_1^{(p)} & \xrightarrow[\cong]{\bar{\alpha}_{K'^p, 1}^p} & T^{(p)}(\mathbf{A}'_1) \\ \downarrow & & \downarrow T^{(p)}(\eta) \\ \widehat{\Lambda}_2^{(p)} & \xrightarrow[\cong]{\bar{\alpha}_{K'^p, 2}^p} & T^{(p)}(\mathbf{A}'_2) \end{array}$$

is a commutative diagram up the action of K'^p , where the left-hand vertical arrow is the isogeny of lattices inside $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty, p}$. (For more detailed discussion, see [Lan13, 1.4.3].)

COROLLARY 3.19. *The integral model $\mathbf{Sh}_{K'}(G'_S)$ defined in Theorem 3.14 gives an integral canonical model $\mathbf{Sh}_{K'_p}(G'_S)$ of $\mathbf{Sh}_{K'}(G'_S)$. Consequently, the quaternionic Shimura variety $\mathbf{Sh}_{K_p}(G_S)$ admits an integral canonical model over $\mathcal{O}_{\bar{\varphi}}$. Similarly, the Shimura varieties $\mathbf{Sh}_{K_p \times K_{E,p}}(G_S \times T_{E, \bar{S}})$ and $\mathbf{Sh}_{K''_p}(G''_S)$ both admit integral canonical models over $\mathcal{O}_{\bar{\varphi}}$. The geometric connected components of these integral canonical models are canonically isomorphic.*

Proof. We first assume that $\mathbf{S}_{\infty} \neq \Sigma_{\infty}$. We need to verify that for any smooth $\mathcal{O}_{\bar{\varphi}}$ -scheme S , any morphism $s_0 : S \otimes_{\mathcal{O}_{\bar{\varphi}}} E_{\bar{S}, \bar{\varphi}} \rightarrow \mathbf{Sh}_{K'_p}(G'_S)$ extends to a morphism $s : S \rightarrow \mathbf{Sh}_{K'_p}(G'_S)$. Explicitly, we have to show that a tuple $(A, \iota, \lambda, \alpha^p \alpha_p)$ over $S \otimes_{\mathcal{O}_{\bar{\varphi}}} E_{\bar{S}, \bar{\varphi}}$ extends to a similar tuple over S . Here $\alpha^p \alpha_p$ is the projective limit in K'^p of level structures $\alpha_{K'^p}^p \alpha_p$ as in Theorem 3.14(c). The same arguments as [Moo98, Corollary 3.8] apply to proving the existence of extension of A, ι, λ , and the prime-to- p level structure α^p . It remains to extend the level structure α_p . Let $\mathbf{Sh}_{\tilde{K}'_p}(G'_S)$ denote the similar moduli space as $\mathbf{Sh}_{K'_p}(G'_S)$ by forgetting the p -level structure α_p . We have seen in the proof of Theorem 3.14 that there is no local deformation of α_p , which means that the forgetful map $\mathbf{Sh}_{K'_p}(G'_S) \rightarrow \mathbf{Sh}_{\tilde{K}'_p}(G'_S)$ is finite and étale. The discussion above shows that $\mathbf{Sh}_{\tilde{K}'_p}(G'_S)$ satisfies the extension property. Hence, there exists a morphism $\tilde{s} : S \rightarrow \mathbf{Sh}_{\tilde{K}'_p}(G'_S)$ such that the square of the following diagram is commutative.

$$\begin{array}{ccc} S \otimes_{\mathcal{O}_{\bar{\varphi}}} E_{\bar{S}, \bar{\varphi}} & \xrightarrow{s_0} & \mathbf{Sh}_{K'_p}(G'_S) \\ \downarrow & \nearrow s & \downarrow \\ S & \xrightarrow{\tilde{s}} & \mathbf{Sh}_{\tilde{K}'_p}(G'_S) \end{array}$$

We have to show that there exists a map s as the dotted arrow that makes the whole diagram commutative. Giving such a map s is equivalent to giving a section of the finite étale cover $S \times_{\mathbf{Sh}_{\tilde{K}'_p}(G'_S)} \mathbf{Sh}_{K'_p}(G'_S) \rightarrow S$ extending the section corresponding to s_0 . Since a section of a finite étale cover of separated schemes is an open and closed immersion, the existence of s follows immediately. The existence of integral canonical models for $\mathbf{Sh}_{K_p}(G_S), \mathbf{Sh}_{K''_p}(G''_S)$, and $\mathbf{Sh}_{K_p \times K_{E,p}}(G_S \times T_{E, \bar{S}})$ follows from Corollary 2.17.

When $S_\infty = \Sigma_\infty$, we need to show that the action of the arithmetic Frobenius $\sigma_{\tilde{\varphi}}$ on the moduli space $\mathbf{Sh}_{K'}(G'_S)$ is given by the reciprocity law as in §2.8. Put $n_{\tilde{\varphi}} = [k_{\tilde{\varphi}} : \mathbb{F}_p]$. Let $\mathfrak{Rec}_{Z'} : \text{Gal}_{E_{\tilde{S}}} \rightarrow Z'(\mathbb{Q})^{\text{cl}} \backslash Z'(\mathbb{A}^\infty) / Z'(\mathbb{Z}_p)$ denote the reciprocity map defined in §2.8, where Z' is the center of G'_S , which is the algebraic group associated to the subgroup of E^\times consisting of elements with norm to F^\times lying in \mathbb{Q}^\times . By definition, $\mathfrak{Rec}_{Z'}(\sigma_{\tilde{\varphi}})$ is the image of $\varpi_{\tilde{\varphi}}^{-1}$ under the composite of

$$\mathfrak{Rec}_{Z', \tilde{\varphi}} : E_{\tilde{S}, \tilde{\varphi}}^\times / \mathcal{O}_{\tilde{\varphi}}^\times \xrightarrow{\text{Rec}_{Z'}(G'_S, \mathfrak{H}_S)} Z'(\mathbb{Q}_p) / Z'(\mathbb{Z}_p)$$

and the natural map $Z'(\mathbb{Q}_p) / Z'(\mathbb{Z}_p) \rightarrow Z'(\mathbb{Q})^{\text{cl}} \backslash Z'(\mathbb{A}^\infty) / Z'(\mathbb{Z}_p)$. Explicitly, one has

$$Z'(\mathbb{Q}_p) = \{((x_{\mathfrak{p}})_{\mathfrak{p} \in \Sigma_p}, y) \mid y \in \mathbb{Q}_p^\times, x_{\mathfrak{p}} \in E_{\mathfrak{p}}^\times, \text{ and } \text{Nm}_{E_{\mathfrak{p}}/F_{\mathfrak{p}}}(x_{\mathfrak{p}}) = y\}.$$

We note that there is no p -adic primes of F of type β and the valuation of y determines the valuation of $x_{\mathfrak{p}}$ for \mathfrak{p} of type β^\sharp . For each prime $\mathfrak{p} \in \Sigma_p$ of type α or α^\sharp , choose a place \mathfrak{q} of E above \mathfrak{p} ; then the map $((x_{\mathfrak{p}})_{\mathfrak{p}}, y) \mapsto (\text{val}_p(y), (\text{val}_p(x_{\mathfrak{q}}))_{\mathfrak{p}})$ defines an isomorphism

$$\xi : Z'(\mathbb{Q}_p) / Z'(\mathbb{Z}_p) \xrightarrow{\cong} \mathbb{Z} \times \prod_{\mathfrak{p} \text{ of type } \alpha \text{ or } \alpha^\sharp} \mathbb{Z},$$

where we have written $x_{\mathfrak{p}} = (x_{\mathfrak{q}}, x_{\bar{\mathfrak{q}}})$ for each prime $\mathfrak{p} \in \Sigma_p$ of type α or α^\sharp . By the definition of $\mathfrak{Rec}_{Z', \tilde{\varphi}}$ in §2.8, using $h'_{\tilde{S}}$, we see that $\xi \circ \mathfrak{Rec}_{Z', \tilde{\varphi}}(\varpi_{\tilde{\varphi}}^{-1})$ is equal to

$$(-n_{\tilde{\varphi}}, (-\#\tilde{S}_{\infty/\bar{\mathfrak{q}}} \cdot n_{\tilde{\varphi}} / f_{\mathfrak{p}})_{\mathfrak{p}}), \tag{3.19.1}$$

where $n_{\tilde{\varphi}} = [k_{\tilde{\varphi}} : \mathbb{F}_p]$ and $f_{\mathfrak{p}}$ is the inertia degree of \mathfrak{p} in F/\mathbb{Q} . (Note that the homomorphism $h_{E, \tilde{S}}$ sends z to \bar{z} with respect to the embeddings $\tilde{\tau} \in \tilde{S}_\infty$.)

On the other hand, $\sigma_{\tilde{\varphi}}$ takes a closed point $x = (A, \iota, \lambda, \alpha_{K'})$ of $\mathbf{Sh}_{K'}(G'_S)_{\overline{\mathbb{F}}_p}$ to $\sigma_{\tilde{\varphi}}(x) = (\sigma_{\tilde{\varphi}}^*(A), \iota', \lambda', \text{Frob}_{\tilde{\varphi}} \circ \alpha_{K'})$, where $\sigma_{\tilde{\varphi}}^*(A)$ denotes the pullback of A via the Frobenius $\sigma_{\tilde{\varphi}} = \sigma^{n_{\tilde{\varphi}}}$ on the residue field $\kappa(x)$, equipped with the induced \mathcal{O}_{D_S} -action and the polarization, and $\text{Frob}_{\tilde{\varphi}} : A \rightarrow \sigma_{\tilde{\varphi}}^*A$ is the relative Frobenius map. For a p -adic prime \mathfrak{p} of F (or of E), denote by $\tilde{\mathcal{D}}(A)_{\mathfrak{p}}$ the covariant Dieudonné module of $A[\mathfrak{p}^\infty]$. We observe that, if \mathfrak{p} is a prime of F of type β , then

$$\tilde{\mathcal{D}}(\sigma_{\tilde{\varphi}}^*(A))_{\mathfrak{p}} = p^{-n_{\tilde{\varphi}}/2} V^{n_{\tilde{\varphi}}} \tilde{\mathcal{D}}(A)_{\mathfrak{p}}$$

and, if \mathfrak{p} is of type α or α^\sharp with \mathfrak{q} a place of E above \mathfrak{p} , then

$$\tilde{\mathcal{D}}(\sigma_{\tilde{\varphi}}^*(A))_{\mathfrak{q}} = p^{-\#\tilde{S}_{\infty/\bar{\mathfrak{q}}} \cdot n_{\tilde{\varphi}} / f_{\mathfrak{p}}} V^{n_{\tilde{\varphi}}} \tilde{\mathcal{D}}(A)_{\mathfrak{q}}.$$

Let $g_p \in Z'(\mathbb{Q}_p)$ be an element such that $\xi(\bar{g}_p)$ is given by (3.19.1), where \bar{g}_p denotes the image of g_p in $Z'(\mathbb{Q}_p) / Z'(\mathbb{Z}_p)$. Then via the isogeny $\text{Frob}_{\tilde{\varphi}} : A \rightarrow \sigma_{\tilde{\varphi}}^*A$, which corresponds to $V^{n_{\tilde{\varphi}}} : \tilde{\mathcal{D}}(A) \rightarrow \tilde{\mathcal{D}}(\sigma_{\tilde{\varphi}}^*A)$, $\tilde{\mathcal{D}}(\sigma_{\tilde{\varphi}}^*A)$ is identified with the lattice $g_p \tilde{\mathcal{D}}(A)$ of $\tilde{\mathcal{D}}(A)[1/p]$. This agrees with the computation of $\mathfrak{Rec}_{Z', \tilde{\varphi}}(\varpi_{\tilde{\varphi}}^{-1})$ above. □

The rest of this section is devoted to understanding how to pass the universal abelian varieties on $\mathbf{Sh}_{K'_p}(G'_S)$ to other Shimura varieties, as well as natural partial Frobenius morphisms among these varieties and their compatibility with the abelian varieties.

3.20 Actions on universal abelian varieties in the unitary case

We need to extend the usual tame Hecke algebra action on the universal abelian variety $\mathbf{A}'_{K'_p}$ over $\mathbf{Sh}_{K'_p}(G'_S)$ to the action of a slightly bigger group $\tilde{G}_S := G'_S(\mathbb{A}^{\infty,p})G''_S(\mathbb{Q})_+^{(p)}$. We follow [Sai09]. Take an element $\tilde{g} \in \tilde{G}_S$ and let $K_1'^p$ and $K_2'^p$ be two open compact subgroups of $G'_S(\mathbb{A}^{\infty,p})$ such that $\tilde{g}^{-1}K_1'^p\tilde{g} \subseteq K_2'^p$ (note that G''_S normalizes G'_S). We put $K_i' = K_i'^p K_p'$ for $i = 1, 2$. Then, starting from the universal abelian variety $\mathbf{A}'_{K'_1}$ together with the tame level structure $\bar{\alpha}_{K_1'^p}^p : \widehat{\Lambda}^{(p)} \xrightarrow{\cong} T^{(p)}(\mathbf{A}'_{K'_1})$, we may obtain an abelian variety \mathbf{B}' over $\mathbf{Sh}_{K'_1}(G'_S)$, together with a prime-to- p quasi-isogeny $\eta : \mathbf{A}'_{K'_1} \rightarrow \mathbf{B}'$ and a tame level structure such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\Lambda}^{(p)} & \xrightarrow[\cong]{\bar{\alpha}_{K_1'^p}^p} & T^{(p)}(\mathbf{A}'_{K'_1}) \\ \downarrow & & \downarrow T^{(p)}(\eta) \\ \widehat{\Lambda}^{(p)} & \xrightarrow[\cong]{\tilde{g}^{-1}} & \tilde{g}\widehat{\Lambda}^{(p)} \xrightarrow{\cong} T^{(p)}(\mathbf{B}') \end{array}$$

where the left vertical arrow is the natural quasi-isogeny as lattices inside $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$. Since $\tilde{g}^{-1}K_1'^p\tilde{g} \subseteq K_2'^p$, we may take the $K_2'^p$ -orbit of the composite of the bottom homomorphism as the tame level structure on \mathbf{B}' .

One can easily transfer other data in the moduli problem of Theorem 3.14 to \mathbf{B}' , *except for the polarization*, for which we make the modification as follows: since $\tilde{g} \in G'_S(\mathbb{A}^{\infty,p})G''_S(\mathbb{Q})_+^{(p)}$, we have $\nu(\tilde{g}) \in \mathcal{O}_{F,(p)}^{\times,>0} \cdot \mathbb{A}_{\mathbb{Q}}^{\infty,p,\times} = \mathcal{O}_{F,(p)}^{\times,>0} \cdot \widehat{\mathbb{Z}}^{(p),\times}$. We can then write $\nu(\tilde{g})$ as the product $\nu_g^+ \cdot u$ for $\nu_g^+ \in \mathcal{O}_{F,(p)}^{\times,>0}$ and $u \in \widehat{\mathbb{Z}}^{(p),\times}$. In fact, ν_g^+ is uniquely determined by this restriction. We take the polarization on \mathbf{B}' to be the composite of a sequence of quasi-isogenies

$$\lambda_{\mathbf{B}'} : \mathbf{B}' \leftarrow \mathbf{A}'_{K'_1} \xrightarrow{\nu_g^+ \lambda_{\mathbf{A}'}} (\mathbf{A}'_{K'_1})^\vee \rightarrow \mathbf{B}'^\vee.$$

Such modification ensures that \mathbf{B}' satisfies condition (c1) of Theorem 3.14.

The moduli problem then implies that $\mathbf{B}' \cong (H_{\tilde{g}})^*(\mathbf{A}'_{K'_2})$ for a uniquely determined morphism $H_{\tilde{g}} : \mathbf{Sh}_{K'_1}(G'_S) \rightarrow \mathbf{Sh}_{K'_2}(G'_S)$. This gives the action of \tilde{G}_S on $\mathbf{Sh}_{K'_p}(G'_S)$. Moreover, we have a quasi-isogeny

$$H_{\tilde{g}}^\# : \mathbf{A}'_{K'_1} \xrightarrow{\eta} \mathbf{B}' \cong (H_{\tilde{g}})^*(\mathbf{A}'_{K'_2})$$

giving rise to an equivariant action of \tilde{G}_S on the universal abelian varieties $\mathbf{A}'_{K'_p}$ over $\mathbf{Sh}_{K'_p}(G'_S)$.

One easily checks that the action of the diagonal $\mathcal{O}_{E,(p)}^\times$ on the Shimura variety $\mathbf{Sh}_{K'_p}(G'_S)$ is trivial and hence we have an action of $\mathcal{G}'_S = \tilde{G}_S/\mathcal{O}_{E,(p)}^{\times,\text{cl}}$ on $\mathbf{Sh}_{K'_p}(G'_S)$. However, the action of $\mathcal{O}_{E,(p)}^\times$ on the universal abelian variety $\mathbf{A}'_{K'_p}$ is *not* trivial. So, the latter does not carry a natural action of \mathcal{G}'_S . So, our earlier framework for Shimura varieties does not apply to the universal abelian varieties directly. However, we observe that, by the construction at the end of § 2.11,

$$\mathbf{Sh}_{K'_p}(G''_S) = \mathbf{Sh}_{K'_p}(G'_S) \times_{\mathcal{G}'_S} \mathcal{G}''_S = \mathbf{Sh}_{K'_p}(G'_S) \times_{\tilde{G}_S} G''_S(\mathbb{A}^{\infty,p}).$$

So,

$$\mathbf{A}''_{K'_p} := \mathbf{A}'_{K'_p} \times_{\mathcal{G}'_S} \mathcal{G}''_S = \mathbf{A}'_{K'_p} \times_{\tilde{G}_S} G''_S(\mathbb{A}^{\infty,p})$$

gives a natural family of abelian variety over $\mathbf{Sh}_{K_p''}(G_{\mathbb{S}}'')$. We will not discuss families of abelian varieties over the quaternionic Shimura variety $\mathbf{Sh}_{K_p}(G_{\mathbb{S}})$ (except when $\mathbb{S} = \emptyset$).

3.21 Automorphic l -adic systems on $\mathbf{Sh}_{K_p''}(G_{\mathbb{S}}'')$ and their geometric interpretation

By a *multiweight*, we mean a tuple $(\underline{k}, w) = ((k_{\tau})_{\tau \in \Sigma_{\infty}}, w) \in \mathbb{N}^{[F:\mathbb{Q}]} \times \mathbb{N}$ such that $k_{\tau} \geq 2$ and $w \equiv k_{\tau} \pmod{2}$ for each τ . We also fix a section of the natural map $\Sigma_{E,\infty} \rightarrow \Sigma_{\infty}$, that is, to fix an extension $\tilde{\tau}$ to E of each real embedding $\tau \in \Sigma_{\infty}$ of F . Use $\tilde{\Sigma}$ to denote the image of this section. In this subsection, we use $\tilde{\tau}$ to denote this chosen lift of τ . We fix a subfield L of $\overline{\mathbb{Q}} \subset \mathbb{C}$ containing all embeddings of E , as the coefficient field.

Let l be a finite place of L over a prime l with $l \neq p$. Fix an isomorphism $\iota_l : \mathbb{C} \simeq \overline{L}_l$. Consider the injection

$$G_{\mathbb{S}}'' \times_{\mathbb{Q}} L = (\text{Res}_{F/\mathbb{Q}}(B_{\mathbb{S}}^{\times}) \times_{\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m} \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m) \times_{\mathbb{Q}} L \hookrightarrow \text{Res}_{E/\mathbb{Q}} D_{\mathbb{S}}^{\times} \times_{\mathbb{Q}} L \cong \prod_{\tau \in \Sigma_{\infty}} \text{GL}_{2,L,\tilde{\tau}} \times \text{GL}_{2,L,\tilde{\tau}^c},$$

where E^{\times} acts on $\text{GL}_{2,L,\tilde{\tau}}$ (respectively $\text{GL}_{2,L,\tilde{\tau}^c}$) through $\tilde{\tau}$ (respectively $\tilde{\tau}^c$). For a multiweight (\underline{k}, w) , we consider the following representation of $G_{\mathbb{S}}'' \times_{\mathbb{Q}} L$:

$$\rho_{\tilde{\mathbb{S}},\tilde{\Sigma}}''(\underline{k},w) = \bigotimes_{\tau \in \tilde{\Sigma}} \rho_{\tau}^{(k_{\tau},w)} \circ \text{pr}_{\tilde{\tau}} \quad \text{for } i = 1, 2 \text{ with } \rho_{\tau}^{(k_{\tau},w)} = \text{Sym}^{k_{\tau}-2} \otimes \det^{(w-k_{\tau})/2},$$

where τ is the restriction of $\tilde{\tau}$ to F , and $\text{pr}_{\tilde{\tau}}$ is the *contragredient* of the natural projection to the $\tilde{\tau}$ -component of $G_{\mathbb{S}}'' \times_{\mathbb{Q}} L \hookrightarrow \text{Res}_{E/\mathbb{Q}} D_{\mathbb{S}}^{\times} \times_{\mathbb{Q}} L$. Note that $\rho_{\tilde{\mathbb{S}},\tilde{\Sigma}}''(\underline{k},w)$ is trivial on the maximal anisotropic \mathbb{R} -split subtorus of the center of $G_{\mathbb{S}}''$, i.e. $\text{Ker}(\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \rightarrow \mathbb{G}_m)$. By [Mil90, ch. III, § 7], $\rho_{\tilde{\mathbb{S}},\tilde{\Sigma}}''(\underline{k},w)$ corresponds to a lisse L_l -sheaf $\mathcal{L}_{\tilde{\mathbb{S}},\tilde{\Sigma}}''(\underline{k},w)$ over the Shimura variety $\mathbf{Sh}_{K_p''}(G_{\mathbb{S}}'')$ compatible as the level structure changes.

We now give a geometric interpretation of this automorphic l -adic sheaf on $\mathbf{Sh}_{K_p''}(G_{\mathbb{S}}'')$. For this, we fix an isomorphism $D_{\mathbb{S}} \simeq M_2(E)$ and let $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{O}_E)$ denote the idempotent element. Let $\mathbf{A}'' = \mathbf{A}_{\mathbb{S},K_p''}''$ denote the natural family of abelian varieties constructed in § 3.20. Let $V(\mathbf{A}'')$ denote the l -adic Tate module of \mathbf{A}'' . We then have a decomposition

$$V(\mathbf{A}'') \otimes_{\mathbb{Q}_l} L_l \cong \bigoplus_{\tau \in \Sigma_{\infty}} (V(\mathbf{A}'')_{\tilde{\tau}} \oplus V(\mathbf{A}'')_{\tilde{\tau}^c}) = \bigoplus_{\tau \in \Sigma_{\infty}} (V(\mathbf{A}'')_{\tilde{\tau}}^{\circ,\oplus 2} \oplus V(\mathbf{A}'')_{\tilde{\tau}^c}^{\circ,\oplus 2}),$$

where $V(\mathbf{A}'')_{\tilde{\tau}}$ (respectively $V(\mathbf{A}'')_{\tilde{\tau}^c}$) is the component where \mathcal{O}_E acts through $\iota_l \circ \tilde{\tau}$ (respectively $\iota_l \circ \tilde{\tau}^c$) and $V(\mathbf{A}'')_{\tilde{\tau}}^{\circ} := \epsilon V(\mathbf{A}'')_{\tilde{\tau}}$ (respectively $V(\mathbf{A}'')_{\tilde{\tau}^c}^{\circ} := \epsilon V(\mathbf{A}'')_{\tilde{\tau}^c}$) is a lisse L_l -sheaf of rank two. For a multiweight (\underline{k}, w) , we put

$$\mathcal{L}_{\tilde{\mathbb{S}},\tilde{\Sigma}}''(\underline{k},w) = \bigotimes_{\tau \in \tilde{\Sigma}} (\text{Sym}^{k_{\tau}-2} V(\mathbf{A}'')_{\tilde{\tau}}^{\circ,\vee} \otimes (\wedge^2 V(\mathbf{A}'')_{\tilde{\tau}}^{\circ,\vee})^{(w-k_{\tau})/2}).$$

Note that the duals on the Tate modules mean that we are essentially taking the relative first étale *cohomology*. The moduli interpretation implies that we have a canonical isomorphism

$$\mathcal{L}_{\tilde{\mathbb{S}},\tilde{\Sigma}}''(\underline{k},w) \cong \mathcal{L}_{\tilde{\Sigma}}(\underline{k},w)(\mathbf{A}_{\mathbb{S}}'').$$

3.22 Twisted partial Frobenius

The action of the twisted partial Frobenius and its compatibility with the GO-strata description will be important to later applications in [TX13]. We first define the twisted partial Frobenius on the universal abelian scheme $\mathbf{A}'_{\tilde{S}} = \mathbf{A}'_{\tilde{S}, K'}$ over the unitary Shimura variety $\mathbf{Sh}_{K'}(G'_{\tilde{S}})$.

Fix $\mathfrak{p} \in \Sigma_p$. We define an action of $\sigma_{\mathfrak{p}}$ on $\Sigma_{E, \infty}$ as follows: for $\tilde{\tau} \in \Sigma_{E, \infty}$, we put

$$\sigma_{\mathfrak{p}}\tilde{\tau} = \begin{cases} \sigma \circ \tilde{\tau} & \text{if } \tilde{\tau} \in \Sigma_{E, \infty/\mathfrak{p}}, \\ \tilde{\tau} & \text{if } \tilde{\tau} \notin \Sigma_{E, \infty/\mathfrak{p}}, \end{cases} \tag{3.22.1}$$

where $\Sigma_{E, \infty/\mathfrak{p}}$ denotes the lifts of places in $\Sigma_{\infty/\mathfrak{p}}$. Note that $\sigma_{\mathfrak{p}}$ induces a natural action on Σ_{∞} , and $\prod_{\mathfrak{p} \in \Sigma_p} \sigma_{\mathfrak{p}} = \sigma$ is the Frobenius action. Let $\sigma_{\mathfrak{p}}\tilde{S}$ denote the image of \tilde{S} under $\sigma_{\mathfrak{p}}$. Note that a prime $\mathfrak{p}' \in \Sigma_p$ has the same type with respect to \tilde{S} or $\sigma_{\mathfrak{p}}\tilde{S}$. We fix an isomorphism $B_{\sigma_{\mathfrak{p}}\tilde{S}} \otimes \mathbb{A}^{\infty} \simeq B_{\tilde{S}} \otimes \mathbb{A}^{\infty}$, which induces in turn an isomorphism $G'_{\sigma_{\mathfrak{p}}\tilde{S}}(\mathbb{A}^{\infty}) \simeq G'_{\tilde{S}}(\mathbb{A}^{\infty})$. We regard K' as an open subgroup of $G'_{\sigma_{\mathfrak{p}}\tilde{S}}(\mathbb{A}^{\infty})$ and have a well-defined unitary Shimura variety $\mathbf{Sh}_{K'}(G'_{\sigma_{\mathfrak{p}}\tilde{S}})$. We also point out that the reflex fields for $G'_{\tilde{S}}$ and for $G'_{\sigma_{\mathfrak{p}}\tilde{S}}$ have the same completion at $\tilde{\varphi}$.

Let S be a locally noetherian $k_{\tilde{\varphi}}$ -scheme and let $(A, \iota, \lambda, \bar{\alpha}_{K'})$ be an S -point on $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_{\tilde{\varphi}}}$. We will define a new S -point $(A', \iota', \lambda', \bar{\alpha}_{K'})$ on $\mathbf{Sh}_{K'}(G'_{\sigma_{\mathfrak{p}}\tilde{S}})_{k_{\tilde{\varphi}}}$ as follows. The kernel of the relative p^2 -Frobenius $\text{Fr}_A: A \rightarrow A^{(p^2/S)}$ carries an action of \mathcal{O}_F , and we denote by Ker_{p^2} its \mathfrak{p} -component. We put $A' = (A/\text{Ker}_{p^2}) \otimes_{\mathcal{O}_F} \mathfrak{p}$ with its induced action by \mathcal{O}_{D_S} . It also comes equipped with a quasi-isogeny η given by the composite

$$\eta : A \longrightarrow A/\text{Ker}_{p^2} \longleftarrow (A/\text{Ker}_{p^2}) \otimes_{\mathcal{O}_F} \mathfrak{p} = A'.$$

It induces canonical isomorphisms of p -divisible groups $A'[\mathfrak{q}^{\infty}] \simeq A[\mathfrak{q}^{\infty}]$ for $\mathfrak{q} \in \Sigma_p$ with $\mathfrak{q} \neq \mathfrak{p}$, and $A'[\mathfrak{p}^{\infty}] \simeq A[\mathfrak{p}^{\infty}]^{(p^2)}$. From this, one can easily check the signature condition for A' . We define the polarization λ' to be the quasi-isogeny defined by the composite

$$A' \xleftarrow{\eta} A \xrightarrow{\lambda} A^{\vee} \xleftarrow{\eta^{\vee}} A'^{\vee}. \tag{3.22.2}$$

We have to check that λ' is a genuine isogeny, and it verifies condition Theorem 3.14(b) on λ' at the prime \mathfrak{p} . By the flatness criterion by fibers, it suffices to do this after base change to every geometric point of S . We may thus suppose that $S = \text{Spec}(k)$ for an algebraically closed field k of characteristic p . Let $\tilde{D}(A)_{\mathfrak{p}}$ be the covariant Dieudonné module of the p -divisible group $A[\mathfrak{p}^{\infty}]$ and define $\tilde{D}(A')_{\mathfrak{p}}$ similarly. By definition, we have

$$\tilde{D}(A')_{\mathfrak{p}} = p\tilde{D}(A/\text{Ker}_{p^2})_{\mathfrak{p}} = pV^{-2}\tilde{D}(A)_{\mathfrak{p}} = p^{-1}F^2\tilde{D}(A)_{\mathfrak{p}},$$

where $pV^{-2}\tilde{D}(A)_{\mathfrak{p}}$ means the inverse image of $\tilde{D}(A)_{\mathfrak{p}}$ under the bijective endomorphism V^2 on $\tilde{D}(A)_{\mathfrak{p}}[1/p]$. Applying the Dieudonné functor to (3.22.2), we get

$$\lambda'_* : \tilde{D}(A')_{\mathfrak{p}} = pV^{-2}\tilde{D}(A)_{\mathfrak{p}} \xleftarrow{\eta_*} \tilde{D}(A)_{\mathfrak{p}} \xrightarrow{\lambda_*} \tilde{D}(A^{\vee})_{\mathfrak{p}} \xleftarrow{\eta_*^{\vee}} \tilde{D}(A'^{\vee})_{\mathfrak{p}} = p^{-1}F^2\tilde{D}(A^{\vee})_{\mathfrak{p}}.$$

Now it is easy to see that λ' is an isogeny, and the condition in Theorem 3.14(b) on λ' follows from that on λ . The tame level structure $\bar{\alpha}'_{K'}$ is given by the composition

$$\widehat{\Lambda}^{(p)} \xrightarrow{\alpha_{K'}} T^{(p)}A \xrightarrow{\cong} T^{(p)}(A/\text{Ker}_{p^2}) \xleftarrow{\cong} T^{(p)}((A/\text{Ker}_{p^2}) \otimes_{\mathcal{O}_F} \mathfrak{p}) = T^{(p)}(A').$$

We are left to define the subgroups $\alpha'_{\mathfrak{p}'}$ for all $\mathfrak{p}' \in \Sigma_p$ of type $\alpha^\#$. The definition is clear for $\mathfrak{p}' \neq \mathfrak{p}$, since $A'[\mathfrak{p}'^\infty]$ is canonically identified with $A[\mathfrak{p}'^\infty]$. Assume thus that $\mathfrak{p}' = \mathfrak{p}$ is of type $\alpha^\#$. In the data of $\alpha'_{\mathfrak{p}} = H'_q \oplus H'_q \subseteq A'[\mathfrak{p}]$, the subgroup H'_q is determined as the orthogonal complement of H'_q under the Weil pairing on $A[\mathfrak{p}]$. Therefore, it suffices to construct H'_q or equivalently an \mathcal{O}_{D_S} -isogeny $f' : A' \rightarrow B' = A'/H'_q$ with kernel in $A'[q]$ of degree $\#k_{\mathfrak{p}}^2$. Let $f : A \rightarrow B = A/H_q$ be the isogeny given by $\alpha_{\mathfrak{p}}$. We write $\text{Ker}_{\mathfrak{p}^2, B}$ for the \mathfrak{p} -component of the kernel of the relative p^2 -Frobenius $B \rightarrow B^{(p^2)}$. It is easy to see that we have a natural isogeny $f_{\mathfrak{p}^2} : A/\text{Ker}_{\mathfrak{p}^2} \rightarrow B/\text{Ker}_{\mathfrak{p}^2, B}$. Then H'_q is defined to be the kernel of

$$f_{\mathfrak{p}^2} \otimes 1 : A' = (A/\text{Ker}_{\mathfrak{p}^2}) \otimes \mathfrak{p} \longrightarrow (B/\text{Ker}_{\mathfrak{p}^2, B}) \otimes \mathfrak{p} =: B',$$

and $\alpha'_{\mathfrak{p}}$ is the direct sum of H'_q and its orthogonal dual H'_q .

To sum up, we obtain a morphism

$$\mathfrak{F}'_{\mathfrak{p}^2} : \mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_{\tilde{\varphi}}} \rightarrow \mathbf{Sh}_{K'}(G'_{\sigma_{\tilde{S}}^2})_{k_{\tilde{\varphi}}}. \tag{3.22.3}$$

In all cases, we call the morphism $\mathfrak{F}'_{\mathfrak{p}^2}$ the *twisted partial Frobenius map* on the unitary Shimura varieties. Moreover, if $\mathbf{A}'_{\tilde{S}}$ and $\mathbf{A}'_{\sigma_{\tilde{S}}^2}$ are respectively the universal abelian schemes over $\mathbf{Sh}_{K'}(G'_{\tilde{S}})$ and $\mathbf{Sh}_{K'}(G'_{\sigma_{\tilde{S}}^2})$, we have the following universal quasi-isogeny:

$$\eta'_{\mathfrak{p}^2} : \mathbf{A}'_{\tilde{S}, k_{\tilde{\varphi}}} \longrightarrow \mathfrak{F}'_{\mathfrak{p}^2,*}(\mathbf{A}'_{\sigma_{\tilde{S}}^2, k_{\tilde{\varphi}}}). \tag{3.22.4}$$

It is clear from the definition that the $(\mathfrak{F}'_{\mathfrak{p}^2}, \eta'_{\mathfrak{p}^2})$ for different $\mathfrak{p} \in \Sigma_p$ commute with each other.

Let $S_p : \mathbf{Sh}_{K'}(G'_{\tilde{S}}) \rightarrow \mathbf{Sh}_{K'}(G'_{\tilde{S}})$ be the automorphism defined by $(A, \iota, \lambda, \bar{\alpha}_{K'}) \mapsto (A, \iota, \lambda, p\bar{\alpha}_{K'})$. It is clear that $S_p^* \mathbf{A}'_{\tilde{S}} \cong \mathbf{A}'_{\tilde{S}}$. Hence, S_p induces an automorphism of the cohomology groups $H_{\text{rig}}^*(\mathbf{Sh}_{K'}(G'_{\tilde{S}}), \mathcal{D}_{\tilde{\Sigma}}^{(k,w)}(\mathbf{A}'_{\tilde{S}, k_0}))$, still denoted by S_p . If

$$F_{\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_{\tilde{\varphi}}}/k_{\tilde{\varphi}}}^2 : \mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_{\tilde{\varphi}}} \longrightarrow \mathbf{Sh}_{K'}(G'_{\sigma_{\tilde{S}}^2})_{k_{\tilde{\varphi}}} \simeq \mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_{\tilde{\varphi}}}^{(p^2)}$$

denotes the relative p^2 -Frobenius, then we have $S_p^{-1} \circ F_{\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_{\tilde{\varphi}}}/k_{\tilde{\varphi}}}^2 = \prod_{\mathfrak{p} \in \Sigma_p} \mathfrak{F}'_{\mathfrak{p}^2}$. Similarly, if $[p] : \mathbf{A}'_{\tilde{S}}^{(p^2)} \rightarrow \mathbf{A}'_{\tilde{S}}^{(p^2)}$ denotes the multiplication by p and

$$F_A^2 : \mathbf{A}'_{\tilde{S}, k_{\tilde{\varphi}}} \rightarrow (F_{\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_{\tilde{\varphi}}}/k_{\tilde{\varphi}}}^2)^*(\mathbf{A}'_{\sigma_{\tilde{S}}^2, k_{\tilde{\varphi}}}) \cong \mathbf{A}'_{\tilde{S}, k_{\tilde{\varphi}}}^{(p^2)}$$

denotes the p^2 -Frobenius homomorphism, we have $[p]^{-1} \circ F_A^2 = \prod_{\mathfrak{p} \in \Sigma_p} \eta'_{\mathfrak{p}^2}$.

Finally, we note that all the discussions above are equivariant with respect to the action of the Galois group and the action of $\tilde{G}_{\tilde{S}} = G''_{\tilde{S}}(\mathbb{Q})^{+, (p)} G'_{\tilde{S}}(\mathbb{A}^{\infty, p}) \simeq G''_{\sigma_{\tilde{S}}^2}(\mathbb{Q})^{+, (p)} G'_{\sigma_{\tilde{S}}^2}(\mathbb{A}^{\infty, p})$ when passing to the limit. (The isomorphism follows from the description of the group $\tilde{G}_{\tilde{S}}$ in (3.12.2).) So, applying $- \times_{\tilde{G}_{\tilde{S}}} G''_{\tilde{S}}(\mathbb{A}^{\infty, p})$ to the construction gives the following proposition.

PROPOSITION 3.23. *Let $\mathbf{A}''_{\tilde{S}}$ denote the natural family of abelian varieties over $\mathbf{Sh}_{K''}(G''_{\tilde{S}})$. We identify the level structure for $G''_{\tilde{S}}$ with that of $G''_{\sigma_{\tilde{S}}^2}$ similarly. Then, for each $\mathfrak{p} \in \Sigma_p$, we have a $G''_{\tilde{S}}(\mathbb{A}^{\infty, p})$ -equivariant natural twisted partial Frobenius morphism and an quasi-isogeny of family of abelian varieties:*

$$\mathfrak{F}''_{\mathfrak{p}^2} : \mathbf{Sh}_{K''}(G''_{\tilde{S}})_{k_{\tilde{\varphi}}} \longrightarrow \mathbf{Sh}_{K''}(G''_{\sigma_{\tilde{S}}^2})_{k_{\tilde{\varphi}}} \quad \text{and} \quad \eta''_{\mathfrak{p}^2} : \mathbf{A}''_{\tilde{S}, k_{\tilde{\varphi}}} \longrightarrow \mathfrak{F}''_{\mathfrak{p}^2,*}(\mathbf{A}''_{\sigma_{\tilde{S}}^2, k_{\tilde{\varphi}}}).$$

This induces a natural $G_{\mathbb{S}}''(\mathbb{A}^{\infty,p})$ -equivariant homomorphism of étale cohomology groups:

$$\begin{CD}
 H_{\text{et}}^*(\mathbf{Sh}_{K_p''}(G_{\sigma_p^2\mathbb{S}}'')_{\mathbb{F}_p}, \mathcal{L}_{\Sigma}^{(k,w)}(\mathbf{A}_{\sigma_p^2\mathbb{S}}'')) @>\tilde{\mathfrak{F}}_{p^2}''>> H_{\text{et}}^*(\mathbf{Sh}_{K_p''}(G_{\mathbb{S}}'')_{\mathbb{F}_p}, \mathcal{L}_{\Sigma}^{(k,w)}(\tilde{\mathfrak{F}}_{p^2}''\mathbf{A}_{\sigma_p^2\mathbb{S}}'')) \\
 @V\Phi_{p^2}VV @VV\eta_{p^2}''V \\
 H_{\text{et}}^*(\mathbf{Sh}_{K_p''}(G_{\mathbb{S}}'')_{\mathbb{F}_p}, \mathcal{L}_{\Sigma}^{(k,w)}(\mathbf{A}_{\mathbb{S}}'')) @. H_{\text{et}}^*(\mathbf{Sh}_{K_p''}(G_{\mathbb{S}}'')_{\mathbb{F}_p}, \mathcal{L}_{\Sigma}^{(k,w)}(\mathbf{A}_{\mathbb{S}}'')).
 \end{CD}$$

Moreover, we have an equality of morphisms

$$\prod_{p \in \Sigma_p} \Phi_{p^2} = S_p^{-1} \circ F^2 : H_{\text{et}}^*(\mathbf{Sh}_{K_p''}(G_{\sigma_p^2\mathbb{S}}'')_{\mathbb{F}_p}, \mathcal{L}_{\Sigma}^{(k,w)}(\mathbf{A}_{\sigma_p^2\mathbb{S}}'')) \longrightarrow H_{\text{et}}^*(\mathbf{Sh}_{K_p''}(G_{\mathbb{S}}'')_{\mathbb{F}_p}, \mathcal{L}_{\Sigma}^{(k,w)}(\mathbf{A}_{\mathbb{S}}'')),$$

where F^2 is the relative p^2 -Frobenius and S_p is the Hecke action given by multiplication by \underline{p}^{-1} . Here \underline{p} is the idele element which is p at all places above p and 1 elsewhere.

Proof. This is clear from the construction. □

3.24 Comparison with the Hilbert modular varieties

When $\mathbb{S} = \emptyset$, we have $G_{\emptyset} = \text{Res}_{F/\mathbb{Q}}(\text{GL}_{2,F})$ and $K_p = \text{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. It is well known that $\text{Sh}_{K_p}(G_{\emptyset}) = \varprojlim_{K^p} \text{Sh}_{K^p K_p}(G_{\emptyset})$ is a projective system of Shimura varieties defined over \mathbb{Q} , and it parametrizes polarized Hilbert–Blumenthal abelian varieties (HBAVs for short) with prime-to- p level structure. Using this moduli interpretation, one can construct an integral canonical model over \mathbb{Z}_p of $\text{Sh}_{K_p}(G_{\emptyset})$ as in [Rap78, Lan13]. By the uniqueness of the integral canonical model, we know that this classical integral model is isomorphic to $\mathbf{Sh}_{K_p}(G_{\emptyset})$ constructed in Corollary 3.19. For this ‘abstract’ isomorphism to be useful in applications, we need to relate the universal HBAV \mathcal{A} on $\mathbf{Sh}_{K_p}(G_{\emptyset})$ and the abelian scheme \mathbf{A}_{\emptyset}'' on $\mathbf{Sh}_{K_p''}(G_{\emptyset}'')$ constructed at the end of § 3.20.

Let $G_{\emptyset}^* \subseteq G_{\emptyset}$ be the inverse image of $\mathbb{G}_{m,\mathbb{Q}} \subseteq T_F = \text{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,\mathbb{Q}})$ via the determinant map $\nu : G_{\emptyset} \rightarrow T_F$. The homomorphism $h_{\emptyset} : \mathbb{C}^{\times} \rightarrow G_{\emptyset}(\mathbb{R})$ factors through $G_{\emptyset}^*(\mathbb{R})$. We can talk about the Shimura variety associated to $(G_{\emptyset}^*, h_{\emptyset})$. We put $K_p^* = K_p \cap G_{\emptyset}^*(\mathbb{Q}_p)$. Then, by Corollary 2.17, $\text{Sh}_{K_p}(G_{\emptyset})$ and $\text{Sh}_{K_p^*}(G_{\emptyset}^*)$ have isomorphic neutral connected components $\text{Sh}_{K_p}(G_{\emptyset})_{\mathbb{Q}_p}^{\circ} \simeq \text{Sh}_{K_p^*}(G_{\emptyset}^*)_{\mathbb{Q}_p}^{\circ}$. The Shimura variety $\text{Sh}_{K_p^*}(G_{\emptyset}^*)$ is of PEL type, and the universal abelian scheme on $\text{Sh}_{K_p^*}(G_{\emptyset}^*)_{\mathbb{Q}_p}^{\circ}$ is identified with that on $\text{Sh}_{K_p}(G_{\emptyset})_{\mathbb{Q}_p}^{\circ}$ via the isomorphism above. Actually, if $K^p \subseteq \text{GL}_2(\mathbb{A}_F^{\infty,p})$ is an open compact subgroup such that $\det(K^p \cap \mathcal{O}_F^{\times}) = \det(K^p) \cap \mathcal{O}_F^{\times,+}$, where $\mathcal{O}_F^{\times,+}$ denotes the set of totally positive units of F , then $\text{Sh}_{K^p K_p}(G_{\emptyset})$ is isomorphic to a finite union of $\text{Sh}_{K^* p K_p^*}(G_{\emptyset}^*)$ for some appropriate tame level structure $K^* p$ (see [Hid04, 4.2.1] or [TX13, Proposition 2.4]).

We now describe the Hilbert moduli problem that defines an integral canonical model of $\text{Sh}_{K_p^*}(G_{\emptyset}^*)$. Let $\widehat{\mathcal{O}}_F^{(p)} = \prod_{v \nmid p\infty} \mathcal{O}_{F_v}$ and put $\widehat{\Lambda}_F^{(p)} = \widehat{\mathcal{O}}_F^{(p)} e_1 \oplus \widehat{\mathcal{O}}_F^{(p)} \mathfrak{d}_F^{-1} e_2$. We endow $\widehat{\Lambda}_F^{(p)}$ with the symplectic form

$$\psi_F(a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2) = \text{Tr}_{F/\mathbb{Q}}(a_2 b_1 - a_1 b_2) \in \widehat{\mathbb{Z}}^{(p)}$$

for $a_1, b_1 \in \widehat{\mathcal{O}}_F^{(p)}$ and $a_2, b_2 \in \mathfrak{d}_F^{-1} \widehat{\mathcal{O}}_F^{(p)}$. It is an elementary fact that every rank-two free $\widehat{\mathcal{O}}_F^{(p)}$ -module together with a perfect $\widehat{\mathbb{Z}}^{(p)}$ -linear \mathcal{O}_F -hermitian symplectic form is isomorphic to $(\widehat{\Lambda}_F^{(p)}, \psi_F)$.

Let $K_p^* = \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) \cap G_\theta^*(\mathbb{Q}_p)$ be as above. For an open compact subgroup K^{*p} , we put $K^* = K^{*p}K_p^*$. Assume that K^{*p} stabilizes the lattice $\widehat{\Lambda}_F^{(p)}$. We consider the functor that associates to each locally noetherian \mathbb{Z}_p -scheme S the set of isomorphism classes of quadruples $(A, \iota, \lambda, \alpha_{K^{*p}})$, where:

- (i) (A, ι) is a HBAV, i.e. an abelian scheme A/S of dimension $[F : \mathbb{Q}]$ equipped with a homomorphism $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}_S(A)$;
- (ii) $\lambda : A \rightarrow A^\vee$ is an \mathcal{O}_F -linear $\mathbb{Z}_{(p)}^\times$ -polarization in the sense of [Lan13, 1.3.2.19];
- (iii) choosing a geometric point \bar{s}_i for each connected component S_i of S , $\alpha_{K^{*p}}$ is a collection of $\pi_1(S_i, \bar{s}_i)$ -invariant K^{*p} -orbits of $\widehat{\mathcal{O}}_F^{(p)}$ -linear isomorphisms $\widehat{\Lambda}_F^{(p)} \xrightarrow{\sim} T^{(p)}(A_{\bar{s}_i})$, sending the symplectic pairing ψ_F on the former to the λ -Weil pairing on the latter for some identification of $\widehat{\mathbb{Z}}^{(p)}$ with $\widehat{\mathbb{Z}}^{(p)}(1)$.

This functor is representable by a quasi-projective and smooth scheme $\mathbf{Sh}_{K^*}(G_\theta^*)$ over \mathbb{Z}_p such that $\mathbf{Sh}_{K^*}(G_\theta^*)_{\mathbb{Q}_p} \simeq \mathrm{Sh}_{K^*}(G_\theta^*)$ [Rap78] and [Lan13, 1.4.1.11]. By the same arguments of [Moo98, Corollary 3.8], it is easy to see that $\mathbf{Sh}_{K_p^*}(G_\theta^*)$ satisfies the extension property § 2.4. This then gives rise to an integral canonical model $\mathbf{Sh}_{K_p^*}(G_\theta)$ of $\mathrm{Sh}_{K_p}(G_\theta)$. We could pull back the universal abelian variety \mathcal{A}^* over $\mathbf{Sh}_{K_p^*}(G_\theta^*)$ to a family of abelian varieties over $\mathbf{Sh}_{K_p}(G_\theta)$ using [Hid04, 4.2.1] cited above. But we prefer to do it more canonically following the same argument as in § 3.20. More precisely, there is a natural equivariant action of $\widetilde{G}^* := G_\theta(\mathbb{Q})_+^{(p)} \cdot G^*(\mathbb{A}^{\infty,p})$ on the universal abelian variety \mathcal{A}^* over $\mathbf{Sh}_{K_p^*}(G_\theta^*)$. Then

$$\mathcal{A} := \mathcal{A}^* \times_{\widetilde{G}^*} \mathrm{GL}_2(\mathbb{A}^{\infty,p}) \tag{3.24.1}$$

gives a natural family of abelian variety over $\mathbf{Sh}_{K_p}(G_\theta)$.

The natural homomorphism $\mathrm{GL}_{2,F} \rightarrow \mathrm{GL}_{2,F} \times_{F^\times} E^\times$ induces a closed immersion of algebraic groups $G_\theta^* \rightarrow G'_\theta$ compatible with the Deligne homomorphisms h_θ and h'_θ . Therefore, one obtains a map of (projective systems of) Shimura varieties $f : \mathrm{Sh}_{K_p^*}(G_\theta^*) \rightarrow \mathrm{Sh}_{K'_p}(G'_\theta)$, which induces an isomorphism of the neutral connected component $\mathrm{Sh}_{K_p^*}(G_\theta^*)_{\mathbb{Q}_p}^{\circ} \simeq \mathrm{Sh}_{K'_p}(G'_\theta)_{\mathbb{Q}_p}^{\circ}$. We will extend f to a map of integral models $\mathbf{Sh}_{K_p^*}(G_\theta^*) \rightarrow \mathbf{Sh}_{K'_p}(G'_\theta)$.

Now let E be a CM extension of F unramified at p as before. In the process of constructing the pairing ψ on D_θ , we may take δ_θ to be $\begin{pmatrix} 0 & -1/\sqrt{\mathfrak{d}} \\ 1/\sqrt{\mathfrak{d}} & 0 \end{pmatrix}$, which is coprime to p , where \mathfrak{d} is the totally negative element chosen in § 3.7. It is easy to check that it satisfies the conditions in Lemma 3.8(1), and the $*$ -involution given by δ_θ on $D_\theta = \mathrm{M}_2(E)$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$ for $a, b, c, d \in E$. The $*$ -hermitian pairing on $\mathrm{M}_2(E)$ is given by

$$\begin{aligned} \psi(v, w) &= \mathrm{Tr}_{\mathrm{M}_2(E)/\mathbb{Q}} \left(v\bar{w} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \quad \text{for } v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \text{ and } w = \begin{pmatrix} a_w & b_w \\ c_w & d_w \end{pmatrix} \in \mathrm{M}_2(E) \\ &= \mathrm{Tr}_{E/\mathbb{Q}}(b_v\bar{a}_w - a_v\bar{b}_w + d_v\bar{c}_w - c_v\bar{d}_w). \end{aligned}$$

In defining the PEL data for G'_θ , we take the \mathcal{O}_{D_θ} -lattice Λ to be $\begin{pmatrix} \mathcal{O}_E & \mathfrak{d}_F^{-1}\mathcal{O}_E \\ \mathcal{O}_E & \mathfrak{d}_F^{-1}\mathcal{O}_E \end{pmatrix}$. Clearly, $\widehat{\Lambda}^{(p)} = \Lambda \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$ satisfies $\widehat{\Lambda}^{(p)} \subseteq \widehat{\Lambda}^{(p),\vee}$ for the bilinear form ψ above. Moreover, if we equip $\Lambda_F^{(p)} \otimes_{\mathcal{O}_F} \mathcal{O}_E$ with the symplectic form $\psi_E = \psi_F(\mathrm{Tr}_{E/F}(\bullet), \mathrm{Tr}_{E/F}(\bullet))$, then $(\widehat{\Lambda}^{(p)}, \psi)$ is isomorphic to $((\widehat{\Lambda}_F^{(p)} \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2}, \psi_E^{\oplus 2})$ as a $*$ -hermitian symplectic $\mathrm{M}_2(\mathcal{O}_E)$ -module.

PROPOSITION 3.25. *For any open compact subgroup K'^p of $G'_\theta(\mathbb{A}^{\infty,p})$, we put $K^{*p} = K'^p \cap G_\theta^*(\mathbb{A}^{\infty,p})$. Then we have a canonical morphism*

$$\mathbf{f} : \mathbf{Sh}_{K^{*p}K_p^*}(G_\theta^*) \rightarrow \mathbf{Sh}_{K'^pK'_p}(G'_\theta)$$

such that, if \mathcal{A} and \mathbf{A}'_\emptyset denote respectively the universal abelian scheme on $\mathbf{Sh}_{K^{*p}K_p^*}(G_\emptyset^*)$ and that on $\mathbf{Sh}_{K'^pK'_p}(G'_\emptyset)$, then we have an isomorphism of abelian schemes $\mathbf{f}^*\mathbf{A}'_\emptyset \cong (\mathcal{A} \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2}$ compatible with the natural action of $M_2(\mathcal{O}_E)$ and polarizations on both sides. By passing to the limit, the morphism \mathbf{f} induces an isomorphism between the integral models of connected Shimura varieties $\mathbf{Sh}_{K_p^*}(G_\emptyset^*)_{\mathbb{Z}^{\text{ur}}} \cong \mathbf{Sh}_{K'_p}(G'_\emptyset)_{\mathbb{Z}^{\text{ur}}}$.

Proof. By Galois descent, it is enough to work over $W(k_0)$ for k_0 in Theorem 3.14. Let S be a locally noetherian $W(k_0)$ -scheme and $x = (A, \iota, \lambda, \alpha_{K^{*p}})$ be an S -valued point of $\mathbf{Sh}_{K^{*p}K_p^*}(G_\emptyset^*)$. We define its image $f(x) = (A', \iota', \lambda', \alpha_{K'^p})$ as follows. We take $A' = (A \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2}$ equipped with the naturally induced action ι' of $M_2(\mathcal{O}_E)$. It is clear that $\text{Lie}(A')_{\tilde{\tau}}$ is an \mathcal{O}_S -module locally free of rank two for all $\tilde{\tau} \in \Sigma_{E,\infty}$. The prime-to- p polarization λ' on A' is defined to be

$$\lambda' : A' \xrightarrow{\sim} (A \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2} \xrightarrow{(\lambda \otimes 1)^{\oplus 2}} (A^\vee \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2} \cong A'^\vee \otimes_{\delta_{E/F}} \rightarrow A'^\vee,$$

where $\delta_{E/F}$ is the different ideal of E over F . We define the K'^p -level structure to be the K'^p -orbit of the isomorphism

$$\alpha_{K'^p} : \widehat{\Lambda}^{(p)} \xrightarrow{\cong} (\widehat{\Lambda}_F^{(p)} \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2} \xrightarrow{\alpha_{K^{*p}}^{\oplus 2}} (T^{(p)}(A_{\bar{s}}) \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2} \simeq T^{(p)}(A'_{\bar{s}}).$$

By the discussion before the proposition, it is clear that $\alpha_{K'^p}$ sends the symplectic form ψ on the left-hand side to the λ' -Weil pairing on the right. This defines the morphism \mathbf{f} from $\mathbf{Sh}_{K^{*p}K_p^*}(G_\emptyset^*)$ to $\mathbf{Sh}_{K'^pK'_p}(G'_\emptyset)$. By looking at the complex uniformization, we note that \mathbf{f} extends the morphism $f : \text{Sh}_{K^{*p}K_p^*}(G_\emptyset^*)_{\mathbb{Q}_p} \rightarrow \text{Sh}_{K'^pK'_p}(G'_\emptyset)_{\mathbb{Q}_p}$ defined previously by group theory. Since both $\mathbf{Sh}_{K_p^*}(G_\emptyset^*)$ and $\mathbf{Sh}_{K'_p}(G'_\emptyset)$ satisfy the extension property § 2.4, it follows that \mathbf{f} induces an isomorphism $\mathbf{Sh}_{K_p^*}(G_\emptyset^*)_{\mathbb{Z}^{\text{ur}}} \simeq \mathbf{Sh}_{K'_p}(G'_\emptyset)_{\mathbb{Z}^{\text{ur}}}$. \square

COROLLARY 3.26. *Let \mathcal{A} denote the universal HBAV over $\mathbf{Sh}_{K_p}(G_\emptyset)$ and \mathbf{A}''_\emptyset be the family of abelian varieties over $\mathbf{Sh}_{K''_p}(G''_\emptyset)$ defined in § 3.20. Then, under the natural morphisms of Shimura varieties*

$$\mathbf{Sh}_{K_p}(G_\emptyset) \xleftarrow{\text{pr}_1} \mathbf{Sh}_{K_p \times_{K_{E,p}}}(G_\emptyset \times T_{E,\emptyset}) \xrightarrow{\alpha} \mathbf{Sh}_{K''_p}(G''_\emptyset), \tag{3.26.1}$$

one has an isomorphism of abelian schemes over $\mathbf{Sh}_{K_p \times_{K_{E,p}}}(G_\emptyset \times T_{E,\emptyset})$

$$\alpha^*\mathbf{A}''_\emptyset \cong (\text{pr}_1^*\mathcal{A} \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2} \tag{3.26.2}$$

compatible with the action of $M_2(\mathcal{O}_E)$ and prime-to- p polarizations.

Proof. This follows from the constructions of \mathcal{A} and \mathbf{A}''_\emptyset and the proposition above. \square

3.27 Comparison of the twisted partial Frobenius

Keep the notation as in § 3.24. The Shimura variety $\mathbf{Sh}_{K^*}(G_\emptyset^*)_{\mathbb{F}_p}$ also admits a twisted partial Frobenius $\Phi_{\mathfrak{p}^2}$ for each $\mathfrak{p} \in \Sigma_p$, which we define as follows. Let S be a locally noetherian \mathbb{F}_p -scheme. Given an S -point $(A, \iota, \lambda, \alpha_{K^{*p}})$ of $\mathbf{Sh}_{K^*}(G_\emptyset^*)_{\mathbb{F}_p}$, we associate a new point $(A', \iota', \lambda', \alpha'_{K^{*p}})$:

- $A' = A/\text{Ker}_{\mathfrak{p}^2} \otimes_{\mathcal{O}_F} \mathfrak{p}$, where $\text{Ker}_{\mathfrak{p}^2}$ is the \mathfrak{p} -component of the kernel of the relative Frobenius homomorphism $\text{Fr}_A^2 : A \rightarrow A^{(p^2)}$; it is equipped with the induced \mathcal{O}_F -action ι' ;
- using the natural quasi-isogeny $\eta : A \rightarrow A'$, λ' is given by the composite of quasi-isogenies $A' \xleftarrow{\eta} A \xrightarrow{\lambda} A^\vee \xrightarrow{\eta^\vee} A'^\vee$ (which is a $\mathbb{Z}_{(p)}^\times$ -isogeny by the same argument as in § 3.22);
- $\bar{\alpha}'_{K^{*p}}$ is the composite $\widehat{\Lambda}_F^{(p)} \xrightarrow{\bar{\alpha}_{K^{*p}}} T^{(p)}(A) \xleftarrow{\eta} T^{(p)}(A')$.

The construction above gives rise to a *twisted partial Frobenius morphism*

$$\mathfrak{F}_{p^2}^* : \mathbf{Sh}_{K^*}(G_\emptyset^*)_{\mathbb{F}_p} \longrightarrow \mathbf{Sh}_{K^*}(G_\emptyset^*)_{\mathbb{F}_p} \quad \text{and} \quad \eta_{p^2} : \mathcal{A}_{\mathbb{F}_p} \rightarrow (\mathfrak{F}_{p^2}^*)^* \mathcal{A}_{\mathbb{F}_p}.$$

Using the formalism of Shimura varieties (Corollary 2.17 and more specifically (3.24.1)), it gives rise to a *twisted partial Frobenius morphism*

$$\mathfrak{F}_{p^2}^\emptyset : \mathbf{Sh}_K(G_\emptyset)_{\mathbb{F}_p} \longrightarrow \mathbf{Sh}_K(G_\emptyset)_{\mathbb{F}_p} \quad \text{and} \quad \eta_{p^2}^\emptyset : \mathcal{A}_{\mathbb{F}_p} \rightarrow (\mathfrak{F}_{p^2}^\emptyset)^* \mathcal{A}_{\mathbb{F}_p}.$$

COROLLARY 3.28. *The twisted partial Frobenius morphism \mathfrak{F}_{p^2}'' on $\mathbf{Sh}_{K_p''}(G_\emptyset'')$ and the twisted partial Frobenius $\mathfrak{F}_{p^2}^\emptyset$ on $\mathbf{Sh}_{K_p}(G_\emptyset)_{\mathbb{F}_p}$ are compatible, in the sense that there exists a morphism $\tilde{\mathfrak{F}}_{p^2}$ so that both squares in the following commutative diagram are Cartesian.*

$$\begin{array}{ccccc} \mathbf{Sh}_{K_p}(G_\emptyset)_{\mathbb{F}_p} & \xleftarrow{\text{pr}_1} & \mathbf{Sh}_{K_p \times K_{E,p}}(G_\emptyset \times T_{E,\emptyset})_{\mathbb{F}_p} & \xrightarrow{\alpha} & \mathbf{Sh}_{K_p''}(G_\emptyset'')_{\mathbb{F}_p} \\ \downarrow \mathfrak{F}_{p^2}^\emptyset & & \downarrow \tilde{\mathfrak{F}}_{p^2} & & \downarrow \mathfrak{F}_{p^2}'' \\ \mathbf{Sh}_{K_p}(G_\emptyset)_{\mathbb{F}_p} & \xleftarrow{\text{pr}_1} & \mathbf{Sh}_{K_p \times K_{E,p}}(G_\emptyset \times T_{E,\emptyset})_{\mathbb{F}_p} & \xrightarrow{\alpha} & \mathbf{Sh}_{K_p''}(G_\emptyset'')_{\mathbb{F}_p} \end{array}$$

Moreover, $\eta_{p^2}^\emptyset$ is compatible with η_{p^2}'' , in the sense that the following diagram commutes.

$$\begin{array}{ccc} \alpha^* \mathbf{A}_{\emptyset, \mathbb{F}_p}'' & \xrightarrow{(3.26.2)} & (\text{pr}_1^* \mathcal{A}_{\mathbb{F}_p} \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2} \\ \downarrow \alpha^*(\eta_{p^2}'') & & \eta_{p^2}^\emptyset \otimes 1 \downarrow \\ \alpha^* \tilde{\mathfrak{F}}_{p^2}''^* \mathbf{A}_{\emptyset, \mathbb{F}_p}'' & \xrightarrow{(3.26.2)} & (\text{pr}_1^* \tilde{\mathfrak{F}}_{p^2}''^* (\mathcal{A}_{\mathbb{F}_p}) \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2} \end{array}$$

Proof. This follows from the definition of the partial Frobenii in various situations and the comparison Proposition 3.25 above. □

4. Goren–Oort stratification

We define an analog of the Goren–Oort stratification on the special fibers of quaternionic Shimura varieties. This is first done for unitary Shimura varieties and then pulled back to the quaternionic ones. Unfortunately, the definition a priori depends on the auxiliary choice of CM field (as well as the signatures $s_{\tilde{\tau}}$). In the case of the Hilbert modular variety, we show that our definition of the GO-strata agrees with Goren–Oort’s original definition in [GO00] (and hence does not depend on the auxiliary choice of data).

4.1 Notation

Keep the notation as in the previous sections. Let k_0 be a finite extension of \mathbb{F}_p containing all residue fields of \mathcal{O}_E of characteristic p . Let $X' := \mathbf{Sh}_{K'}(G_\mathbb{S}')_{k_0}$ denote the base change to k_0 of the Shimura variety $\mathbf{Sh}_{K'}(G_\mathbb{S}')$ considered in Theorem 3.14.

Recall that $\mathfrak{e} \in \mathcal{O}_{D_s,p}$ corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ when identifying $\mathcal{O}_{D_s,p}$ with $M_2(\mathcal{O}_{E,p})$. For an abelian scheme A over a locally noetherian k_0 -scheme S carrying an action of \mathcal{O}_{D_s} , we have the reduced module of invariant differential 1-forms $\omega_{A/S}^\circ$, the reduced Lie algebra $\text{Lie}(A/S)^\circ$, and the reduced de Rham homology $H_1^{\text{dR}}(A/S)^\circ$ defined in § 3.16. Their $\tilde{\tau}$ -components $\omega_{A/S, \tilde{\tau}}^\circ$,

$\text{Lie}(A/S)_{\tilde{\tau}}^{\circ}$, and $H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ}$ for $\tilde{\tau} \in \Sigma_{E,\infty}$ fit in an exact sequence, called the *reduced Hodge filtration*,

$$0 \rightarrow \omega_{A^\vee/S,\tilde{\tau}}^{\circ} \rightarrow H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ} \rightarrow \text{Lie}(A/S)_{\tilde{\tau}}^{\circ} \rightarrow 0.$$

Let $A^{(p)}$ denote the base change of A via the absolute Frobenius on S . The Verschiebung $\text{Ver} : A^{(p)} \rightarrow A$ and the Frobenius morphism $\text{Fr} : A \rightarrow A^{(p)}$ induce respectively maps of coherent sheaves on S :

$$F_A : H_1^{\text{dR}}(A/S)^{\circ,(p)} \rightarrow H_1^{\text{dR}}(A/S)^{\circ} \quad \text{and} \quad V_A : H_1^{\text{dR}}(A/S)^{\circ} \rightarrow H_1^{\text{dR}}(A/S)^{\circ,(p)},$$

which are compatible with the action of \mathcal{O}_E . Here, for a coherent \mathcal{O}_S -module, $M^{(p)}$ denotes the base change $M \otimes_{\mathcal{O}_S, F_{\text{abs}}} \mathcal{O}_S$. If there is no confusion, we drop the subscript A from the notation and simply write F and V for the two maps. (Although the letter F also stands for the totally real field, we think that this should not cause confusion.) Moreover, we have

$$\text{Ker}(F) = \text{Im}(V) = (\omega_{A^\vee/S}^{\circ})^{(p)} \quad \text{and} \quad \text{Im}(F) = \text{Ker}(V) \cong \text{Lie}(A^{(p)}/S)^{\circ}.$$

Let $(A, \iota, \lambda, \alpha_{K'})$ be an S -valued point of $X' = \mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0}$. By Kottwitz' determinant condition 3.14(a), for each $\tilde{\tau} \in \Sigma_{E,\infty}$, $\text{Lie}(A/S)_{\tilde{\tau}}^{\circ}$ is a locally free \mathcal{O}_S -module of rank $s_{\tilde{\tau}}$. (The numbers $s_{\tilde{\tau}}$ are defined as in §3.4.) By duality, this implies that $\omega_{A^\vee/S,\tilde{\tau}}^{\circ}$ is locally free of rank $s_{\tilde{\tau}^c} = 2 - s_{\tilde{\tau}}$. Moreover, when $\tau \in \Sigma_{\infty/\mathfrak{p}}$ with \mathfrak{p} not of type β^{\sharp} , the universal polarization λ induces an isomorphism of locally free \mathcal{O}_S -modules

$$\omega_{A^\vee/S,\tilde{\tau}}^{\circ} \cong \omega_{A/S,\tilde{\tau}^c}^{\circ}. \tag{4.1.1}$$

4.2 Essential Frobenius and essential Verschiebung

We now define two very important morphisms: essential Frobenius and essential Verschiebung; we will often encounter later their variants for crystalline homology and Dieudonné modules, for which we shall simply refer to the similar construction given here.

Let $(A, \iota, \lambda, \alpha_{K'})$ be as above. For $\tilde{\tau} \in \Sigma_{E,\infty}$ lifting a place $\tau \in \mathbf{S}_{\infty}$, we define the *essential Frobenius* to be

$$F_{\text{es}} = F_{A,\text{es},\tilde{\tau}} : (H_1^{\text{dR}}(A/S)_{\sigma^{-1}\tilde{\tau}}^{\circ})^{(p)} = H_1^{\text{dR}}(A^{(p)}/S)_{\tilde{\tau}}^{\circ} \longrightarrow H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ}$$

$$x \longmapsto \begin{cases} F(x) & \text{when } s_{\sigma^{-1}\tilde{\tau}} = 1 \text{ or } 2, \\ V^{-1}(x) & \text{when } s_{\sigma^{-1}\tilde{\tau}} = 0. \end{cases} \tag{4.2.1}$$

Note that in the latter case, the morphism $V : H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ} \xrightarrow{\sim} H_1^{\text{dR}}(A^{(p)}/S)_{\tilde{\tau}}^{\circ}$ is an isomorphism by Kottwitz' determinant condition.

Similarly, we define the *essential Verschiebung* to be

$$V_{\text{es}} = V_{A,\text{es},\tilde{\tau}} : H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ} \longrightarrow H_1^{\text{dR}}(A^{(p)}/S)_{\tilde{\tau}}^{\circ} = (H_1^{\text{dR}}(A/S)_{\sigma^{-1}\tilde{\tau}}^{\circ})^{(p)}$$

$$x \longmapsto \begin{cases} V(x) & \text{when } s_{\sigma^{-1}\tilde{\tau}} = 0 \text{ or } 1, \\ F^{-1}(x) & \text{when } s_{\sigma^{-1}\tilde{\tau}} = 2. \end{cases} \tag{4.2.2}$$

Here, in the latter case, the morphism $F : H_1^{\text{dR}}(A^{(p)}/S)_{\tilde{\tau}}^{\circ} \rightarrow H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ}$ is an isomorphism.

When no confusion arises, we may suppress the subscript A and/or $\tilde{\tau}$ from $F_{A,\text{es},\tilde{\tau}}$ and $V_{A,\text{es},\tilde{\tau}}$.

Thus, if $s_{\sigma^{-1}\tilde{\tau}} = 0$ or 2 , both $F_{\text{es},\tilde{\tau}} : H_1^{\text{dR}}(A^{(p)}/S)_{\tilde{\tau}}^{\circ} \rightarrow H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ}$ and $V_{\text{es},\tilde{\tau}} : H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ} \rightarrow H_1^{\text{dR}}(A^{(p)}/S)_{\tilde{\tau}}^{\circ}$ are isomorphisms and both $F_{\text{es},\tilde{\tau}}V_{\text{es},\tilde{\tau}}$ and $V_{\text{es},\tilde{\tau}}F_{\text{es},\tilde{\tau}}$ are isomorphisms. When $s_{\sigma^{-1}\tilde{\tau}} = 1$, we usually prefer to write the usual Frobenius and Verschiebung.

We will also use composites of Frobenii and Verschiebungs:

$$V_{\text{es},\tilde{\tau}}^n : H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ} \xrightarrow{V_{\text{es},\tilde{\tau}}} H_1^{\text{dR}}(A^{(p)}/S)_{\tilde{\tau}}^{\circ} \xrightarrow{V_{\text{es},\sigma^{-1}\tilde{\tau}}^{(p)}} \dots \xrightarrow{V_{\text{es},\sigma^{1-n}\tilde{\tau}}^{(p^{n-1})}} H_1^{\text{dR}}(A^{(p^n)}/S)_{\tilde{\tau}}^{\circ}, \quad (4.2.3)$$

$$F_{\text{es},\tilde{\tau}}^n : H_1^{\text{dR}}(A^{(p^n)}/S)_{\tilde{\tau}}^{\circ} \xrightarrow{F_{\text{es},\sigma^{1-n}\tilde{\tau}}^{(p^{n-1})}} H_1^{\text{dR}}(A^{(p^{n-1})}/S)_{\tilde{\tau}}^{\circ} \xrightarrow{F_{\text{es},\sigma^{2-n}\tilde{\tau}}^{(p^{n-2})}} \dots \xrightarrow{F_{\text{es},\tilde{\tau}}} H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ}. \quad (4.2.4)$$

Suppose now that $S = \text{Spec}(k)$ is the spectrum of a perfect field of characteristic $p > 0$. Let $\tilde{\mathcal{D}}_A$ denote the covariant Dieudonné module of $A[p^\infty]$. We have a canonical decomposition $\tilde{\mathcal{D}}_A = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} \tilde{\mathcal{D}}_{A,\tilde{\tau}}$. We put $\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ = \mathfrak{e} \cdot \tilde{\mathcal{D}}_{A,\tilde{\tau}}$. Then we define the essential Frobenius and essential Verschiebung

$$F_{\text{es}} = F_{A,\text{es},\tilde{\tau}} : \tilde{\mathcal{D}}_{A,\sigma^{-1}\tilde{\tau}}^\circ \rightarrow \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ \quad \text{and} \quad V_{\text{es}} = V_{A,\text{es},\tilde{\tau}} : \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ \rightarrow \tilde{\mathcal{D}}_{A,\sigma^{-1}\tilde{\tau}}^\circ$$

in the same way as in (4.2.1) and (4.2.2) for $H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^\circ$. The morphisms $F_{A,\text{es},\tilde{\tau}}$ and $V_{A,\text{es},\tilde{\tau}}$ on $H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^\circ$ can be recovered from those on $\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ by reduction modulo p .

Notation 4.3. For $\tau \in \Sigma_\infty - \mathbf{S}_\infty$, we define $n_\tau = n_{\tau,\mathbf{S}} \geq 1$ to be the integer such that $\sigma^{-1}\tau, \dots, \sigma^{-n_\tau+1}\tau \in \mathbf{S}_\infty$ and $\sigma^{-n_\tau}\tau \notin \mathbf{S}_\infty$.

4.4 Partial Hasse invariants

For each $\tilde{\tau}$ lifting a place $\tau \in \Sigma_\infty - \mathbf{S}_\infty$, we must have $s_{\tilde{\tau}} = 1$. So, in the definition of $V_{\text{es},\tilde{\tau}}^{n_\tau}$ in (4.2.3), all morphisms are isomorphisms except the last one. Similarly, in the definition of $F_{\text{es},\tilde{\tau}}^{n_\tau}$ in (4.2.4), all morphisms are isomorphisms except the first one. It is clear that $V_{\text{es},\tilde{\tau}}^{n_\tau} F_{\text{es},\tilde{\tau}}^{n_\tau} = F_{\text{es},\tilde{\tau}}^{n_\tau} V_{\text{es},\tilde{\tau}}^{n_\tau} = 0$, coming from the composition of $V_{\sigma^{n_\tau-1}\tilde{\tau}}^{(p^{n_\tau-1})}$ and $F_{\sigma^{n_\tau-1}\tilde{\tau}}^{(p^{n_\tau-1})}$ in both ways. Note also that the cokernels of $V_{\text{es},\tilde{\tau}}^{n_\tau}$ and $F_{\text{es},\tilde{\tau}}^{n_\tau}$ are both locally free $\mathcal{O}_{X'}$ -modules of rank one.

The restriction of $V_{\text{es},\tilde{\tau}}^{n_\tau}$ to the line bundle $\omega_{A^\vee/S,\tilde{\tau}}^\circ$ induces a homomorphism

$$h_{\tilde{\tau}}(A) : \omega_{A^\vee/S,\tilde{\tau}}^\circ \longrightarrow \omega_{A^\vee,(p^{n_\tau})/S,\tilde{\tau}}^\circ = (\omega_{A^\vee/S,\sigma^{-n_\tau}\tilde{\tau}}^\circ)^{\otimes p^{n_\tau}}.$$

Applied to the universal case, this gives rise to a global section

$$h_{\tilde{\tau}} \in \Gamma(X', (\omega_{\mathbf{A}^\vee/X',\sigma^{-n_\tau}\tilde{\tau}}^\circ)^{\otimes p^{n_\tau}} \otimes (\omega_{\mathbf{A}^\vee/X',\tilde{\tau}}^\circ)^{\otimes (-1)}), \quad (4.4.1)$$

where \mathbf{A}' is the universal abelian scheme over $X' = \mathbf{Sh}_{K'}(G'_\mathbb{S})_{k_0}$. We call $h_{\tilde{\tau}}$ the $\tilde{\tau}$ -partial Hasse invariant. With $\tilde{\tau}$ replaced by $\tilde{\tau}^c$ everywhere, we can define similarly a partial Hasse invariant $h_{\tilde{\tau}^c}$. They are analogs of the partial Hasse invariants in the unitary case.

LEMMA 4.5. *Let $(A, \iota, \lambda, \bar{\alpha}_{K'})$ be an S -valued point of X' as above. Then the following statements are equivalent for $\tilde{\tau}$ lifting $\tau \in \Sigma_\infty - \mathbf{S}_\infty$:*

- (i) we have $h_{\tilde{\tau}}(A) = 0$;
- (ii) the image of $F_{\text{es},\tilde{\tau}}^{n_\tau} : H_1^{\text{dR}}(A^{(p^{n_\tau})}/S)_{\tilde{\tau}}^\circ \rightarrow H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^\circ$ is $\omega_{A^\vee/S,\tilde{\tau}}^\circ$;
- (iii) we have $h_{\tilde{\tau}^c}(A) = 0$;
- (iv) the image of $F_{\text{es},\tilde{\tau}^c}^{n_\tau} : H_1^{\text{dR}}(A^{(p^{n_\tau})}/S)_{\tilde{\tau}^c}^\circ \rightarrow H_1^{\text{dR}}(A/S)_{\tilde{\tau}^c}^\circ$ is $\omega_{A^\vee/S,\tilde{\tau}^c}^\circ$.

Proof. The equivalences (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4) follow from the fact that the image of F coincides with the kernel of V . We prove now (2) \Leftrightarrow (4). Let $\mathfrak{p} \in \Sigma_p$ be the prime above p so

that $\tau \in \Sigma_{\infty/\mathfrak{p}}$. Since $\Sigma_{\infty/\mathfrak{p}} \neq \mathbf{S}_{\infty/\mathfrak{p}}$, \mathfrak{p} can not be of type β^\sharp by Hypothesis 3.3. We consider the following diagram:

$$\begin{array}{ccc} H_1^{\mathrm{dR}}(A^{(p^{n\tau})}/S)_{\tilde{\tau}}^\circ & \times & H_1^{\mathrm{dR}}(A^{(p^{n\tau})}/S)_{\tilde{\tau}^c}^\circ \xrightarrow{\langle \cdot, \cdot \rangle} \mathcal{O}_S \\ \begin{array}{c} \uparrow V_{\mathrm{es}, \tilde{\tau}}^{n\tau} \\ \left(\begin{array}{c} \uparrow F_{\mathrm{es}, \tilde{\tau}}^{n\tau} \\ \downarrow \end{array} \right) \\ \downarrow F_{\mathrm{es}, \tilde{\tau}^c}^{n\tau} \end{array} & & \begin{array}{c} \uparrow V_{\mathrm{es}, \tilde{\tau}^c}^{n\tau} \\ \left(\begin{array}{c} \uparrow F_{\mathrm{es}, \tilde{\tau}^c}^{n\tau} \\ \downarrow \end{array} \right) \\ \downarrow \end{array} \\ H_1^{\mathrm{dR}}(A/S)_{\tilde{\tau}}^\circ & \times & H_1^{\mathrm{dR}}(A/S)_{\tilde{\tau}^c}^\circ \xrightarrow{\langle \cdot, \cdot \rangle} \mathcal{O}_S, \end{array}$$

where the pairings $\langle \cdot, \cdot \rangle$ are induced by the polarization λ , and they are perfect because \mathfrak{p} is not of type β^\sharp . We have $\langle F_{\mathrm{es}, \tilde{\tau}}^{n\tau} x, y \rangle = \langle x, V_{\mathrm{es}, \tilde{\tau}^c}^{n\tau} y \rangle^{\sigma^{n\tau}}$. It follows that

$$(\omega_{A^\vee/S, \tilde{\tau}}^\circ)^\perp = \omega_{A^\vee/S, \tilde{\tau}^c}^\circ \quad \text{and} \quad \mathrm{Im}(F_{\mathrm{es}, \tilde{\tau}}^{n\tau})^\perp = \mathrm{Im}(F_{\mathrm{es}, \tilde{\tau}^c}^{n\tau}),$$

where \perp means the orthogonal complement under $\langle \cdot, \cdot \rangle$. Therefore, we have

$$(2) \quad \omega_{A^\vee/S, \tilde{\tau}}^\circ = \mathrm{Im}(F_{\mathrm{es}, \tilde{\tau}}^{n\tau}) \iff (\omega_{A^\vee/S, \tilde{\tau}}^\circ)^\perp = \mathrm{Im}(F_{\mathrm{es}, \tilde{\tau}}^{n\tau})^\perp \iff (4) \quad \omega_{A^\vee/S, \tilde{\tau}^c}^\circ = \mathrm{Im}(F_{\mathrm{es}, \tilde{\tau}^c}^{n\tau}). \quad \square$$

DEFINITION 4.6. We fix a section $\tau \mapsto \tilde{\tau}$ of the natural restriction map $\Sigma_{E, \infty} \rightarrow \Sigma_\infty$. Let $\mathbf{T} \subset \Sigma_\infty - \mathbf{S}_\infty$ be a subset. We put $\mathbf{Sh}_{K'}(G'_S)_{k_0, \emptyset} = X'$, and $X'_T := \mathbf{Sh}_{K'}(G'_S)_{k_0, T}$ to be the closed subscheme of $\mathbf{Sh}_{K'}(G'_S)_{k_0}$ defined as the vanishing locus of $\{h_{\tilde{\tau}} : \tau \in T\}$. Passing to the limit, we put

$$\mathbf{Sh}_{K'_p}(G'_S)_{k_0, T} := \varprojlim_{K'^p} \mathbf{Sh}_{K'^p K'_p}(G'_S)_{k_0, T}.$$

We call $\{\mathbf{Sh}_{K'}(G'_S)_{k_0, T} : T \subset \Sigma_\infty - \mathbf{S}_\infty\}$ (respectively $\{\mathbf{Sh}_{K'_p}(G'_S)_{k_0, T} : T \subset \Sigma_\infty - \mathbf{S}_\infty\}$) the *Goren-Oort stratification* (or GO-stratification for short) of $\mathbf{Sh}_{K'}(G'_S)_{k_0}$ (respectively $\mathbf{Sh}_{K'_p}(G'_S)_{k_0}$).

By Lemma 4.5, the GO-strata X'_T do not depend on the choice of the section $\tau \mapsto \tilde{\tau}$.

PROPOSITION 4.7. For any subset $T \subseteq \Sigma_\infty - \mathbf{S}_\infty$, the closed GO-stratum $X'_T \subseteq X'$ is smooth of codimension $\#T$, and the tangent bundle $\mathcal{T}_{X'_T}$ is the subbundle

$$\bigoplus_{\tau \in \Sigma_\infty - (\mathbf{S}_\infty \cup T)} (\mathrm{Lie}(\mathbf{A}')_{\tilde{\tau}}^\circ \otimes \mathrm{Lie}(\mathbf{A}')_{\tilde{\tau}^c}^\circ)|_{X'_T} \subseteq \bigoplus_{\tau \in \Sigma_\infty - \mathbf{S}_\infty} (\mathrm{Lie}(\mathbf{A}')_{\tilde{\tau}}^\circ \otimes \mathrm{Lie}(\mathbf{A}')_{\tilde{\tau}^c}^\circ)|_{X'_T},$$

where the latter is identified with the restriction to X'_T of the tangent bundle of X' computed in Corollary 3.17. Moreover, X'_T is proper if $\mathbf{S}_\infty \cup T$ is non-empty.

Proof. We follow the same strategy as in [Hel12, Proposition 3.4]. First, the same argument as [Hel12, Lemma 3.7] proves the non-emptiness of X'_T . We now proceed as in the proof of Corollary 3.17. Let $S_0 \hookrightarrow S$ be a closed immersion of locally noetherian k_0 -schemes whose ideal of definition \mathcal{I} satisfies $\mathcal{I}^2 = 0$. Consider an S_0 -valued point $x_0 = (A_0, \iota_0, \lambda_0, \bar{\alpha}_{K'})$ of X'_T . To prove the smoothness of X'_T , it suffices to show that, locally for the Zariski topology on S_0 , there exists $x \in X'_T(S)$ lifting x_0 . By Lemma 4.5, we have, for every $\tau \in T$,

$$\omega_{A_0^\vee/S_0, \tilde{\tau}}^\circ = F_{\mathrm{es}, \tilde{\tau}}^{n\tau}(H_1^{\mathrm{dR}}(A_0^{(p^{n\tau})}/S_0)_{\tilde{\tau}}^\circ).$$

The reduced ‘crystalline homology’ $H_1^{\mathrm{cris}}(A_0/S_0)_{S_0}^\circ$ is equipped with natural operators F and V , lifting the corresponding operators on $H_1^{\mathrm{dR}}(A_0/S_0)^\circ$. We define the composite of essential Frobenius

$$\tilde{F}_{\mathrm{es}, \tilde{\tau}}^{n\tau} : H_1^{\mathrm{cris}}(A_0^{(p^{n\tau})}/S_0)_{S, \tilde{\tau}}^\circ \rightarrow H_1^{\mathrm{cris}}(A_0/S_0)_{S, \tilde{\tau}}^\circ$$

in the same manner as $F_{\text{es},\bar{\tau}}^{n_\tau}$ on $H_1^{\text{dR}}(A_0^{(p^{n_\tau})}/S_0)_{\bar{\tau}}^\circ$ in Notation 4.2. Let $\tilde{\omega}_{A_0^\vee/S_0,\bar{\tau}}^\circ$ denote the image of $\tilde{F}_{\text{es},\bar{\tau}}^{n_\tau}$ for $\tau \in \mathbb{T}$. This is a local direct factor of $H_1^{\text{cris}}(A_0/S_0)_{S,\bar{\tau}}^\circ$ that lifts $\omega_{A_0^\vee/S_0,\bar{\tau}}^\circ$. As in the proof of Theorem 3.14, specifying a deformation $x \in X'(S)$ of x_0 to S is equivalent to giving a local direct summand $\omega_{S,\bar{\tau}}^\circ \subseteq H_1^{\text{cris}}(A_0/S_0)_{S,\bar{\tau}}^\circ$ that lifts $\omega_{A_0^\vee/S_0,\bar{\tau}}^\circ$ for each $\tau \in \Sigma_\infty - \mathbb{S}_\infty$. By Lemma 4.5, such a deformation x lies in $X'_\mathbb{T}$ if and only if $\omega_{S,\bar{\tau}}^\circ = \tilde{\omega}_{A_0^\vee/S_0,\bar{\tau}}^\circ$ for all $\tau \in \mathbb{T}$. Therefore, to give a deformation of x_0 to S in $X'_\mathbb{T}$, we just need to specify the liftings $\omega_{S,\bar{\tau}}^\circ$ of $\omega_{A_0^\vee/S_0,\bar{\tau}}^\circ$ for $\tau \in \Sigma_\infty - (\mathbb{S}_\infty \cup \mathbb{T})$. The set-valued sheaf of liftings $\omega_{S,\bar{\tau}}^\circ$ for $\tau \in \Sigma_\infty - (\mathbb{S}_\infty \cup \mathbb{T})$ forms a torsor under the group

$$\text{Hom}_{\mathcal{O}_{S_0}}(\omega_{A_0^\vee/S_0,\bar{\tau}}^\circ, \text{Lie}(A_0)_{\bar{\tau}}^\circ) \otimes_{\mathcal{O}_{S_0}} \mathcal{I} \simeq \text{Lie}(A_0)_{\bar{\tau}}^\circ \otimes_{\mathcal{O}_{S_0}} \text{Lie}(A_0)_{\bar{\tau}^c}^\circ \otimes_{\mathcal{O}_{S_0}} \mathcal{I}.$$

Here, in the last isomorphism, we have used (4.1.1). The statement for the tangent bundle of $X'_\mathbb{T}$ now follows immediately.

It remains to prove the properness of $X'_\mathbb{T}$ when $\mathbb{S}_\infty \cup \mathbb{T}$ is non-empty. The arguments are similar to those in [Hel12, Proposition 3.4]. We use the valuative criterion of properness. Let R be a discrete valuation ring containing $\overline{\mathbb{F}}_p$ and L be its fraction field. Let $x_L = (A_L, \iota, \bar{\alpha}_{K'})$ be an L -valued point of $X'_\mathbb{T}$. We have to show that x_L extends to an R -valued point $x_R \in X'_\mathbb{T}$ up to a finite extension of L . By Grothendieck’s semi-stable reduction theorem, we may assume that, up to a finite extension of L , the Néron model A_R of A_L over R has a semi-stable reduction. Let \bar{A} be the special fiber of A_R and $\mathbb{T} \subset \bar{A}$ be its torus part. Since the Néron model is canonical, the action of \mathcal{O}_{D_S} extends uniquely to A_R and hence to \mathbb{T} . The rational cocharacter group $X_*(\mathbb{T})_{\mathbb{Q}} := \text{Hom}(\mathbb{G}_m, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a \mathbb{Q} -vector space of dimension at most $\dim(\bar{A}) = 4g = \frac{1}{2} \dim_{\mathbb{Q}}(D_S)$, and equipped with an induced action of $D_S \cong M_2(E)$. By the classification of $M_2(E)$ -modules, $X_*(\mathbb{T})_{\mathbb{Q}}$ is either 0 or isomorphic to $E^{\oplus 2}$. In the latter case, we have $X_*(\mathbb{T})_{\mathbb{Q}} \otimes L \cong \text{Lie}(A_L)$, and the trace of the action of $b \in E$ on $X_*(\mathbb{T})_{\mathbb{Q}}$ is $2 \sum_{\bar{\tau} \in \Sigma_E} \bar{\tau}(b)$, which implies that $\mathbb{S}_\infty = \emptyset$. Therefore, if $\mathbb{S}_\infty \neq \emptyset$, \mathbb{T} has to be trivial and A_R is an abelian scheme over R with generic fiber A_L . The polarization λ and level structure $\bar{\alpha}_{K'}$ extend uniquely to A_R due to the canonicity of the Néron model. We thus obtain a point $x_R \in X'(R)$ extending x_L . Since $X'_\mathbb{T} \subseteq X'$ is a closed subscheme, we see easily that $x_R \in X'_\mathbb{T}$. Now consider the case $\mathbb{S}_\infty = \emptyset$ but \mathbb{T} is non-empty. If $X_*(\mathbb{T})_{\mathbb{Q}} \cong E^{\oplus 2}$, then the abelian part of \bar{A} is trivial. Since the action of Verschiebung on $\omega_{\mathbb{T}}$ is an isomorphism, the point x_L cannot lie in any $X'_\mathbb{T}$ with \mathbb{T} non-empty. Therefore, if $\mathbb{T} \neq \emptyset$, \mathbb{T} must be trivial, and we conclude as in the case $\mathbb{S}_\infty \neq \emptyset$. \square

Remark 4.8. It seems that $X'_\mathbb{T}$ is still proper if \mathbb{S} is non-empty. But we do not know a convincing algebraic argument.

4.9 GO-stratification of connected Shimura varieties

From the definition, it is clear that the GO-stratification on $\mathbf{Sh}_{K'_p}(G'_S)_{k_0}$ is compatible with the action (as described in § 3.20) of the group \mathcal{G}'_S (introduced in § 3.12). By Corollary 2.13, for each $\mathbb{T} \subseteq \Sigma_\infty - \mathbb{S}_\infty$, there is a natural scheme

$$\mathbf{Sh}_{K'_p}(G'_S)_{\overline{\mathbb{F}}_p,\mathbb{T}}^\circ \subseteq \mathbf{Sh}_{K'_p}(G'_S)_{\overline{\mathbb{F}}_p}^\circ$$

equivariant for the action of \mathcal{E}_{G,k_0} . We call them the *Goren–Oort stratification* for the connected Shimura variety.

Using the identification of connected Shimura variety in Corollary 3.11 together with Corollary 2.13, we obtain the *Goren–Oort strata* $\mathbf{Sh}_{K_p}(G_{\mathbf{S}})_{k_0, \mathbf{T}} \subseteq \mathbf{Sh}_{K_p}(G_{\mathbf{S}})_{k_0}$ and $\mathbf{Sh}_{K_p''}(G_{\mathbf{S}}'')_{k_0, \mathbf{T}} \subseteq \mathbf{Sh}_{K_p''}(G_{\mathbf{S}}'')_{k_0}$ for each subset $\mathbf{T} \subseteq \Sigma_{\infty} - \mathbf{S}_{\infty}$. Explicitly, for the latter case, we have

$$\mathbf{Sh}_{K_p''}(G_{\mathbf{S}}'')_{k_0, \mathbf{T}} := \mathbf{Sh}_{K_p'}(G_{\mathbf{S}}')_{k_0, \mathbf{T}} \times_{\tilde{G}_{\mathbf{S}}} G_{\mathbf{S}}''(\mathbb{A}^{\infty, p}).$$

Alternatively, in terms of the natural family of abelian varieties $\mathbf{A}_{\mathbf{S}}''$, the stratum $\mathbf{Sh}_{K_p''}(G_{\mathbf{S}}'')_{k_0, \mathbf{T}}$ is the common zero locus of partial Hasse invariants

$$h_{\tilde{\tau}} : \omega_{\mathbf{A}_{\mathbf{S}, k_0}''}^{\circ} / \mathbf{Sh}_{K_p''}(G_{\mathbf{S}}'')_{k_0, \tilde{\tau}} \longrightarrow (\omega_{\mathbf{A}_{\mathbf{S}, k_0}''}^{\circ} / \mathbf{Sh}_{K_p''}(G_{\mathbf{S}}'')_{k_0, \sigma^{-n_{\tau}} \tilde{\tau}})^{\otimes p^{n_{\tau}}}$$

for all $\tilde{\tau}$ lifting $\tau \in \mathbf{T}$.

THEOREM 4.10. *When $\mathbf{S} = \emptyset$, the GO-stratification on $\mathbf{Sh}_{K_p}(G_{\emptyset})_{k_0}$ defined above agrees with the original definition given in [GO00]. Moreover, for each subset $\mathbf{T} \subseteq \Sigma_{\infty}$, under the morphisms (3.26.1), we have*

$$\mathbf{pr}_1^*(\mathbf{Sh}_{K_p}(G_{\emptyset})_{k_0, \mathbf{T}}) = \alpha^*(\mathbf{Sh}_{K_p''}(G_{\emptyset}'')_{k_0, \mathbf{T}}),$$

where $\mathbf{Sh}_{K_p}(G_{\emptyset})_{k_0, \mathbf{T}}$ denotes the GO-stratum for \mathbf{T} defined in [GO00].

Proof. Put $X = \mathbf{Sh}_{K_p}(G_{\emptyset})_{k_0}$ for simplicity. By Proposition 3.25, we have an isomorphism of abelian varieties $\alpha^* \mathbf{A}_{\emptyset}'' = (\mathbf{pr}_1^*(\mathcal{A}) \otimes_{\mathcal{O}_F} \mathcal{O}_E)^{\oplus 2}$ on $\mathbf{Sh}_{K_p \times K_{E,p}}(G_{\emptyset} \times T_{E, \emptyset})$. Let $\omega_{\mathbf{A}_{k_0}^{\vee}/X} = \bigoplus_{\tau \in \Sigma_{\infty}} \omega_{\mathbf{A}_{k_0}^{\vee}/X, \tau}$ be the canonical decomposition, where $\omega_{\mathbf{A}_{k_0}^{\vee}/X, \tau}$ is the local direct factor on which \mathcal{O}_F acts via $\iota_p \circ \tau : \mathcal{O}_F \rightarrow \mathbb{Z}_p^{\text{ur}} \rightarrow \overline{\mathbb{F}}_p$. Then we have a canonical isomorphism of line bundles over $\mathbf{Sh}_{K_p \times K_{E,p}}(G_{\emptyset} \times T_{E, \emptyset})_{k_0}$

$$\alpha^* \omega_{\mathbf{A}_{k_0}^{\vee}/X, \tilde{\tau}} \simeq \mathbf{pr}_1^* \omega_{\mathbf{A}_{k_0}^{\vee}/X, \tau}$$

for either lift $\tilde{\tau} \in \Sigma_{E, \infty}$ of τ . Via these identifications, the (pullback of) the partial Hasse invariant $\alpha^*(h_{\tilde{\tau}})$ defined in (4.4.1) coincides with the pullback via \mathbf{pr}_1 of the partial Hasse invariant $h_{\tau} \in \Gamma(X, \omega_{\mathcal{A}/X, \sigma^{-1}\tau}^{\otimes p} \otimes \omega_{\mathcal{A}/X, \tau}^{\otimes -1})$ defined in [GO00]. Therefore, for any $\mathbf{T} \subset \Sigma_{\infty}$, the pullback along \mathbf{pr}_1 of the GO-stratum $X_{\mathbf{T}} \subseteq X$ defined by the vanishing of $\{h_{\tau} : \tau \in \mathbf{T}\}$ is the same as the pullback along α of the GO-stratum defined by $\{h_{\tilde{\tau}} : \tau \in \mathbf{T}\}$. \square

Remark 4.11. It would be interesting to know, in general, whether the GO-strata on quaternionic Shimura varieties depend on the auxiliary choice of CM field E .

To understand the ‘action’ of the twisted partial Frobenius on the GO-strata, we need the following lemma.

LEMMA 4.12. *Let $x = (A, \iota, \lambda, \bar{\alpha}_{K'})$ be a point of X' with values in a locally noetherian k_0 -scheme S , and $\mathfrak{F}'_{\mathfrak{p}^2}(x) = (A', \iota', \lambda', \bar{\alpha}'_{K'})$ be the image of x under the twisted partial Frobenius at \mathfrak{p} § 3.22 (which lies on another Shimura variety). Then $h_{\tilde{\tau}}(x) = 0$ if and only if $h_{\sigma_{\tilde{\tau}}^2}(\mathfrak{F}'_{\mathfrak{p}^2}(x)) = 0$.*

Proof. The statement is clear if $\tilde{\tau} \notin \Sigma_{E, \infty/\mathfrak{p}}$, since $\mathfrak{F}'_{\mathfrak{p}^2}$ induces a canonical isomorphism of p -divisible groups $A[\mathfrak{q}^{\infty}] \simeq A'[\mathfrak{q}^{\infty}]$ for $\mathfrak{q} \in \Sigma_p$ with $\mathfrak{q} \neq \mathfrak{p}$. Consider the case $\tilde{\tau} \in \Sigma_{E, \infty/\mathfrak{p}}$. We claim that there exists an isomorphism

$$H_1^{\text{dR}}(A'/S)_{\tilde{\tau}}^{\circ} \cong (H_1^{\text{dR}}(A/S)_{\sigma^{-2}\tilde{\tau}}^{\circ})^{(p^2)}$$

compatible with the action of F and V on both sides with $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$ varying. By Lemma 4.5, the lemma follows from the claim immediately. It thus remains to prove the claim. Note that the \mathfrak{p} -component of the de Rham homology

$$H_1^{\text{dR}}(A'/S)_{\mathfrak{p}} := \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}} H_1^{\text{dR}}(A'/S)_{\tilde{\tau}}^{\circ, \oplus 2}$$

is canonically isomorphic to the evaluation at the trivial pd-thickening $S \hookrightarrow S$ of the reduced covariant Dieudonné crystal of $A'[\mathfrak{p}^\infty]$, which we denote by $\mathcal{D}(A'[\mathfrak{p}^\infty])_S$. By definition of $\mathfrak{F}'_{\mathfrak{p}^2}$, the p -divisible group $A'[\mathfrak{p}^\infty] \cong (A/\text{Ker}_{\mathfrak{p}^2})[\mathfrak{p}^\infty]$ is isomorphic to the quotient of $A[\mathfrak{p}^\infty]$ by its kernel of p^2 -Frobenius $A[\mathfrak{p}^\infty] \rightarrow (A[\mathfrak{p}^\infty])^{(p^2)}$. Therefore, by functoriality of Dieudonné crystals, one has $\mathcal{D}(A'[\mathfrak{p}^\infty])_S = \mathcal{D}(A[\mathfrak{p}^\infty])_S^{(p^2)}$, whence the claim. \square

One deduces immediately the following corollary.

COROLLARY 4.13. *For $\overline{\text{Sh}}_{\mathfrak{S}} = \mathbf{Sh}_{K'_p}(G'_{\mathfrak{S}})_{k_0}$ and $\mathbf{Sh}_{K''_p}(G''_{\mathfrak{S}})_{k_0}$, the twisted partial Frobenius map $\mathfrak{F}'_{\mathfrak{p}^2} : \overline{\text{Sh}}_{\mathfrak{S}} \rightarrow \overline{\text{Sh}}_{\sigma_{\mathfrak{p}^2}\mathfrak{S}}$ takes the subvariety $\overline{\text{Sh}}_{\mathfrak{S},\mathfrak{T}}$ to $\overline{\text{Sh}}_{\sigma_{\mathfrak{p}^2}\mathfrak{S},\sigma_{\mathfrak{p}^2}\mathfrak{T}}$ for each $\mathfrak{T} \subseteq \Sigma_\infty - \mathfrak{S}_\infty$.*

5. The global geometry of the GO-strata: Helm’s isogeny trick

In this section, we will prove that each closed GO-stratum of the special fiber of the unitary Shimura variety defined in Definition 4.6 is a $(\mathbb{P}^1)^N$ -bundle over the special fiber of another unitary Shimura variety for some appropriate integer N . This then allows us to deduce the similar result for the case of quaternionic Shimura varieties.

This section is largely inspired by Helm’s pioneering work [Hel12], where he considered the case when p splits in E_0/\mathbb{Q} and \mathfrak{S} is ‘sparse’ (we refer to [Hel12] for the definition of a sparse subset; essentially, this means that, for any $\tau \in \Sigma_\infty$, τ and $\sigma\tau$ cannot belong to \mathfrak{S} simultaneously).

5.1 The associated quaternionic Shimura data for a GO-stratum

We first introduce the recipe for describing general GO-strata. We recommend first reading the light version of the same recipe in the special case of Hilbert modular varieties, as explained in the introduction 1.5, before diving into the general but more complicated definition below.

Keep the notation as in the previous sections. Let \mathfrak{T} be a subset of $\Sigma_\infty - \mathfrak{S}_\infty$. Our main theorem will say that the Goren–Oort stratum $\mathbf{Sh}_K(G_{\mathfrak{S}})_{\overline{\mathbb{F}}_p,\mathfrak{T}}$ is a $(\mathbb{P}^1)^N$ -bundle over $\mathbf{Sh}_{K_{\mathfrak{T}}}(G_{\mathfrak{S}(\mathfrak{T})})_{\overline{\mathbb{F}}_p}$ for some $N \in \mathbb{Z}_{\geq 0}$, some even subset $\mathfrak{S}(\mathfrak{T})$ of places of F , and an open compact subgroup $K_{\mathfrak{T}} \subseteq G_{\mathfrak{S}(\mathfrak{T})}(\mathbb{A}^\infty)$.

We describe the set $\mathfrak{S}(\mathfrak{T})$ now. For each prime $\mathfrak{p} \in \Sigma_p$, we put $\mathfrak{T}/\mathfrak{p} = \mathfrak{T} \cap \Sigma_{\infty/\mathfrak{p}}$. We define first a subset $\mathfrak{T}'/\mathfrak{p} \subseteq \Sigma_{\infty/\mathfrak{p}} \cup \{\mathfrak{p}\}$ containing $\mathfrak{T}/\mathfrak{p}$ which depends on the types of \mathfrak{p} as in § 3.2 and we put

$$\mathfrak{T}' = \coprod_{\mathfrak{p} \in \Sigma_p} \mathfrak{T}'/\mathfrak{p} \quad \text{and} \quad \mathfrak{S}(\mathfrak{T}) = \mathfrak{S} \sqcup \mathfrak{T}'. \tag{5.1.1}$$

We separate the discussion into several cases.

- If \mathfrak{p} is of type α^\sharp or type β^\sharp for $\mathbf{Sh}_K(G_{\mathfrak{S}})$, we put $\mathfrak{T}'/\mathfrak{p} = \emptyset$.
- If \mathfrak{p} is of type α for \mathfrak{S} , i.e. $(\Sigma_{\infty/\mathfrak{p}} - \mathfrak{S}_{\infty/\mathfrak{p}})$ has even cardinality, we distinguish two cases.

* (Case $\alpha 1$) $\mathfrak{T}/\mathfrak{p} \subsetneq \Sigma_{\infty/\mathfrak{p}} - \mathfrak{S}_{\infty/\mathfrak{p}}$. We write $\mathfrak{S}_{\infty/\mathfrak{p}} \cup \mathfrak{T}/\mathfrak{p} = \coprod C_i$ as a disjoint union of chains. Here, by a chain C_i , we mean that there exist $\tau_i \in \mathfrak{S}_{\infty/\mathfrak{p}} \cup \mathfrak{T}/\mathfrak{p}$ and an integer

$m_i \geq 0$ such that $C_i = \{\sigma^{-a}\tau_i : 0 \leq a \leq m_i\}$ belong to $S_{\infty/\mathfrak{p}} \cup T/\mathfrak{p}$ and $\sigma\tau_i, \sigma^{-m_i-1}\tau_i \notin (S_{\infty/\mathfrak{p}} \cup T/\mathfrak{p})$. We put $T'/\mathfrak{p} = \coprod_i C'_i$, where

$$C'_i := \begin{cases} C_i \cap T/\mathfrak{p} & \text{if } \#(C_i \cap T/\mathfrak{p}) \text{ is even,} \\ (C_i \cap T/\mathfrak{p}) \cup \{\sigma^{-m_i-1}\tau_i\} & \text{if } \#(C_i \cap T/\mathfrak{p}) \text{ is odd.} \end{cases}$$

For example, if $\Sigma_{\infty/\mathfrak{p}} = \{\tau_0, \sigma^{-1}\tau_0, \dots, \sigma^{-9}\tau_0\}$, $S_{\infty/\mathfrak{p}} = \{\sigma^{-2}\tau_0, \sigma^{-6}\tau_0\}$, and $T/\mathfrak{p} = \{\sigma^{-3}\tau_0, \sigma^{-5}\tau_0, \sigma^{-7}\tau_0\}$, then $S_{\infty/\mathfrak{p}} \cup T/\mathfrak{p}$ is separated into two chains $C_1 = \{\sigma^{-2}\tau_0, \sigma^{-3}\tau_0\}$ and $C_2 = \{\sigma^{-5}\tau_0, \sigma^{-6}\tau_0, \sigma^{-7}\tau_0\}$. We have $T'/\mathfrak{p} = \{\sigma^{-3}\tau_0, \sigma^{-4}\tau_0, \sigma^{-5}\tau_0, \sigma^{-7}\tau_0\}$. An alternative way to understand the partition of T/\mathfrak{p} is to view it as a subset of $\Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}}$ with the cycle structure inherited from $\Sigma_{\infty/\mathfrak{p}}$. Then $C_i \cap T/\mathfrak{p}$ is just to group elements of T/\mathfrak{p} into connected subchains.

- * (Case $\alpha 2$) $T/\mathfrak{p} = \Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}}$. We put $T'/\mathfrak{p} = T/\mathfrak{p}$.
- If \mathfrak{p} is of type β for S , i.e. $(\Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}})$ has odd cardinality and B_S splits at \mathfrak{p} . We distinguish two cases.
 - * (Case $\beta 1$) $T/\mathfrak{p} \subsetneq \Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}}$. In this case, we define T'/\mathfrak{p} using the same rule as in Case $\alpha 1$.
 - * (Case $\beta 2$) $T/\mathfrak{p} = \Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}}$. We put $T'_\mathfrak{p} = T/\mathfrak{p} \cup \{\mathfrak{p}\}$.

In either case, we put $T'_{\infty/\mathfrak{p}} = T'/\mathfrak{p} \cap \Sigma_{\infty}$. It is equal to T'/\mathfrak{p} unless in case $\beta 2$.

It is easy to see that each $T'_{\mathfrak{p}}$ has even cardinality. Therefore, $S(T)$ is also an even set, and it defines a quaternion algebra $B_{S(T)}$ over F . Note that $S(T)$ still satisfies Hypothesis 3.3.

Let $G_{S(T)} = \text{Res}_{F/\mathbb{Q}}(B_{S(T)}^\times)$ be the algebraic group over \mathbb{Q} associated to $B_{S(T)}^\times$. We fix an isomorphism $B_S \otimes_F F_l \simeq B_{S(T)} \otimes_F F_l$ whenever $\{l\} \cap S = \{l\} \cap S(T)$. We define an open compact subgroup $K_T = K_T^p K_{T,p} \subseteq G_{S(T)}(\mathbb{A}^\infty)$ determined by K as follows.

- We put $K_T^p = K^p$. This makes sense, because $B_S \otimes_F F_l \simeq B_{S(T)} \otimes_F F_l$ for any finite place l prime to p .
- For $K_{T,p} = \prod_{\mathfrak{p} \in \Sigma_p} K_{T,\mathfrak{p}}$, we take $K_{T,\mathfrak{p}} = K_{\mathfrak{p}}$, unless we are in case $\alpha 2$ or $\beta 2$.
 - * If \mathfrak{p} is of type $\alpha 2$ for $\text{Sh}_K(G_S)$, we have $B_{S(T)} \otimes_F F_{\mathfrak{p}} \simeq B_S \otimes_F F_{\mathfrak{p}} \simeq M_2(\mathcal{O}_{F_{\mathfrak{p}}})$. We take $K_{T,\mathfrak{p}} = K_{\mathfrak{p}}$ if $T/\mathfrak{p} = (\Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}}) = \emptyset$, and $K_{T,\mathfrak{p}} = \text{Iw}_{\mathfrak{p}}$ if $T/\mathfrak{p} \neq \emptyset$.
 - * If we are in case $\beta 2$ (and β^\sharp), $B_{S(T)}$ is ramified at \mathfrak{p} . We take $K_{T,\mathfrak{p}} = \mathcal{O}_{B_{F_{\mathfrak{p}}}}^\times$, where $\mathcal{O}_{B_{F_{\mathfrak{p}}}}$ is the unique maximal order of the division algebra over $F_{\mathfrak{p}}$ with invariant $1/2$.

The level K_T fits into the framework considered in §3.2. We thus obtain a quaternionic Shimura variety $\text{Sh}_{K_T}(G_{S(T)})$, and its integral model $\mathbf{Sh}_{K_T}(G_{S(T)})$ is given by Corollary 3.19. Note that:

- if we are in case $\alpha 1$ above, then \mathfrak{p} is of type α for the Shimura variety $\text{Sh}_{K_T}(G_{S(T)})$;
- if we are in case $\alpha 2$ above, then \mathfrak{p} is of type α^\sharp for $\text{Sh}_{K_T}(G_{S(T)})$ unless \mathfrak{p} is of type α for $\text{Sh}_K(G_S)$ and $T/\mathfrak{p} = \Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}} = \emptyset$, in which case \mathfrak{p} remains of type α for $\text{Sh}_{K_T}(G_{S(T)})$;
- if we are in case $\beta 1$ above, then \mathfrak{p} is of type β for $\text{Sh}_{K_T}(G_{S(T)})$;
- if we are in case $\beta 2$ or β^\sharp above, then \mathfrak{p} is of type β^\sharp for $\text{Sh}_{K_T}(G_{S(T)})$.

THEOREM 5.2. *For a subset $T \subseteq \Sigma_{\infty} - S_{\infty}$, the GO-stratum $\mathbf{Sh}_K(G_S)_{\overline{\mathbb{F}}_p, T}$ is isomorphic to a $(\mathbb{P}^1)^{I_T}$ -bundle over $\mathbf{Sh}_{K_T}(G_{S(T)})_{\overline{\mathbb{F}}_p}$, where $S(T)$ is as described above and the index set is given by*

$$I_T = S(T)_{\infty} - (S_{\infty} \cup T) = \bigcup_{\mathfrak{p} \in \Sigma_p} (T'_{\infty/\mathfrak{p}} - T/\mathfrak{p}).$$

Moreover, this isomorphism is compatible with the action of $G_S(\mathbb{A}^{\infty,p})$, when taking the limit over open compact subgroups $K^p \subseteq G_S(\mathbb{A}^{\infty,p})$.

As an example, the situation $I_T = \emptyset$ happens exactly when $T'_{\infty/\mathfrak{p}} = T/\mathfrak{p}$ for every \mathfrak{p} , which means that, for each \mathfrak{p} , either T/\mathfrak{p} is divided into a union of chains of an *even* number of ‘consecutive’ elements of $\Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}}$, or $T/\mathfrak{p} = \Sigma_{\infty/\mathfrak{p}}$.

Theorem 5.2 will follow from the analogous statements (Theorem 5.8 and Corollary 5.9) in the unitary case. But note Remark 5.10.

5.3 The signatures at infinity for the unitary Shimura varieties

In order to describe the unitary Shimura data associated to $\text{Sh}_{K_T}(G_S(T))$ as in §§ 3.4 and 3.5, we need to pick a lift $\tilde{S}(T)$ of the set $S(T)$ to embeddings of E . More precisely, we will define a subset $\tilde{S}(T)_{\infty} = \coprod_{\mathfrak{p} \in \Sigma_p} \tilde{S}(T)_{\infty/\mathfrak{p}}$, where $\tilde{S}(T)_{\infty/\mathfrak{p}}$ consists of exactly one lift $\tilde{\tau} \in \Sigma_{E,\infty}$ for each $\tau \in S(T)_{\infty/\mathfrak{p}}$. Then we put $\tilde{S}(T) = (S(T), \tilde{S}(T)_{\infty})$. So, we just need to assign such choices of lifts.

- When $\tau \in S(T)_{\infty/\mathfrak{p}}$ belongs to $S_{\infty/\mathfrak{p}}$, we choose its lift $\tilde{\tau} \in \Sigma_{E,\infty}$ to be the one that belongs to \tilde{S} .

We now specify our choices of the lifts in $\tilde{S}(T)_{\infty/\mathfrak{p}}$ for the elements of $T'_{\infty/\mathfrak{p}}$, which are collectively denoted by $\tilde{T}'_{\mathfrak{p}}$. We separate into cases and use freely the notation from § 5.1. There is nothing to do if \mathfrak{p} is of type α^{\sharp} or type β^{\sharp} (for S).

- \mathfrak{p} is of type α (for S). In this case, \mathfrak{p} splits into two primes \mathfrak{q} and \mathfrak{q}^c in E . For a place $\tau \in \Sigma_{\infty/\mathfrak{p}}$, we use $\tilde{\tau}$ to denote its lift to $\Sigma_{E,\infty}$, which corresponds to the p -adic place \mathfrak{q} .

- * (Case $\alpha 1$) For a chain $C_i = \{\sigma^{-a}\tau_i, 0 \leq a \leq m_i\} \subseteq S_{\infty/\mathfrak{p}} \cup T/\mathfrak{p}$ and the corresponding set $C'_i = \{\sigma^{-a_1}\tau_i, \dots, \sigma^{-a_{r_i}}\tau_i\}$ as defined in 5.1 with some $0 \leq a_1 < \dots < a_{r_i} \leq m_i + 1$ (note that r_i is always even by construction), we put

$$\tilde{C}'_i = \{\sigma^{-a_1}\tilde{\tau}_i, \sigma^{-a_2}\tilde{\tau}_i^c, \sigma^{-a_3}\tilde{\tau}_i, \dots, \sigma^{-a_{r_i}}\tilde{\tau}_i^c\}.$$

Set $\tilde{T}'_{\mathfrak{p}} = \coprod_i \tilde{C}'_i$.

- * (Case $\alpha 2$) We need to fix $\tau_0 \in T/\mathfrak{p} = \Sigma_{\infty/\mathfrak{p}} - S_{\infty/\mathfrak{p}}$ and write T/\mathfrak{p} as $\{\sigma^{-a_1}\tau_0, \dots, \sigma^{-a_{2r}}\tau_0\}$ for integers $0 = a_1 < \dots < a_{2r} \leq f_{\mathfrak{p}} - 1$. We put

$$\tilde{T}'_{\mathfrak{p}} = \{\sigma^{-a_1}\tilde{\tau}_0, \sigma^{-a_2}\tilde{\tau}_0^c, \sigma^{-a_3}\tilde{\tau}_0, \dots, \sigma^{-a_{2r}}\tilde{\tau}_0^c\}.$$

- \mathfrak{p} is of type β (for S). In this case, \mathfrak{p} is inert in E/F , and we do not have a canonical choice for the lift $\tilde{\tau}$ of an embedding τ .

- * (Case $\beta 1$) In this case, we fix a partition of the preimage of C'_i under the map $\Sigma_{E,\infty/\mathfrak{p}} \rightarrow \Sigma_{\infty/\mathfrak{p}}$ into two chains $\tilde{C}''_i \amalg \tilde{C}''_i{}^c$, where

$$\tilde{C}''_i = \{\sigma^{-a_1}\tilde{\tau}_i, \dots, \sigma^{-a_{r_i}}\tilde{\tau}_i\} \quad \text{and} \quad \tilde{C}''_i{}^c = \{\sigma^{-a_1}\tilde{\tau}_i^c, \dots, \sigma^{-a_{r_i}}\tilde{\tau}_i^c\}.$$

Here the choice of $\tilde{\tau}_i$ is arbitrary, and r_i is always even by construction. We put

$$\tilde{C}'_i := \{\sigma^{-a_1}\tilde{\tau}_i, \sigma^{-a_2}\tilde{\tau}_i^c, \sigma^{-a_3}\tilde{\tau}_i, \dots, \sigma^{-a_{r_i}}\tilde{\tau}_i^c\}.$$

Finally, we set $\tilde{T}'_{\mathfrak{p}} = \coprod_i \tilde{C}'_i$.

* (Case β_2) We fix an element $\tilde{\tau}_0 \in \Sigma_{E,\infty/\mathfrak{p}}$. Then the preimage of $T'_{/\mathfrak{p}}$ under the natural map $\Sigma_{E,\infty/\mathfrak{p}} \rightarrow \Sigma_{\infty/\mathfrak{p}}$ can be written as $\{\sigma^{-a_1}\tilde{\tau}_0, \dots, \sigma^{-a_{2r}}\tilde{\tau}_0\}$ (where $r = \#(\Sigma_{\infty} - \mathbf{S}_{\infty})$ is odd) with $0 = a_1 < \dots < a_{2r} \leq 2f_{\mathfrak{p}} - 1$ and $a_{r+i} = a_i + f_{\mathfrak{p}}$ for all i . We put

$$\tilde{T}'_{/\mathfrak{p}} = \{\sigma^{-a_1}\tilde{\tau}_0, \sigma^{-a_3}\tilde{\tau}_0, \sigma^{-a_5}\tilde{\tau}_0, \dots, \sigma^{-a_{2r-1}}\tilde{\tau}_0\}.$$

Since r is odd, this consists of exactly one lift of each element of $T'_{\infty/\mathfrak{p}}$.

Now we can assign integers $s_{T,\tilde{\tau}}$ according to $\tilde{\mathbf{S}}(T)$:

- if $\tau \in \Sigma_{\infty} - \mathbf{S}(T)_{\infty}$, we have $s_{\tilde{\tau}} = 1$ for all lifts $\tilde{\tau}$ of τ ;
- if $\tau \in \mathbf{S}(T)_{\infty}$ and $\tilde{\tau}$ is the lift in $\tilde{\mathbf{S}}(T)_{\infty}$, we have $s_{T,\tilde{\tau}} = 0$ and $s_{T,\tilde{\tau}^c} = 2$.

We put $\tilde{T}' = \bigcup_{\mathfrak{p} \in \Sigma_p} \tilde{T}'_{/\mathfrak{p}}$ and \tilde{T}'^c , the complex conjugations of the elements in \tilde{T}' .

Now we compare the PEL data for the Shimura varieties for G'_S and $G'_{\tilde{\mathbf{S}}(T)}$. We fix an isomorphism $\theta_T : D_S \rightarrow D_{S(T)}$ that induces an isomorphism between $\mathcal{O}_{D_S,\mathfrak{p}}$ and $\mathcal{O}_{D_{S(T)},\mathfrak{p}}$ for each $\mathfrak{p} \in \Sigma_p$, where $\mathcal{O}_{D_S,\mathfrak{p}}$ and $\mathcal{O}_{D_{S(T)},\mathfrak{p}}$ are respectively fixed maximal orders of $D_S \otimes_F F_{\mathfrak{p}}$ and $D_{S(T)} \otimes_F F_{\mathfrak{p}}$ as in § 3.7.

LEMMA 5.4. *Let $\delta_S \in (D_S^{\text{sym}})^{\times}$ be an element satisfying Lemma 3.8(1). Then there exists an element $\delta_{S(T)} \in (D_{S(T)}^{\text{sym}})^{\times}$ satisfying the same condition with \mathbf{S} replaced by $\mathbf{S}(T)$ such that, if $*_S : l \mapsto \delta_S^{-1}\bar{l}\delta_S$ and $*_{S(T)} : l \mapsto \delta_{S(T)}^{-1}\bar{l}\delta_{S(T)}$ denote the involutions on D_S and $D_{S(T)}$ induced by δ_S and $\delta_{S(T)}$, respectively, then θ_T induces an isomorphism of algebras with positive involutions $(D_S, *_S) \xrightarrow{\sim} (D_{S(T)}, *_{S(T)})$.*

Proof. We choose first an arbitrary $\delta'_{S(T)} \in (D_{S(T)}^{\text{sym}})^{\times}$ satisfying Lemma 3.8(1). Let $*'_{S(T)}$ denote the involution $l \mapsto (\delta'_{S(T)})^{-1}\bar{l}\delta'_{S(T)}$ on $D_{S(T)}$. By the Skolem–Noether theorem, there exists $g \in D_{S(T)}^{\times}$ such that $\theta_T(x)^{*'_{S(T)}} = g\theta_T(x^{*S})g^{-1}$ for all $x \in D_S$. Since both $*^2_S$ and $*^2_{S(T)}$ are the identity, we get $g^{*'_{S(T)}} = g\mu$ for some $\mu \in E^{\times}$ with $\bar{\mu}\mu = 1$. By Hilbert 90, we can write $\mu = \lambda/\bar{\lambda}$ for some $\lambda \in E^{\times}$. Up to replacing g by $g\lambda$, we may assume that $g^{*'_{S(T)}} = g$ or, equivalently, $\overline{\delta'_{S(T)}g} = \delta'_{S(T)}g$ and hence $\delta'_{S(T)}g \in (D_{S(T)}^{\text{sym}})^{\times}$. Note that we still have the freedom to modify g by an element of F^{\times} without changing $*_{S(T)}$. We claim that, up to such a modification on g , $\delta_{S(T)} = \delta'_{S(T)}g$ will answer the question. Indeed, by construction, θ_T is an $*$ -isomorphism, i.e. $\theta_T(x)^{*_{S(T)}} = \theta_T(x^{*S})$. Note that θ_T sends $\mathcal{O}_{D_S,\mathfrak{p}}$ isomorphically to $\mathcal{O}_{D_{S(T)},\mathfrak{p}}$ for every $\mathfrak{p} \in \Sigma_p$, and both lattices are invariant under the involutions $*_S$ and $*'_{S(T)}$, respectively. So, up to modifying g by an element of F^{\times} , we may assume that $g \in \mathcal{O}_{D_{S(T)},\mathfrak{p}}^{\times}$ for all $\mathfrak{p} \in \Sigma_p$. Then it is clear that $\delta_{S(T)}$ satisfies Lemma 3.8(1)(a), since so does $\delta'_{S(T)}$ by assumption. It remains to prove that, up to multiplying g by an element of $\mathcal{O}_{F,(p)}^{\times}$, the hermitian form

$$(v, w) \mapsto \psi_{\delta_{S(T)}}(v, wh'_{\tilde{\mathbf{S}}(T)}(\mathbf{i})) = \text{Tr}_{D_{S(T),\mathbb{R}}/\mathbb{R}}(\sqrt{v}wh'_{\tilde{\mathbf{S}}(T)}(\mathbf{i})^{-1}\bar{w}\delta_{S(T)})$$

on $D_{S(T),\mathbb{R}} := D_{S(T)} \otimes_{\mathbb{Q}} \mathbb{R}$ is positive definite, where $\psi_{\delta_{S(T)}}$ is the $*_{S(T)}$ -hermitian alternating form on $D_{S(T)}$ defined as in § 3.7. Since the elements δ_S and $\delta'_{S(T)}$ satisfy similar positivity conditions by assumption, we get two semi-simple \mathbb{R} -algebras with positive involutions $(D_{S,\mathbb{R}}, *_S)$ and $(D_{S(T),\mathbb{R}}, *_{S(T)})$. By [Kot92, Lemma 2.11], there exists an element $b \in D_{S(T),\mathbb{R}}^{\times}$ such that $b\theta_T(x^{*S})b^{-1} = (b\theta_T(x)b^{-1})^{*'_{S(T)}}$. It follows that $g = b^{*'_{S(T)}}b\lambda$ with $\lambda \in (F \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$. Up to multiplying g by an element of $\mathcal{O}_{F,(p)}^{\times}$, we may assume that λ is totally positive so that $\lambda = \xi^2$ with $\xi \in (F \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$. Then, up

to replacing b by $b\xi$, we have $g = b^{*\prime_{\mathfrak{S}(\mathbf{T})}}b$. Then the positivity of the form $\psi_{\delta_{\mathfrak{S}(\mathbf{T})}}$ follows immediately from the positivity of $\psi_{\delta'_{\mathfrak{S}(\mathbf{T})}}$ and the fact that $\psi_{\delta_{\mathfrak{S}(\mathbf{T})}}(v, wh'_{\mathfrak{S}(\mathbf{T})}(\mathbf{i})) = \psi_{\delta'_{\mathfrak{S}(\mathbf{T})}}(bv, bwh'_{\mathfrak{S}(\mathbf{T})}(\mathbf{i}))$. \square

LEMMA 5.5. *We keep the choice of $\delta_{\mathfrak{S}(\mathbf{T})}$ as in Lemma 5.4. Then there exists an isomorphism $\Theta_{\mathbf{T}} : D_{\mathfrak{S}}(\mathbb{A}^{\infty,p}) \rightarrow D_{\mathfrak{S}(\mathbf{T})}(\mathbb{A}^{\infty,p})$ of skew $*$ -hermitian spaces compatible with the actions of $D_{\mathfrak{S}}$ and $D_{\mathfrak{S}(\mathbf{T})}$, respectively.*

Proof. This could be done explicitly. We however prefer a sneaky quick proof. Under Morita equivalence, we are essentially working with two-dimensional hermitian spaces and the associated unitary groups. It is well known that, over a non-archimedean local field, there are exactly two hermitian spaces and the associated unitary groups are not isomorphic (see e.g. [Min11, 3.2.1]). In our situation, we know that $G'_{\mathfrak{S},1,v} \cong G'_{\mathfrak{S}(\mathbf{T}),1,v}$ for any place $v \nmid p\infty$. It follows that the associated hermitian spaces at v are isomorphic. The lemma follows. \square

COROLLARY 5.6. *The isomorphisms $\theta_{\mathbf{T}}$ and $\Theta_{\mathbf{T}}$ induce an isomorphism $\theta'_{\mathbf{T}} : \mathcal{G}'_{\mathfrak{S}} \xrightarrow{\cong} \mathcal{G}'_{\mathfrak{S}(\mathbf{T})}$. Moreover, $\theta'_{\mathbf{T}} \times \text{id}$ takes the subgroup $\mathcal{E}_{G,\mathfrak{S},k_0} \subset \mathcal{G}'_{\mathfrak{S}} \times \text{Gal}_{k_0}$ to the subgroup $\mathcal{E}_{G,\mathfrak{S}(\mathbf{T}),k_0} \subset \mathcal{G}'_{\mathfrak{S}(\mathbf{T})} \times \text{Gal}_{k_0}$.*

Proof. The first statement follows from the description of the two groups in (3.12.1) and (3.12.2) and the interpretation of these groups as certain automorphic groups of the skew $*$ -hermitian spaces. The second statement follows from the description of both subgroups in §3.12 and the observation that the choice of signatures in §5.3 ensures that the reciprocity maps $\mathfrak{R}ec_{k_0}$ for both Shimura data are the same at p . \square

5.7 Level structure of $\text{Sh}_{K'_{\mathbf{T}}}(G'_{\mathfrak{S}(\mathbf{T})})$

We now specify the level structure $K'_{\mathbf{T}} \subseteq G'_{\mathfrak{S}(\mathbf{T})}(\mathbb{A}^{\infty})$.

- For the prime-to- p level, since $\Theta_{\mathbf{T}}$ induces an isomorphism $G'_{\mathfrak{S}(\mathbf{T})}(\mathbb{A}^{\infty,p}) \simeq G'_{\mathfrak{S}}(\mathbb{A}^{\infty,p})$, the subgroup $K'^p \subseteq G'_{\mathfrak{S}}(\mathbb{A}^{\infty,p})$ corresponds to a subgroup $K'_{\mathbf{T}}{}^p \subseteq G'_{\mathfrak{S}(\mathbf{T})}(\mathbb{A}^{\infty,p})$.
- For $K'_{\mathbf{T},p}$, we take it as the open compact subgroup of $G'_{\mathfrak{S}(\mathbf{T})}(\mathbb{Q}_p)$ corresponding to $K_{\mathbf{T},p} \subseteq G'_{\mathfrak{S}}(\mathbb{Q}_p)$ by the rule in §3.9. According to the discussion there, it suffices to choose a chain of lattices $\Lambda_{\mathbf{T},\mathfrak{p}}^{(1)} \subset \Lambda_{\mathbf{T},\mathfrak{p}}^{(2)}$ in $D_{\mathfrak{S}(\mathbf{T})} \otimes_F F_{\mathfrak{p}}$ for each $\mathfrak{p} \in \Sigma_p$. Using the isomorphism $\theta_{\mathbf{T}}$, we can identify $D_{\mathfrak{S}(\mathbf{T})} \otimes_F F_{\mathfrak{p}}$ with $D_{\mathfrak{S}} \otimes_F F_{\mathfrak{p}}$ and hence with $M_2(E \otimes_F F_{\mathfrak{p}})$.
 - * For $\mathfrak{p} \in \Sigma_p$ with $K_{\mathbf{T},\mathfrak{p}} = K_{\mathfrak{p}}$, we take $\Lambda_{\mathbf{T},\mathfrak{p}}^{(1)} \subseteq \Lambda_{\mathbf{T},\mathfrak{p}}^{(2)}$ to be the same as the chain $\Lambda_{\mathfrak{p}}^{(1)} \subseteq \Lambda_{\mathfrak{p}}^{(2)}$ for defining $K'_{\mathfrak{p}} \subset G'_{\mathfrak{S}}(\mathbb{Q}_p)$.
 - * For $\mathfrak{p} \in \Sigma_p$ with $K_{\mathbf{T},\mathfrak{p}} \neq K_{\mathfrak{p}}$, $K_{\mathbf{T},\mathfrak{p}}$ is either the Iwahori subgroup of $\text{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$ or $\mathcal{O}_{B_{F_{\mathfrak{p}}}}^{\times}$. We take then $\Lambda_{\mathbf{T},\mathfrak{p}}^{(1)} \subsetneq \Lambda_{\mathbf{T},\mathfrak{p}}^{(2)}$ to be the corresponding lattice as in §3.9 that defines the Iwahori level at \mathfrak{p} .

We also specify the lattices we use for both Shimura varieties: if $\Lambda_{\mathfrak{S}}$ denotes the chosen lattice of $D_{\mathfrak{S}}$, we choose the lattice of $D_{\mathfrak{S}(\mathbf{T})}$ to be $\Lambda_{\mathfrak{S}(\mathbf{T})} = \theta_{\mathbf{T}}(\Lambda_{\mathfrak{S}})$. With these data, we have a unitary Shimura variety $\text{Sh}_{K'_{\mathbf{T}}}(G'_{\mathfrak{S}(\mathbf{T})})$ over the reflex field $E_{\mathfrak{S}(\mathbf{T})}$, which is the field corresponding to the Galois group fixing the subset $\tilde{\mathfrak{S}}(\mathbf{T}) \subseteq \Sigma_{E,\infty}$. To construct an integral model of $\text{Sh}_{K'_{\mathbf{T}}}(G'_{\mathfrak{S}(\mathbf{T})})$, we need to choose an order $\mathcal{O}_{D_{\mathfrak{S}(\mathbf{T})}}$. Let $\mathcal{O}_{D_{\mathfrak{S}}}$ be the order stable under $*$ and maximal at p used to define the integral model $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}})$. We put $\mathcal{O}_{D_{\mathfrak{S}(\mathbf{T})}} = \theta_{\mathbf{T}}(\mathcal{O}_{D_{\mathfrak{S}}})$. For any $\mathfrak{p} \in \Sigma_p$, both $\mathcal{O}_{D_{\mathfrak{S},\mathfrak{p}}}$ and $\mathcal{O}_{D_{\mathfrak{S}(\mathbf{T}),\mathfrak{p}}}$ can be identified with $M_2(\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_{\mathfrak{p}}})$.

We have now all the PEL data needed for Theorem 3.14, which ensures that $\text{Sh}_{K'_T}(G'_{\mathbb{S}(\mathbb{T})})$ admits an integral model $\mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(\mathbb{T})})$ over $W(k_0)$. Using $\mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(\mathbb{T})})$, we can construct an integral model $\mathbf{Sh}_{K_T}(G_{\mathbb{S}(\mathbb{T})})$ of the quaternionic Shimura variety $\text{Sh}_{K_T}(G_{\mathbb{S}(\mathbb{T})})$.

THEOREM 5.8. *For a subset $\mathbb{T} \subseteq \Sigma_\infty - \mathbb{S}_\infty$, let $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0, \mathbb{T}} \subseteq \mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$ denote the GO-stratum defined in Definition 4.6. Let $I_{\mathbb{T}}$ be as in Theorem 5.2. Then we have the following.*

- (1) (Description) $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0, \mathbb{T}}$ is isomorphic to a $(\mathbb{P}^1)^{I_{\mathbb{T}}}$ -bundle over $\mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(\mathbb{T})})_{k_0}$.
- (2) (Compatibility of abelian varieties) Let $\pi_{\mathbb{T}} : \mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0, \mathbb{T}} \rightarrow \mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(\mathbb{T})})_{k_0}$ denote the projection of the $(\mathbb{P}^1)^{I_{\mathbb{T}}}$ -bundle in (1). The abelian schemes $\mathbf{A}'_{\mathbb{S}, k_0}$ and $\pi_{\mathbb{T}}^* \mathbf{A}'_{\mathbb{S}(\mathbb{T}), k_0}$ over $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$ are isogenous, where $\mathbf{A}'_{\mathbb{S}, k_0}$ and $\mathbf{A}'_{\mathbb{S}(\mathbb{T}), k_0}$ denote respectively the universal abelian varieties over $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0, \mathbb{T}}$ and over $\mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(\mathbb{T})})_{k_0}$.
- (3) (Compatibility with Hecke action) Taking the limit over the open compact subgroups $K'^p \subseteq G'_{\mathbb{S}}(\mathbb{A}^{\infty, p})$, the isomorphism as well as the isogeny of abelian varieties is compatible with the action of the Hecke correspondence given by $\tilde{G}_{\mathbb{S}} = G''_{\mathbb{S}}(\mathbb{Q})^{+, (p)} G'_{\mathbb{S}}(\mathbb{A}^{\infty, p}) \cong \tilde{G}_{\mathbb{S}(\mathbb{T})}$.
- (4) (Compatibility with partial Frobenius) The description in (1) is compatible with the action of the twisted partial Frobenius (§ 3.22) in the sense that we have a commutative diagram

$$\begin{array}{ccccc}
 & & \tilde{\mathfrak{F}}'_{p^2, \mathbb{S}} & & \\
 & & \curvearrowright & & \\
 \mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0, \mathbb{T}} & \xrightarrow{\xi^{\text{rel}}} & \tilde{\mathfrak{F}}'^*_{p^2, \mathbb{S}(\mathbb{T})}(\mathbf{Sh}_{K'}(G'_{\sigma_p^2 \mathbb{S}})_{k_0, \sigma_p^2 \mathbb{T}}) & \xrightarrow{\tilde{\mathfrak{F}}'^*_{p^2, \mathbb{S}(\mathbb{T})}} & \mathbf{Sh}_{K'}(G'_{\sigma_p^2 \mathbb{S}})_{k_0, \sigma_p^2 \mathbb{T}} \\
 & \searrow \pi_{\mathbb{T}} & \downarrow & & \downarrow \pi_{\sigma_p^2 \mathbb{T}} \\
 & & \mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(\mathbb{T})})_{k_0} & \xrightarrow{\tilde{\mathfrak{F}}'_{p^2, \mathbb{S}(\mathbb{T})}} & \mathbf{Sh}_{K'_T}(G'_{\sigma_p^2 \mathbb{S}(\mathbb{T})})_{k_0}
 \end{array}$$

where the square is Cartesian, we added subscripts to the partial Frobenius to indicate the corresponding base scheme, and the morphism ξ^{rel} is a morphism whose restriction to each fiber $\pi_{\mathbb{T}}^{-1}(x) = (\mathbb{P}^1_x)^{I_{\mathbb{T}}}$ is the product of the relative p^2 -Frobenius of the \mathbb{P}^1 indexed by $I_{\mathbb{T}} \cap \Sigma_{\infty/p} = \mathbb{T}'_{\infty/p} - \mathbb{T}/p$, and the identity on the other \mathbb{P}^1_x .

The proof of this theorem will occupy the rest of this section and concludes in § 5.24. We first state a corollary.

COROLLARY 5.9. (i) *The Goren–Oort stratum $\mathbf{Sh}_{K'_p}(G'_{\mathbb{S}})_{\mathbb{F}_p, \mathbb{T}}^{\circ}$ is isomorphic to a $(\mathbb{P}^1)^{I_{\mathbb{T}}}$ -bundle over $\mathbf{Sh}_{K'_p}(G'_{\mathbb{S}(\mathbb{T})})_{\mathbb{F}_p}^{\circ}$, equivariant for the action of $\mathcal{E}_{G, \mathbb{S}, \tilde{\varphi}} \cong \mathcal{E}_{G, \mathbb{S}(\mathbb{T}), \tilde{\varphi}}$ (which are identified as in Corollary 5.6) with trivial action on the fibers.*

(ii) *The GO-stratum $\mathbf{Sh}_{K''_p}(G''_{\mathbb{S}})_{k_0, \mathbb{T}}$ is isomorphic to a $(\mathbb{P}^1)^{I_{\mathbb{T}}}$ -bundle over $\mathbf{Sh}_{K''_p}(G''_{\mathbb{S}(\mathbb{T})})_{k_0}$, such that the natural projection $\pi_{\mathbb{T}} : \mathbf{Sh}_{K''_p}(G''_{\mathbb{S}})_{k_0, \mathbb{T}} \rightarrow \mathbf{Sh}_{K''_p}(G''_{\mathbb{S}(\mathbb{T})})_{k_0}$ is equivariant for the tame Hecke action.*

(iii) *The abelian schemes $\mathbf{A}''_{\mathbb{S}, k_0}$ and $\pi_{\mathbb{T}}^*(\mathbf{A}''_{\mathbb{S}(\mathbb{T}), k_0})$ over $\mathbf{Sh}_{K''_p}(G''_{\mathbb{S}})_{k_0, \mathbb{T}}$ are isogenous.*

(iv) The following diagram is commutative:

$$\begin{array}{ccccc}
 & & \mathfrak{F}''_{p^2, \tilde{\mathfrak{S}}} & & \\
 & \xrightarrow{\quad \quad \quad} & \xrightarrow{\quad \quad \quad} & \xrightarrow{\quad \quad \quad} & \\
 \mathbf{Sh}_{K_p''}(G''_{\tilde{\mathfrak{S}}})_{k_0, \mathbb{T}} & \xrightarrow{\xi^{\text{rel}}} & \mathfrak{F}''_{p^2, \tilde{\mathfrak{S}}(\mathbb{T})}(\mathbf{Sh}_{K_p''}(G''_{\sigma_p^2 \tilde{\mathfrak{S}}})_{k_0, \sigma_p^2 \mathbb{T}}) & \xrightarrow{\mathfrak{F}''_{p^2, \tilde{\mathfrak{S}}(\mathbb{T})}} & \mathbf{Sh}_{K_p''}(G''_{\sigma_p^2 \tilde{\mathfrak{S}}})_{k_0, \sigma_p^2 \mathbb{T}} \\
 & \searrow \pi_{\mathbb{T}} & \downarrow & & \downarrow \pi_{\tilde{\mathfrak{S}}(\mathbb{T})} \\
 & & \mathbf{Sh}_{K_{\mathbb{T}, p}''}(G''_{\tilde{\mathfrak{S}}(\mathbb{T})})_{k_0} & \xrightarrow{\mathfrak{F}''_{p^2, \tilde{\mathfrak{S}}(\mathbb{T})}} & \mathbf{Sh}_{K_{\sigma_p^2 \mathbb{T}, p}''}(G''_{\sigma_p^2(\tilde{\mathfrak{S}}(\mathbb{T}))})_{k_0}
 \end{array}$$

where the square is Cartesian, $\mathfrak{F}''_{p^2, \tilde{\mathfrak{S}}}$ and $\mathfrak{F}''_{p^2, \tilde{\mathfrak{S}}(\mathbb{T})}$ denote the twisted partial Frobenii (§ 3.22) on $\mathbf{Sh}_{K_p''}(G''_{\tilde{\mathfrak{S}}})_{k_0}$ and $\mathbf{Sh}_{K_{\mathbb{T}, p}''}(G''_{\tilde{\mathfrak{S}}(\mathbb{T})})_{k_0}$, respectively, and ξ^{rel} is a morphism whose restriction to each fiber $\pi_{\mathbb{T}}^{-1}(x) = (\mathbb{P}^1_x)^{I_{\mathbb{T}}}$ is the product of the relative p^2 -Frobenius of the \mathbb{P}^1_x indexed by $I_{\mathbb{T}} \cap \Sigma_{\infty/p} = \mathbb{T}'_{\infty/p} - \mathbb{T}/p$, and the identity on the other \mathbb{P}^1_x .

Proof. This is an immediate consequence of Corollary 5.9 above. The claims regarding the universal abelian varieties follow from the explicit construction of $\mathbf{A}''_{\tilde{\mathfrak{S}}}$ and $\mathbf{A}''_{\tilde{\mathfrak{S}}(\mathbb{T})}$ in § 3.20. \square

Remark 5.10. We emphasize that the analog of Corollary 5.9(ii) for quaternionic Shimura varieties only holds over $\overline{\mathbb{F}}_p$. This is because the subgroups $\mathcal{E}_{G, \mathfrak{S}, \varphi}$ and $\mathcal{E}_{G, \mathfrak{S}(\mathbb{T}), \varphi}$, although abstractly isomorphic, sit in $\mathcal{G}_{\mathfrak{S}} \times \text{Gal}_{k_0} \cong \mathcal{G}_{\mathfrak{S}(\mathbb{T})} \times \text{Gal}_{k_0}$ as *different* subgroups. The two Deligne homomorphisms are different.

The rest of this section is devoted to the proof of Theorem 5.8, which concludes in § 5.24.

5.11 Signature changes

The basic idea of proving Theorem 5.8 is to find a quasi-isogeny between the two universal abelian varieties $\mathbf{A}_{\tilde{\mathfrak{S}}}$ and $\mathbf{B} := \mathbf{A}_{\tilde{\mathfrak{S}}(\mathbb{T})}$ (over an appropriate base). We view this quasi-isogeny as two genuine isogenies $\mathbf{A}_{\tilde{\mathfrak{S}}} \xrightarrow{\phi} \mathbf{C} \xleftarrow{\phi'} \mathbf{B}$ for some abelian variety \mathbf{C} . Each isogeny is characterized by the set of places $\tilde{\tau} \in \Sigma_{E, \infty}$ where the isogeny does not induce an isomorphism of the $\tilde{\tau}$ -components of the de Rham cohomology of the abelian varieties. We define these two subsets $\tilde{\Delta}(\mathbb{T})^+$ and $\tilde{\Delta}(\mathbb{T})^-$ of $\Sigma_{E, \infty}$ now, as follows. As before, $\tilde{\Delta}(\mathbb{T})^{\pm} = \coprod_{\mathfrak{p} \in \Sigma_p} \tilde{\Delta}(\mathbb{T})^{\pm}_{/\mathfrak{p}}$ for subsets $\tilde{\Delta}(\mathbb{T})^{\pm}_{/\mathfrak{p}} \subseteq \Sigma_{E, \infty/p}$. When \mathfrak{p} is of type $\alpha^{\#}$ or $\beta^{\#}$ for \mathfrak{S} , we set $\tilde{\Delta}(\mathbb{T})^{\pm}_{/\mathfrak{p}} = \emptyset$. For the other two types, we use the notation in § 5.3 in the corresponding cases (in particular, our convention on $\tilde{\tau}$ and the a_j).

– (Case $\alpha 1$) Put

$$\tilde{C}_i^- := \bigcup_{\substack{j \text{ odd} \\ 1 \leq j \leq r_i}} \{\sigma^{-\ell} \tilde{\tau}_i : a_j \leq \ell \leq a_{j+1} - 1\}.$$

We set $\tilde{\Delta}(\mathbb{T})^-_{/\mathfrak{p}} = \prod_i \tilde{C}_i^-$ and $\tilde{\Delta}(\mathbb{T})^+_{/\mathfrak{p}} = (\tilde{\Delta}(\mathbb{T})^-_{/\mathfrak{p}})^c$.

– (Case $\alpha 2$) Put

$$\tilde{\Delta}(\mathbb{T})^-_{/\mathfrak{p}} := \bigcup_{1 \leq i \leq r} \{\sigma^{-l} \tilde{\tau}_0 : a_{2i-1} \leq l < a_{2i}\} \quad \text{and} \quad \tilde{\Delta}(\mathbb{T})^+_{/\mathfrak{p}} := (\tilde{\Delta}(\mathbb{T})^-_{/\mathfrak{p}})^c.$$

– (Case $\beta 1$) Put

$$\tilde{C}_i^- := \bigcup_{\substack{j \text{ odd} \\ 1 \leq j \leq r_i}} \{\sigma^{-\ell} \tilde{\tau}_i : a_j \leq \ell \leq a_{j+1} - 1\}.$$

We set $\tilde{\Delta}(\mathbb{T})_{/p}^- = \coprod_i \tilde{C}_i^-$ and $\tilde{\Delta}(\mathbb{T})_{/p}^+ = (\tilde{\Delta}(\mathbb{T})_{/p}^-)^c$. (Formally, this is the same recipe as in case $\alpha 1$, but the choice of $\tilde{\tau}_i$ is less determined; see § 5.3.)

– (Case $\beta 2$) Put

$$\tilde{\Delta}(\mathbb{T})_{/p}^- := \bigcup_{1 \leq i \leq r} \{ \sigma^{-l} \tilde{\tau}_0 : a_{2i-1} \leq l < a_{2i} \}.$$

Unlike in all other cases, we put $\tilde{\Delta}(\mathbb{T})_{/p}^+ = \emptyset$.

We always have $\tilde{\Delta}(\mathbb{T})^+ \cap \tilde{\Delta}(\mathbb{T})^- = \emptyset$.

Notation 5.12. We use \mathbb{T}_E (respectively \mathbb{T}'_E) to denote the preimage of \mathbb{T} (respectively \mathbb{T}') under the map $\Sigma_{E,\infty} \rightarrow \Sigma_\infty$.

The following two lemmas follow from the definition by a case-by-case check.

LEMMA 5.13. *For each $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+$ (respectively $\tilde{\Delta}(\mathbb{T})^-$), let n be the unique positive integer such that $\tilde{\tau}, \sigma^{-1}\tilde{\tau}, \dots, \sigma^{1-n}\tilde{\tau}$ all belong to $\tilde{\Delta}(\mathbb{T})^+$ (respectively $\tilde{\Delta}(\mathbb{T})^-$) but $\sigma^{-n}\tilde{\tau}$ does not. Then, for this n , $\sigma^{-n}\tilde{\tau} \in \mathbb{T}'_E$. Moreover, if $\tilde{\tau}$ also belongs to \mathbb{T}'_E , then n equals the number n_τ introduced in § 4.4.*

LEMMA 5.14. (1) *If both $\tilde{\tau}$ and $\sigma\tilde{\tau}$ belong to $\tilde{\Delta}(\mathbb{T})^+$ (respectively $\tilde{\Delta}(\mathbb{T})^-$), then $\tilde{\tau}|_F$ belongs to \mathbb{S}_∞ .*

(2) *If $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^-$ but $\sigma\tilde{\tau} \notin \tilde{\Delta}(\mathbb{T})^-$, then $\tilde{\tau} \in \tilde{\mathbb{T}}'$.*

(3) *If $\tilde{\tau} \notin \tilde{\Delta}(\mathbb{T})^-$ but $\sigma\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^-$, then $\tilde{\tau} \in \tilde{\mathbb{T}}'^c$.*

5.15 Description of the strata $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0, \mathbb{T}}$ via isogenies

To simplify the notation, we put $X' = \mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$ and $X'_T = \mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0, \mathbb{T}}$ for a subset $\mathbb{T} \subseteq \Sigma_\infty - \mathbb{S}_\infty$. We will first prove statement (1) of Theorem 5.8. Following the idea of Helm [Hel12], we introduce auxiliary moduli spaces Y'_T and Z'_T and establish isomorphisms

$$X'_T \xleftarrow[\cong]{\eta_1} Y'_T \xrightarrow[\cong]{\eta_2} Z'_T, \tag{5.15.1}$$

where Z'_T is a $(\mathbb{P}^1)^{I_T}$ -bundle over the special fiber of $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$.

Recall that we have fixed an isomorphism $\theta_T : (D_S, *_S) \xrightarrow{\sim} (D_{\mathbb{S}(\mathbb{T})}, *_{\mathbb{S}(\mathbb{T})})$ of simple algebras over E with positive involution, and put $\mathcal{O}_{D_{\mathbb{S}(\mathbb{T})}} = \theta_T(\mathcal{O}_{D_S})$. To ease the notation, we identify $\mathcal{O}_{D_{\mathbb{S}(\mathbb{T})}}$ with \mathcal{O}_{D_S} via θ_T , and denote them by \mathcal{O}_D when there is no confusion.

We start now to describe Y'_T : it is the moduli space over k_0 which attaches to a locally noetherian k_0 -scheme S the set of isomorphism classes of $(A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_{K'}, C, \iota_C; \phi_A, \phi_B)$, where we have the following.

- (i) $(A, \iota_A, \lambda_A, \alpha_{K'})$ is an element in $X'_T(S)$.
- (ii) $(B, \iota_B, \lambda_B, \beta_{K'})$ is an element in $\mathbf{Sh}_{K'}(G'_{\mathbb{S}(\mathbb{T})})(S)$.
- (iii) C is an abelian scheme over S of dimension $4g$, equipped with an embedding $\iota_C : \mathcal{O}_D \rightarrow \text{End}_S(C)$.
- (iv) $\phi_A : A \rightarrow C$ is an \mathcal{O}_D -isogeny whose kernel is killed by p , such that the induced map

$$\phi_{A,*,\tilde{\tau}} : H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^\circ \rightarrow H_1^{\text{dR}}(C/S)_{\tilde{\tau}}^\circ$$

is an isomorphism for $\tilde{\tau} \in \Sigma_{E,\infty}$ unless $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+$, in which case we require that

$$\text{Ker}(\phi_{A,*,\tilde{\tau}}) = \text{Im}(F_{A,\text{es},\tilde{\tau}}^n), \tag{5.15.2}$$

where n is as determined in Lemma 5.13 and $F_{A,es,\tilde{\tau}}^n$ is defined in (4.2.4). (When τ itself belongs to T' , the number n equals n_τ introduced in § 4.4. In this case, condition (5.15.2) is equivalent to saying that $\text{Ker}(\phi_{A,*,\tilde{\tau}}) = \omega_{A^\vee/S,\tilde{\tau}}^\circ$)

- (v) $\phi_B : B \rightarrow C$ is an \mathcal{O}_D -isogeny whose kernel is killed by p such that $\phi_{B,*,\tilde{\tau}} : H_1^{\text{dR}}(B/S)_{\tilde{\tau}}^\circ \rightarrow H_1^{\text{dR}}(C/S)_{\tilde{\tau}}^\circ$ is an isomorphism for $\tilde{\tau} \in \Sigma_{E,\infty}$ unless $\tilde{\tau} \in \tilde{\Delta}(T)^-$, in which case we require that $\text{Im}(\phi_{B,*,\tilde{\tau}})$ is equal to $\phi_{A,*,\tilde{\tau}}(\text{Im}(F_{A,es,\tilde{\tau}}^n))$, where n is as determined in Lemma 5.13. (Note that as $\tilde{\tau} \notin \tilde{\Delta}(T)^+$, $\phi_{A,*,\tilde{\tau}}$ is an isomorphism by (iv).)
- (vi) The tame level structures are compatible, i.e. $T^{(p)}(\phi_A) \circ \alpha_{K'^p}^p = T^{(p)}(\phi_B) \circ \alpha_{K'^p}^p$ as maps from $\widehat{\Lambda}_S^{(p)} \cong \widehat{\Lambda}_{S(T)}^{(p)}$ to $T^{(p)}(C)$, modulo K'^p , if we identify the two lattices naturally as in § 5.7.
- (vii) If \mathfrak{p} is a prime of type α^\sharp for the original quaternionic Shimura variety $\text{Sh}_K(G_S)$, then $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ are compatible, i.e. $\phi_A(\alpha_{\mathfrak{p}}) = \phi_B(\beta_{\mathfrak{p}})$, where $\alpha_{\mathfrak{p}} \subseteq A[\mathfrak{p}]$ denotes the closed finite flat group scheme given by Theorem 3.14(c2). (Note that ϕ_A and ϕ_B induce isomorphisms between $A[\mathfrak{p}]$, $C[\mathfrak{p}]$, and $B[\mathfrak{p}]$ for a prime \mathfrak{p} of type α^\sharp for S .)
- (viii) Let \mathfrak{p} be a prime in Case $\alpha 2$, splitting into $\mathfrak{q}\bar{\mathfrak{q}}$ in E . Then $\beta_{\mathfrak{p}} = H_{\mathfrak{q}} \oplus H_{\bar{\mathfrak{q}}}$. If $\phi_{\mathfrak{q}} : B \rightarrow B'_{\mathfrak{q}} = B/H_{\mathfrak{q}}$ is the canonical isogeny, then the kernel of the induced map $\phi_{\mathfrak{q},*} : H_1^{\text{dR}}(B/S)_{\tilde{\tau}}^\circ \rightarrow H_1^{\text{dR}}(B'_{\mathfrak{q}}/S)_{\tilde{\tau}}^\circ$ coincides with that of $\phi_{B,*} : H_1^{\text{dR}}(B/S)_{\tilde{\tau}}^\circ \rightarrow H_1^{\text{dR}}(C/S)_{\tilde{\tau}}^\circ$ for all $\tilde{\tau} \in \tilde{\Delta}(T)_{/\mathfrak{p}}^-$.
- (ix) We have the following commutative diagram.

$$\begin{array}{ccccc}
 A & \xrightarrow{\phi_A} & C & \xleftarrow{\phi_B} & B \\
 \lambda_A \downarrow & & & & \downarrow \lambda_B \\
 A^\vee & \xleftarrow{\phi_A^\vee} & C^\vee & \xrightarrow{\phi_B^\vee} & B^\vee
 \end{array}$$

Remark 5.16. Compared to [Hel12], our moduli problem appears to be more complicated. This is because we allow places above p of F to be inert in the CM extension E . It is clear that B is quasi-isogenous to A . So, when S is the spectrum of a perfect field k , the covariant Dieudonné module \tilde{D}_B is a $W(k)$ -lattice in $\tilde{D}_A[1/p]$. The complicated conditions (v) and (vi) can be better understood by looking at \tilde{D}_B (see the proof of Proposition 5.17 below).

PROPOSITION 5.17. *The natural forgetful functor*

$$\eta_1 : (A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_{K'_T}, C, \iota_C; \phi_A, \phi_B) \mapsto (A, \iota_A, \lambda_A, \alpha_{K'})$$

induces an isomorphism $\eta_1 : Y'_T \rightarrow X'_T$.

Proof. By the general theory of moduli spaces of abelian schemes due to Mumford, Y'_T is representable by an k_0 -scheme of finite type. Hence, to prove the proposition, it suffices to show that the natural map $Y'_T \rightarrow X'_T$ induces a bijection on closed points and the tangent spaces at each closed point. The proposition will thus follow from Lemmas 5.18 and 5.20 below. This is a long and tedious book-keeping check, essentially following the ideas of [Hel12, Proposition 4.4]. \square

LEMMA 5.18. *Let $x = (A, \iota_A, \lambda_A, \alpha_{K'}) \in X'_T(k)$ for a perfect field k . Then there exist unique $(B, \iota_B, \lambda_B, \beta_{K'_T}, C, \iota_C; \phi_A, \phi_B)$ such that $(A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_{K'_T}, C, \iota_C; \phi_A, \phi_B) \in Y'_T(k)$.*

Proof. We first recall some notation regarding Dieudonné modules. Let $\tilde{\mathcal{D}}_A$ denote the covariant Dieudonné module of $A[p^\infty]$. Then $\mathcal{D}_A := \tilde{\mathcal{D}}_A/p$ is the covariant Dieudonné module of $A[p]$. Given the action of $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq M_2(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ on A , we have direct sum decompositions

$$\tilde{\mathcal{D}}_A^\circ := \epsilon \tilde{\mathcal{D}}_A = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ, \quad \mathcal{D}_A^\circ := \epsilon \mathcal{D}_A = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} \mathcal{D}_{A,\tilde{\tau}}^\circ,$$

where ϵ denotes the idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By the theory of Dieudonné modules, we have canonical isomorphisms

$$H_1^{\text{cris}}(A/k)_{W(k)} \cong \tilde{\mathcal{D}}_A, \quad H_1^{\text{dR}}(A/k) \cong \mathcal{D}_A,$$

compatible with all the structures. For $\tilde{\tau} \in \Sigma_{E,\infty}$, we have the Hodge filtration $0 \rightarrow \omega_{A^\vee,\tilde{\tau}}^\circ \rightarrow \mathcal{D}_{A,\tilde{\tau}}^\circ \rightarrow \text{Lie}(A)_{\tilde{\tau}}^\circ \rightarrow 0$. We use $\tilde{\omega}_{A^\vee,\tilde{\tau}}^\circ$ to denote the preimage of $\omega_{A^\vee,\tilde{\tau}}^\circ \subseteq \mathcal{D}_{A,\tilde{\tau}}^\circ$ under the reduction map $\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ \rightarrow \mathcal{D}_{A,\tilde{\tau}}^\circ$.

We first construct C from A . For each $\tilde{\tau} \in \Sigma_{E,\infty}$, we define a $W(k)$ -module $M_{\tilde{\tau}}^\circ$ with $\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ \subseteq M_{\tilde{\tau}}^\circ \subseteq p^{-1}\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ as follows. We put $M_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ unless $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+$. In the exceptional case, let n be the integer as determined in Lemma 5.13 (or as in property (v) of Y'_T above), and put

$$M_{\tilde{\tau}}^\circ := p^{-1}F_{A,\text{es}}^n(\tilde{\mathcal{D}}_{A,\sigma^{-n}\tilde{\tau}}^\circ),$$

where $F_{A,\text{es}}^n$ is the n -iteration of the essential Frobenius on $\tilde{\mathcal{D}}_A^\circ$ defined in Notation 4.2.

If $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+ \cap \mathbb{T}_E$, then the number n for $\tilde{\tau}$ in Lemma 5.13 coincides with n_τ introduced in § 4.4. Since the partial Hasse invariant $h_{\tilde{\tau}}(A)$ vanishes for any $\tilde{\tau} \in \mathbb{T}_E$ by the definition of X'_T , we see that $M_{\tilde{\tau}}^\circ = p^{-1}\tilde{\omega}_{A,\tilde{\tau}}^\circ$ for $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+ \cap \mathbb{T}_E$.

We now check that, for any $\tilde{\tau} \in \Sigma_{E,\infty}$,

$$F_A(M_{\sigma^{-1}\tilde{\tau}}^\circ) \subseteq M_{\tilde{\tau}}^\circ \quad \text{and} \quad V_A(M_{\tilde{\tau}}^\circ) \subseteq M_{\sigma^{-1}\tilde{\tau}}^\circ. \tag{5.18.1}$$

Note that we are using the genuine but not essential Frobenius and Verschiebung here. We distinguish several cases.

- $\tilde{\tau}, \sigma^{-1}\tilde{\tau} \notin \tilde{\Delta}(\mathbb{T})^+$. Then $M_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ and $M_{\sigma^{-1}\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{A,\sigma^{-1}\tilde{\tau}}^\circ$. Hence, (5.18.1) is clear.
- $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+$ and $\sigma^{-1}\tilde{\tau} \notin \tilde{\Delta}(\mathbb{T})^+$. Then we have $M_{\sigma^{-1}\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{A,\sigma^{-1}\tilde{\tau}}^\circ$ and $M_{\tilde{\tau}}^\circ = p^{-1}F_A(\tilde{\mathcal{D}}_{A,\sigma^{-1}\tilde{\tau}}^\circ)$. Hence, $F_A(M_{\sigma^{-1}\tilde{\tau}}^\circ) \subseteq M_{\tilde{\tau}}^\circ$ is trivial, and $V_A(M_{\tilde{\tau}}^\circ) = M_{\sigma^{-1}\tilde{\tau}}^\circ$.
- $\tilde{\tau}, \sigma^{-1}\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+$. Let n be the positive integer for $\tilde{\tau}$ as in Lemma 5.13. Then we have

$$M_{\tilde{\tau}}^\circ = p^{-1}F_{A,\text{es}}^n(\tilde{\mathcal{D}}_{A,\sigma^{-n}\tilde{\tau}}^\circ) = F_{A,\text{es}}(p^{-1}F_{A,\text{es}}^{n-1}(\tilde{\mathcal{D}}_{A,\sigma^{-n}\tilde{\tau}}^\circ)) = F_{A,\text{es}}(M_{\sigma^{-1}\tilde{\tau}}^\circ).$$

The inclusions (5.18.1) are clear from this.

- $\tilde{\tau} \notin \tilde{\Delta}(\mathbb{T})^+$ and $\sigma^{-1}\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+$. In this case, $\sigma^{-1}\tilde{\tau}$ must be in \mathbb{T}_E by Lemma 5.13. Hence, we have $M_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ and $M_{\sigma^{-1}\tilde{\tau}}^\circ = p^{-1}\tilde{\omega}_{A,\sigma^{-1}\tilde{\tau}}^\circ$ as remarked above. We thus see that $F_A(M_{\sigma^{-1}\tilde{\tau}}^\circ) = M_{\tilde{\tau}}^\circ$ and $V_A(M_{\tilde{\tau}}^\circ) = pM_{\sigma^{-1}\tilde{\tau}}^\circ$.

Consequently, if we put $M^\circ = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} M_{\tilde{\tau}}^\circ$ and $M = (M^\circ)^{\oplus 2}$, then M is a Dieudonné module, and $\tilde{\mathcal{D}}_A \subseteq M \subseteq p^{-1}\tilde{\mathcal{D}}_A$ with induced F and V on M . Consider the quotient $M/\tilde{\mathcal{D}}_A$. It corresponds to a finite subgroup scheme K of $A[p]$ stable under the action of \mathcal{O}_D by the covariant Dieudonné theory. We put $C = A/K$ and let $\phi_A : A \rightarrow C$ denote the natural quotient, so that the induced map $\phi_{A,*} : \tilde{\mathcal{D}}_A \rightarrow \tilde{\mathcal{D}}_C$ is identified with the natural inclusion $\tilde{\mathcal{D}}_A \hookrightarrow M$. The morphisms F_C and V_C on $\tilde{\mathcal{D}}_C$ are induced from those on $\tilde{\mathcal{D}}_A[1/p]$. It is clear that C is equipped with a natural action ι_C by \mathcal{O}_D , and ϕ_A satisfies conditions (iii) and (iv) for the moduli space Y'_T .

Conversely, if C exists, the conditions (iii) and (iv) imply that $\tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ$ has to coincide with $M_{\tilde{\tau}}^\circ$. Therefore, C is uniquely determined by A . We finally remark that, by construction, $\tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ/\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ is isomorphic to k if $\tilde{\tau} \in \tilde{\Delta}(\mathbf{T})^+$ and trivial otherwise.

We now construct the abelian variety B and the isogeny $\phi_B : B \rightarrow C$. Similarly to the construction for C , we will first define a $W(k)$ -lattice $N^\circ = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} N_{\tilde{\tau}}^\circ \subseteq \tilde{\mathcal{D}}_C^\circ$, with $N_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ$ unless $\tilde{\tau} \in \tilde{\Delta}(\mathbf{T})^-$. In the exceptional case, we put $N_{\tilde{\tau}}^\circ = F_{A,\text{es}}^n(\tilde{\mathcal{D}}_{C,\sigma^{-n\tilde{\tau}}}^\circ)$, where n is the positive integer given by Lemma 5.13. Here we view $\tilde{\mathcal{D}}_{C,\sigma^{-n\tilde{\tau}}}^\circ$ as a lattice of $\tilde{\mathcal{D}}_{A,\sigma^{-n\tilde{\tau}}}^\circ[1/p]$, so that $F_{A,\text{es}}^n(\tilde{\mathcal{D}}_{C,\sigma^{-n\tilde{\tau}}}^\circ)$ makes sense. Note once again that, if $\tilde{\tau} \in \tilde{\Delta}(\mathbf{T})^- \cap \mathbf{T}_E$, then n equals n_τ defined in 4.4, and we have $N_{\tilde{\tau}}^\circ = \tilde{\omega}_{C,\tilde{\tau}}^\circ \simeq \tilde{\omega}_{A,\tilde{\tau}}^\circ$, since $h_{\tilde{\tau}}(A)$ vanishes. We now check that N° is stable under F_C and V_C , i.e. $F_C(N_{\sigma^{-1}\tilde{\tau}}^\circ) \subseteq N_{\tilde{\tau}}^\circ$ and $V_C(N_{\tilde{\tau}}^\circ) \subseteq N_{\sigma^{-1}\tilde{\tau}}^\circ$ for all $\tilde{\tau} \in \Sigma_{E,\infty}$. The same arguments for M above work verbatim in this case (with $\tilde{\Delta}(\mathbf{T})^-$ in place of $\tilde{\Delta}(\mathbf{T})^+$). Again, we point out that, by construction, $\tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ/\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ is isomorphic to k if $\tilde{\tau} \in \tilde{\Delta}(\mathbf{T})^-$ and trivial otherwise.

Therefore, $N = (N^\circ)^{\oplus 2}$ is a Dieudonné module such that the inclusions $p\tilde{\mathcal{D}}_C \subseteq N \subseteq \tilde{\mathcal{D}}_C$ respect the Frobenius and Verschiebung actions. In particular, the Dieudonné submodule $N/p\tilde{\mathcal{D}}_C \subset \tilde{\mathcal{D}}_C/p\tilde{\mathcal{D}}_C$ is the covariant Dieudonné module of a closed finite subgroup scheme $H \subset C[p]$ stable under the action of \mathcal{O}_D . We put $B = C/H$ and define $\phi_B : B \rightarrow C$ to be the isogeny such that the composite $C \rightarrow B = C/H \xrightarrow{\phi_B} C$ is the multiplication by p . Then the induced morphism $\phi_{B,*} : \tilde{\mathcal{D}}_B \rightarrow \tilde{\mathcal{D}}_C$ is identified with the natural inclusion $N \subseteq \tilde{\mathcal{D}}_C$. It is clear that B is equipped with a natural action by \mathcal{O}_D , and the condition (v) for the moduli space Y'_T is satisfied. Conversely, if the abelian variety B exists, then condition (v) implies that $\tilde{\mathcal{D}}_B^\circ$ has to be N° defined above. This means that B is uniquely determined by C and thus by A .

To see condition (ix) of the moduli space Y'_T , we consider the quasi-isogeny

$$\lambda_B : B \xrightarrow{\phi_B} C \xleftarrow{\phi_A} A \xrightarrow{\lambda_A} A^\vee \xleftarrow{\phi_A^\vee} C^\vee \xrightarrow{\phi_B^\vee} B^\vee.$$

We have to show that λ_B is a genuine isogeny and verify that it satisfies conditions (b2) and (b3) in Theorem 3.14 for the Shimura variety $\mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(\mathbf{T})})$. It is equivalent to proving that, when viewing $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ as a $W(k)$ -lattice of $\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ[1/p]$ via the quasi-isogeny $B \xrightarrow{\phi_B} C \xleftarrow{\phi_A} A$, the perfect alternating pairing

$$\langle \cdot, \cdot \rangle_{\lambda_A,\tilde{\tau}} : \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ[1/p] \times \tilde{\mathcal{D}}_{A,\tilde{\tau}^c}^\circ[1/p] \rightarrow W(k)[1/p]$$

for $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$ induces a perfect pairing of $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ \times \tilde{\mathcal{D}}_{B,\tilde{\tau}^c}^\circ \rightarrow W(k)$ if \mathfrak{p} is not of type β^\sharp for $\mathbf{S}(\mathbf{T})$, and induces an inclusion $\tilde{\mathcal{D}}_{B,\tilde{\tau}^c}^\circ \subset \tilde{\mathcal{D}}_{B,\tilde{\tau}}^{\circ,\vee}$ with quotient equal to k if \mathfrak{p} is of type β^\sharp for $\mathbf{S}(\mathbf{T})$. We discuss this case by case.

- If \mathfrak{p} is of type β^\sharp for \mathbf{S} , then both ϕ_A and ϕ_B induce isomorphisms on the \mathfrak{p} -divisible groups and the statement is clear in this case.
- If \mathfrak{p} is in Case $\beta 2$, $\tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^+ = \emptyset$. By the construction of B , we have $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ unless $\tilde{\tau} \in \tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^-$; in the latter case, $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ = \tilde{F}_{\text{es},\tilde{\tau}}^n(\tilde{\mathcal{D}}_{A,\sigma^{-n\tilde{\tau}}}^\circ)$ is a submodule of $\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ with quotient isomorphic to k . Note that $\Sigma_{E,\infty/\mathfrak{p}} = \tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^- \amalg (\tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^-)^c$. This implies that the pairing $\langle \cdot, \cdot \rangle_{\lambda_A,\tilde{\tau}}$ induces an inclusion $\tilde{\mathcal{D}}_{B,\tilde{\tau}^c}^\circ \subset \tilde{\mathcal{D}}_{B,\tilde{\tau}}^{\circ,\vee}$ with quotient equal to k .
- In all other cases, we have $\tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^+ = (\tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^-)^c$. So,

$$\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ = \begin{cases} \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ & \text{if } \tilde{\tau} \notin (\tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^+ \cup \tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^-), \\ p^{-1}F_{A,\text{es}}^n(\tilde{\mathcal{D}}_{A,\sigma^{-n\tilde{\tau}}}^\circ) & \text{if } \tilde{\tau} \in \tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^+, \\ F_{A,\text{es}}^n(\tilde{\mathcal{D}}_{A,\sigma^{-n\tilde{\tau}}}^\circ) & \text{if } \tilde{\tau} \in \tilde{\Delta}(\mathbf{T})_{/\mathfrak{p}}^-. \end{cases} \tag{5.18.2}$$

It is clear that $\langle \cdot, \cdot \rangle_{\lambda_A}$ induces a perfect pairing on $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ \times \tilde{\mathcal{D}}_{B,\tilde{\tau}^c}^\circ$ if $\tilde{\tau} \notin (\tilde{\Delta}(\mathbb{T})_{/\mathfrak{p}}^+ \cup \tilde{\Delta}(\mathbb{T})_{/\mathfrak{p}}^-)$. If $\tilde{\tau} \in (\tilde{\Delta}(\mathbb{T})_{/\mathfrak{p}}^+ \cup \tilde{\Delta}(\mathbb{T})_{/\mathfrak{p}}^-)$, the perfect duality between $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ and $\tilde{\mathcal{D}}_{B,\tilde{\tau}^c}^\circ$ follows from the equality

$$\langle p^{-1}F_{A,es}^n u, F_{A,es}^n v \rangle_{\lambda_A, \tilde{\tau}} = \langle u, v \rangle_{\lambda_A, \sigma^{-n}\tilde{\tau}}$$

for all $u \in \tilde{\mathcal{D}}_{A, \sigma^{-n}\tilde{\tau}}^\circ$ and $v \in \tilde{\mathcal{D}}_{A, \sigma^{-n}\tilde{\tau}^c}^\circ$.

This completes the verification of condition (viii) of the moduli space $Y_{\mathbb{T}}'$ and conditions (b2) and (b3) in Theorem 3.14 for $\mathbf{Sh}_{K_{\mathbb{T}}'}(G'_{\mathbb{S}(\mathbb{T})})$. It is also clear that λ_B induces the involution $*_{\mathbb{S}(\mathbb{T})}$ on $\mathcal{O}_D = \mathcal{O}_{D_{\mathbb{S}(\mathbb{T})}}$.

We now check that the abelian variety B has the correct signature required by the moduli space $\mathbf{Sh}_{K_{\mathbb{T}}'}(G'_{\mathbb{S}(\mathbb{T})})$. For convenience of future reference, we put this into the following lemma.

LEMMA 5.19. *In the setup above, that is, knowing*

$$\dim \text{Coker}(\phi_{A,*,\tilde{\tau}}) = \delta_{\tilde{\Delta}(\mathbb{T})^+}(\tilde{\tau}), \quad \dim \text{Coker}(\phi_{B,*,\tilde{\tau}}) = \delta_{\tilde{\Delta}(\mathbb{T})^-}(\tilde{\tau}),$$

where $\delta_{\bullet}(\cdot)$ is 1 if $\cdot \in \bullet$ and is 0 if $\cdot \notin \bullet$, we have $\dim \omega_{B^{\vee}/k, \tilde{\tau}}^\circ = 2 - s_{\mathbb{T}, \tilde{\tau}}$ for all $\tilde{\tau} \in \Sigma_{E, \infty}$ if and only if $\dim \omega_{A^{\vee}/k, \tilde{\tau}}^\circ = 2 - s_{\tilde{\tau}}$ for all $\tilde{\tau} \in \Sigma_{E, \infty}$, with the numbers $s_{\mathbb{T}, \tilde{\tau}}$ defined as in § 5.3.

Proof. This is a simple dimension count. We prove the sufficiency, and the necessity follows by reversing the argument. Using the signature condition for the Shimura variety $X_{\mathbb{T}}'$, we have

$$s_{\tilde{\tau}} = \dim_k (\tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ / V(\tilde{\mathcal{D}}_{A, \sigma\tilde{\tau}}^\circ)).$$

Comparing this with the abelian variety B , we have

$$\dim_k \frac{\tilde{\mathcal{D}}_{B, \tilde{\tau}}^\circ}{V(\tilde{\mathcal{D}}_{B, \sigma\tilde{\tau}}^\circ)} = \dim_k \frac{\tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ}{V(\tilde{\mathcal{D}}_{A, \sigma\tilde{\tau}}^\circ)} - \left(\dim_k \frac{\tilde{\mathcal{D}}_{C, \tilde{\tau}}^\circ}{\tilde{\mathcal{D}}_{B, \tilde{\tau}}^\circ} - \dim_k \frac{\tilde{\mathcal{D}}_{C, \tilde{\tau}}^\circ}{\tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ} \right) + \left(\dim_k \frac{\tilde{\mathcal{D}}_{C, \sigma\tilde{\tau}}^\circ}{\tilde{\mathcal{D}}_{B, \sigma\tilde{\tau}}^\circ} - \dim_k \frac{\tilde{\mathcal{D}}_{C, \sigma\tilde{\tau}}^\circ}{\tilde{\mathcal{D}}_{A, \sigma\tilde{\tau}}^\circ} \right);$$

here we used the fact that the quotient $\tilde{\mathcal{D}}_{C, \sigma\tilde{\tau}}^\circ / \tilde{\mathcal{D}}_{B, \sigma\tilde{\tau}}^\circ$ has the same dimension as $V(\tilde{\mathcal{D}}_{C, \sigma\tilde{\tau}}^\circ) / V(\tilde{\mathcal{D}}_{B, \sigma\tilde{\tau}}^\circ)$ and the same for A in place of B because V is equivariant. Using our construction of the abelian varieties B and C , we deduce that

$$\dim_k (\tilde{\mathcal{D}}_{B, \tilde{\tau}}^\circ / V(\tilde{\mathcal{D}}_{B, \sigma\tilde{\tau}}^\circ)) = s_{\tilde{\tau}} - (\delta_{\tilde{\Delta}(\mathbb{T})^-}(\tilde{\tau}) - \delta_{\tilde{\Delta}(\mathbb{T})^+}(\tilde{\tau})) + (\delta_{\tilde{\Delta}(\mathbb{T})^-}(\sigma\tilde{\tau}) - \delta_{\tilde{\Delta}(\mathbb{T})^+}(\sigma\tilde{\tau})). \tag{5.19.1}$$

Using the definition of $\tilde{\Delta}(\mathbb{T})^\pm$, one checks case-by-case that the expression (5.19.1) is equal to $s_{\mathbb{T}, \tilde{\tau}}$. We will only indicate the proof when $\tilde{\tau} \in \Sigma_{E, \infty/\mathfrak{p}}$ for \mathfrak{p} in Case $\alpha 1$, and leave the other cases as an exercise for the interested reader. Indeed, under the notation from § 5.11, when $\mathfrak{p} \in \Sigma_p$ is of type $\alpha 1$, $\tilde{\Delta}(\mathbb{T})_{/\mathfrak{p}}^\pm = \coprod_i \tilde{C}_i^\pm$. Then

$$\delta_{\tilde{\Delta}(\mathbb{T})^+}(\tilde{\tau}) - \delta_{\tilde{\Delta}(\mathbb{T})^+}(\sigma\tilde{\tau}) = \begin{cases} 1 & \text{if } \tilde{\tau} \text{ is one of } \sigma^{-a_1}\tilde{\tau}_i^c, \sigma^{-a_3}\tilde{\tau}_i^c, \dots, \\ -1 & \text{if } \tilde{\tau} \text{ is one of } \sigma^{-a_2}\tilde{\tau}_i^c, \sigma^{-a_4}\tilde{\tau}_i^c, \dots, \\ 0 & \text{otherwise;} \end{cases}$$

$$\text{and } \delta_{\tilde{\Delta}(\mathbb{T})^-}(\tilde{\tau}) - \delta_{\tilde{\Delta}(\mathbb{T})^-}(\sigma\tilde{\tau}) = \begin{cases} 1 & \text{if } \tilde{\tau} \text{ is one of } \sigma^{-a_1}\tilde{\tau}_i, \sigma^{-a_3}\tilde{\tau}_i, \dots, \\ -1 & \text{if } \tilde{\tau} \text{ is one of } \sigma^{-a_2}\tilde{\tau}_i, \sigma^{-a_4}\tilde{\tau}_i, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Putting these two formulas together and using the notation from § 5.3, we have

$$(\delta_{\tilde{\Delta}(\mathbb{T})^+}(\tilde{\tau}) - \delta_{\tilde{\Delta}(\mathbb{T})^+}(\sigma\tilde{\tau})) - (\delta_{\tilde{\Delta}(\mathbb{T})^-}(\tilde{\tau}) - \delta_{\tilde{\Delta}(\mathbb{T})^-}(\sigma\tilde{\tau})) = \delta_{\tilde{\mathbb{T}}'}(\tilde{\tau}^c) - \delta_{\tilde{\mathbb{T}}'}(\tilde{\tau}).$$

This implies that (5.19.1) is equal to $s_{\mathbb{T}, \tilde{\tau}}$ and concludes the proof of Lemma 5.19. □

We now continue our proof of Lemma 5.18 (as part of the proof of Proposition 5.17). To fulfill condition (vi) of the moduli space Y'_T , the tame level structure on B is chosen and determined as the composite

$$\beta_{K'_T}^p : \widehat{\Lambda}_{\mathfrak{S}(T)}^{(p)} \xrightarrow{\theta_T^{-1}} \widehat{\Lambda}_{\mathfrak{S}}^{(p)} \xrightarrow{\alpha} T^{(p)}(A) \xrightarrow{T^{(p)}(\phi_A)} T^{(p)}(C) \xrightarrow{T^{(p)}(\phi_B)^{-1}} T^{(p)}(B),$$

where both $T^{(p)}(\phi_A)$ and $T^{(p)}(\phi_B)$ are isomorphisms because ϕ_A and ϕ_B are p -isogenies.

It remains to show that there exists a unique collection of subgroups β_p satisfying Theorem 3.14(c2) for $\mathbf{Sh}_{K'_T}(G'_{\mathfrak{S}(T)})$ and properties (vii) and (viii) of Y'_T . So, the corresponding prime $\mathfrak{p} \in \Sigma_p$ is either of type α^\sharp for \mathfrak{S} or in the Case $\alpha 2$ of §5.1. In the former case, we have $T/\mathfrak{p} = \emptyset$, which forces $\Delta(T)_{\mathfrak{p}}^\pm = \emptyset$ by definition. So, the induced morphisms $\phi_{A,\mathfrak{p}} : A[\mathfrak{p}^\infty] \rightarrow C[\mathfrak{p}^\infty]$ and $\phi_{B,\mathfrak{p}} : B[\mathfrak{p}^\infty] \rightarrow C[\mathfrak{p}^\infty]$ are isomorphisms. Now condition (vii) of the moduli space Y'_T determines that the level structure β_p is taken to be $\phi_{B,\mathfrak{p}}^{-1}(\phi_{A,\mathfrak{p}}(\alpha_p))$.

If \mathfrak{p} is in Case $\alpha 2$ of §5.1, the prime \mathfrak{p} splits into two primes \mathfrak{q} and $\bar{\mathfrak{q}}$ in E . Using the polarization λ_B , we just need to show that there exists a unique subgroup $H_q \subseteq B[\mathfrak{q}]$ satisfying condition (vii). Since $s_{T,\tilde{\tau}} = 0$ or 2 for $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$, both $F_{B,es,\tilde{\tau}}$ and $V_{B,es,\tilde{\tau}}$ are isomorphisms. We define a one-dimensional k -vector subspace $\mathcal{D}_{H_q}^\circ \subseteq \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ/p\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ for each $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{q}}$ as follows.

- If $\tilde{\tau} \in \tilde{\Delta}(T)_{\mathfrak{p}}^-$, then $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ is contained in $\tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ \cong \tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ with quotient isomorphic to k . Put $\mathcal{D}_{H_q}^\circ = p\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ/p\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$.
- If $\tilde{\tau} \notin \tilde{\Delta}(T)_{\mathfrak{p}}^-$, let $n \in \mathbb{N}$ be the least positive integer such that $\sigma^{-n}\tilde{\tau} \in \tilde{\Delta}(T)_{\mathfrak{p}}^-$ (such n exists because $\tilde{\tau}_0$ in §5.3 belongs to $\tilde{\Delta}(T)_{\mathfrak{p}}^-$). Put $\mathcal{D}_{H_q,\tilde{\tau}}^\circ = F_{B,es}^n(\mathcal{D}_{H_q,\sigma^{-n}\tilde{\tau}}^\circ)$.

Put $\mathcal{D}_{H_q} = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{q}}} \mathcal{D}_{H_q,\tilde{\tau}}^{\circ,\oplus 2}$. Using the vanishing of the partial Hasse invariants $\{h_{\tilde{\tau}}(A) : \tilde{\tau} \in T_{E,\infty/\mathfrak{p}}\}$, one checks easily that $\mathcal{D}_{H_q} \subseteq \mathcal{D}_{B[\mathfrak{q}]}$ is a Dieudonné submodule. We define $H_q \subseteq B[\mathfrak{q}]$ as the finite subgroup scheme corresponding to \mathcal{D}_{H_q} by covariant Dieudonné theory. Then \mathcal{D}_{H_q} is canonically identified with the kernel of the induced map

$$\phi_{q,*} : \mathcal{D}_B = H_1^{\text{dR}}(B/k) \rightarrow \mathcal{D}_{B/H_q} = H_1^{\text{dR}}((B/H_q)/k).$$

Therefore, H_q satisfies condition (viii) of the moduli space Y'_T . This shows the existence of H_q . For the uniqueness, the condition (viii) forces the choice of $\mathcal{D}_{H_q,\tilde{\tau}}^\circ$ for $\tilde{\tau} \in \tilde{\Delta}(T)_{\mathfrak{p}}^-$ and the stability under F_B and V_B forces the choice at the other $\tilde{\tau}$. This concludes the proof that $Y'_T \rightarrow X'_T$ induces a bijection on closed points. \square

LEMMA 5.20. *The map $\eta_1 : Y'_T \rightarrow X'_T$ induces an isomorphism of tangent spaces at every closed point.*

Proof. Let $y = (A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_{K'_T}, C, \iota_C; \phi_A, \phi_B)$ be a closed point of Y'_T with values in a perfect field k , and $x = (A, \iota_A, \lambda_A, \alpha_{K'})$ be its image in X'_T . We have to show that $Y'_T \rightarrow X'_T$ induces an isomorphism of k -vector spaces between tangent spaces: $T_{Y'_T,y} \xrightarrow{\cong} T_{X'_T,x}$.

Set $\mathbb{I} = \text{Spec}(k[\epsilon]/\epsilon^2)$. By deformation theory, $T_{X'_T,x}$ is identified with the \mathbb{I} -valued points $x_{\mathbb{I}} = (A_{\mathbb{I}}, \iota_{A,\mathbb{I}}, \lambda_{A,\mathbb{I}}, \alpha_{K',\mathbb{I}})$ of X'_T with reduction $x \in X'_T(k)$ modulo ϵ . In the proof of Proposition 4.7, we have seen that giving an $x_{\mathbb{I}}$ is equivalent to giving, for each $\tilde{\tau} \in \Sigma_{E,\infty}$, a direct factor $\omega_{A^\vee,\mathbb{I},\tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(A/k)_{\mathbb{I},\tilde{\tau}}^\circ$ that lifts $\omega_{A^\vee,\tilde{\tau}}^\circ \subseteq H_1^{\text{dR}}(A/k)_{\tilde{\tau}}^\circ$ and satisfies the following properties.

- (a) If $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$ with \mathfrak{p} not of type β^\sharp for \mathbf{S} , then $\omega_{A^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ and $\omega_{A^\vee, \mathbb{I}, \tilde{\tau}^c}^\circ$ are orthogonal complements of each other under the perfect pairing

$$H_1^{\text{cris}}(A/k)_{\mathbb{I}, \tilde{\tau}}^\circ \times H_1^{\text{cris}}(A/k)_{\mathbb{I}, \tilde{\tau}^c}^\circ \rightarrow k[\epsilon]/\epsilon^2$$

induced by the polarization λ_A .

- (b) If $\tilde{\tau} \in \tilde{\mathbf{S}}_\infty$, then $\omega_{A^\vee, \mathbb{I}, \tilde{\tau}}^\circ = 0$ and $\omega_{A^\vee, \mathbb{I}, \tilde{\tau}^c}^\circ = H_1^{\text{cris}}(A/k)_{\mathbb{I}, \tilde{\tau}^c}^\circ$.
- (c) If $\tilde{\tau}$ restricts to $\tau \in \mathbf{T}$, then $\omega_{A^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ has to be $F_{A, \text{es}}^{n_\tau}(H_1^{\text{cris}}(A^{(p^{n_\tau})}/k)_{\mathbb{I}, \tilde{\tau}}^\circ)$, where n_τ is as introduced in § 4.4 and $F_{A, \text{es}}^{n_\tau}$ on the crystalline homology are defined in the same way as $F_{A, \text{es}}^{n_\tau}$ on the de Rham homology as in Notation 4.2. Since we are in characteristic p , we have $F_{A, \text{es}}^{n_\tau}(H_1^{\text{cris}}(A^{(p^{n_\tau})}/k)_{\mathbb{I}, \tilde{\tau}}^\circ) = \omega_{A^\vee, \tilde{\tau}}^\circ \otimes k[\epsilon]/\epsilon^2$.

Note also that the crystal nature of $H_1^{\text{cris}}(A/k)$ implies that there is a canonical isomorphism

$$H_1^{\text{cris}}(A/k)_{\mathbb{I}} \cong H_1^{\text{dR}}(A/k) \otimes_k k[\epsilon]/\epsilon^2.$$

We have to show that, given such an $x_{\mathbb{I}}$, or equivalently given the liftings $\omega_{A^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ as above, there exist unique $(B_{\mathbb{I}}, \iota_{B, \mathbb{I}}, \lambda_{B, \mathbb{I}}, \beta_{K'_T, \mathbb{I}}, C_{\mathbb{I}}, \iota_{C, \mathbb{I}}; \phi_{A, \mathbb{I}}, \phi_{B, \mathbb{I}})$ over \mathbb{I} deforming $(B, \iota_B, \lambda_B, \beta_{K'_T}, C, \iota_C; \phi_A, \phi_B)$ such that $(A_{\mathbb{I}}, \iota_{A, \mathbb{I}}, \lambda_{A, \mathbb{I}}, \alpha_{K', \mathbb{I}}, B_{\mathbb{I}}, \iota_{B, \mathbb{I}}, \lambda_{B, \mathbb{I}}, \beta_{K'_T, \mathbb{I}}, C_{\mathbb{I}}, \iota_{C, \mathbb{I}}; \phi_{A, \mathbb{I}}, \phi_{B, \mathbb{I}})$ is an \mathbb{I} -valued point of Y'_T .

We start with $C_{\mathbb{I}}$. To show its existence, it suffices to construct, for each $\tilde{\tau} \in \Sigma_{E,\infty}$, a direct factor $\omega_{C^\vee, \mathbb{I}, \tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(C/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ that lifts $\omega_{C^\vee, \tilde{\tau}}^\circ \subseteq \mathcal{D}_{C, \tilde{\tau}}^\circ \cong H_1^{\text{dR}}(C/k)_{\tilde{\tau}}^\circ$.

- When neither $\tilde{\tau}$ nor $\sigma\tilde{\tau}$ belongs to $\tilde{\Delta}(\mathbf{T})^+$, $\phi_{A, *, ?} : H_1^{\text{dR}}(A/k)_{?}^\circ \xrightarrow{\sim} H_1^{\text{dR}}(C/k)_{?}^\circ$ is an isomorphism for $? = \tilde{\tau}, \sigma\tilde{\tau}$. We take $\omega_{C^\vee, \mathbb{I}, \tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(C/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ to be the image of $\omega_{A^\vee, \mathbb{I}, \tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(A/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ under the induced morphism $\phi_{A, *, \tilde{\tau}}^{\text{cris}}$ on the crystalline homology.
- When either one of $\tilde{\tau}$ and $\sigma\tilde{\tau}$ belongs to $\tilde{\Delta}(\mathbf{T})^+$, an easy dimension count argument similar to Lemma 5.19 (using Lemma 5.14) shows that $\omega_{C^\vee, \tilde{\tau}}^\circ$ is either 0 or of rank two. So, there is a unique obvious such lift $\omega_{C^\vee, \mathbb{I}, \tilde{\tau}}^\circ$.

This finishes the construction of $\omega_{C^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ for all $\tilde{\tau}$; hence, one gets a deformation $C_{\mathbb{I}}$ of C , carrying a natural action of \mathcal{O}_D . It is clear that the map $\phi_{A, *}^{\text{cris}} : H_1^{\text{cris}}(A/k)_{\mathbb{I}, \tilde{\tau}}^\circ \rightarrow H_1^{\text{cris}}(C/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ sends $\omega_{A^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ to $\omega_{C^\vee, \mathbb{I}, \tilde{\tau}}^\circ$. Hence, ϕ_A deforms to an \mathcal{O}_D -equivariant isogeny of abelian schemes $\phi_{A, \mathbb{I}} : A_{\mathbb{I}} \rightarrow C_{\mathbb{I}}$ by [Lan13, 2.1.6.9].

We check now that $\phi_{A_{\mathbb{I}}}$ satisfies condition (iv) of the moduli space Y'_T . We note that the map $\phi_{A_{\mathbb{I}}, *} : H_1^{\text{dR}}(A_{\mathbb{I}}/\mathbb{I}) \rightarrow H_1^{\text{dR}}(C_{\mathbb{I}}/\mathbb{I})$ is canonically identified with $\phi_{A, *}^{\text{cris}} : H_1^{\text{cris}}(A/k)_{\mathbb{I}} \rightarrow H_1^{\text{cris}}(C/k)_{\mathbb{I}}$ by crystalline theory, which is in turn isomorphic to the base change of $\phi_{A, *} : H_1^{\text{dR}}(A/k) \rightarrow H_1^{\text{dR}}(C/k)$ via $k \hookrightarrow k[\epsilon]/\epsilon^2$. Let $\tilde{\tau} \in \tilde{\Delta}(\mathbf{T})^+$. Since the Frobenius on $k[\epsilon]/\epsilon^2$ factors as

$$k[\epsilon]/\epsilon^2 \rightarrow k \xrightarrow{x \mapsto x^p} k \hookrightarrow k[\epsilon]/\epsilon^2,$$

we see that

$$F_{A, \text{es}}^n(H_1^{\text{dR}}(A_{\mathbb{I}}^{(p^n)}/\mathbb{I})_{\tilde{\tau}}^\circ) = F_{A, \text{es}}^n(H_1^{\text{dR}}(A^{(p^n)}/k)_{\tilde{\tau}}^\circ) \otimes_k k[\epsilon]/\epsilon^2.$$

Hence, the kernel of $\phi_{A_{\mathbb{I}}, *, \tilde{\tau}} : H_1^{\text{dR}}(A_{\mathbb{I}}/\mathbb{I})_{\tilde{\tau}}^\circ \rightarrow H_1^{\text{dR}}(C_{\mathbb{I}}/\mathbb{I})_{\tilde{\tau}}^\circ$ coincides with $F_{A, \text{es}}^n(H_1^{\text{dR}}(A_{\mathbb{I}}^{(p^n)}/\mathbb{I})_{\tilde{\tau}}^\circ)$, since it is the case after reduction modulo ϵ . This shows that $\phi_{A_{\mathbb{I}}}$ satisfies the condition (iv). Conversely, it is clear that, if $C_{\mathbb{I}}$ and $\phi_{A_{\mathbb{I}}}$ satisfy the condition (iv), then they have to be of the form as above.

We show now that there exists a unique deformation $(B_{\mathbb{I}}, \phi_{B_{\mathbb{I}}})$ over \mathbb{I} of (B, ϕ_B) satisfying condition (vi) of the moduli space Y'_T . To construct $B_{\mathbb{I}}$, one has to specify, for each $\tilde{\tau} \in \Sigma_{E,\infty}$, a

subbundle $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(B/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ lifting $\omega_{B^\vee, \tilde{\tau}}^\circ \subseteq H_1^{\text{dR}}(B/k)_{\tilde{\tau}}^\circ$. Similar to the discussion above, we have the following.

- If neither $\tilde{\tau}$ nor $\sigma\tilde{\tau}$ belongs to $\tilde{\Delta}(\mathbb{T})^-$, then $\phi_{B, *, ?} : H_1^{\text{dR}}(B/k)_{?}^\circ \xrightarrow{\sim} H_1^{\text{dR}}(C/k)_{?}^\circ$ is an isomorphism for $? = \tilde{\tau}, \sigma\tilde{\tau}$. We take $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(B/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ to be the image of $\omega_{C^\vee, \mathbb{I}, \tilde{\tau}}^\circ \subseteq H_1^{\text{cris}}(C/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ under the induced morphism $\phi_{B, *, \tilde{\tau}}^{-1}$ on the crystalline homology.
- If at least one of $\tilde{\tau}$ and $\sigma\tilde{\tau}$ belongs to $\tilde{\Delta}(\mathbb{T})^-$, then an easy dimension count argument similar to Lemma 5.19 (using Lemma 5.14) shows that $\omega_{B^\vee, \tilde{\tau}}^\circ$ is either 0 or of rank two. There is a unique obvious such lift $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}}^\circ$.

This defines $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ for all $\tilde{\tau} \in \Sigma_{E, \infty}$. Hence, one gets a deformation $B_{\mathbb{I}}$ of B over $k[\epsilon]/\epsilon^2$. It is immediate from the construction that the action of \mathcal{O}_D lifts to $B_{\mathbb{I}}$, and $\phi_{B, *, \tilde{\tau}}^{\text{cris}} : H_1^{\text{cris}}(B/k)_{\mathbb{I}, \tilde{\tau}}^\circ \rightarrow H_1^{\text{cris}}(C/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ sends $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ to $\omega_{C^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ for all $\tilde{\tau} \in \Sigma_{E, \infty}$. Hence, $\phi_B : B \rightarrow C$ deforms to an \mathcal{O}_D -equivariant isogeny $\phi_{B_{\mathbb{I}}} : B_{\mathbb{I}} \rightarrow C_{\mathbb{I}}$. In the same way as for $\phi_{A_{\mathbb{I}}}$, we prove that $\phi_{B_{\mathbb{I}}}$ satisfies the condition (v) of the moduli space $Y_{\mathbb{T}}'$, and conversely the condition (v) determines $B_{\mathbb{I}}$ uniquely.

Let $\langle \cdot, \cdot \rangle_{\lambda_B} : H_1^{\text{cris}}(B/k)_{\mathbb{I}}^\circ \times H_1^{\text{cris}}(B/k)_{\mathbb{I}}^\circ \rightarrow k[\epsilon]/\epsilon^2$ be the pairing induced by the polarization λ_B . To prove that λ_B deforms (necessarily uniquely) to a polarization $\lambda_{B_{\mathbb{I}}}$ on $B_{\mathbb{I}}$, it suffices to check that $\langle \cdot, \cdot \rangle_{\lambda_B}^{\text{cris}}$ vanishes on $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}}^\circ \times \omega_{B^\vee, \mathbb{I}, \tilde{\tau}^c}^\circ$ for all $\tilde{\tau} \in \Sigma_{E, \infty}$ (cf. [Lan13, 2.1.6.9, 2.2.2.2, and 2.2.2.6]).

- If $\tau = \tilde{\tau}|_F$ lies in $\mathbf{S}(\mathbb{T})_\infty$, this is trivial, because one of $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}}^\circ$ and $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}^c}^\circ$ is equal to 0 and the other one is equal to $H_1^{\text{cris}}(B/k)_{\mathbb{I}, \tilde{\tau}}^\circ$ by construction.
- If $\tau = \tilde{\tau}|_F$ is not in $\mathbf{S}(\mathbb{T})_\infty$, then the natural isomorphism $H_1^{\text{cris}}(B/k)_{\mathbb{I}, \star}^\circ \cong H_1^{\text{cris}}(A/k)_{\mathbb{I}, \star}^\circ$ sends $\omega_{B^\vee, \mathbb{I}, \star}^\circ$ to $\omega_{A^\vee, \mathbb{I}, \star}^\circ$ for $\star = \tilde{\tau}, \tilde{\tau}^c$. The vanishing of $\langle \cdot, \cdot \rangle_{\lambda_B}$ on $\omega_{B^\vee, \mathbb{I}, \tilde{\tau}}^\circ \times \omega_{B^\vee, \mathbb{I}, \tilde{\tau}^c}^\circ$ follows from the similar statement with B replaced by A .

Therefore, we see that λ_B deforms to a polarization $\lambda_{B_{\mathbb{I}}}$ on $B_{\mathbb{I}}$. Since $\lambda_{B_{\mathbb{I}}, \star}^{\text{dR}} : H_1^{\text{dR}}(B/\mathbb{I}) \rightarrow H_1^{\text{dR}}(B^\vee/\mathbb{I})$ is canonically identified with $\lambda_{B, \star}^{\text{cris}} : H_1^{\text{cris}}(B/k)_{\mathbb{I}} \rightarrow H_1^{\text{cris}}(B^\vee/k)_{\mathbb{I}}$, which is in turn identified with the base change of $\lambda_{B, \star}^{\text{dR}}$ via $k \hookrightarrow k[\epsilon]/\epsilon^2$, it is clear that condition (ii) regarding the polarization is preserved by the deformation $\lambda_{B_{\mathbb{I}}}$.

It remains to prove that $\beta_{K_{\mathbb{T}}'}$ deforms to $\beta_{K_{\mathbb{T}, \mathbb{I}}}'$. The deformation of the tame level structure is automatic; the deformation of the subgroup at p -adic places of type α^\sharp and α_2 is also unique, by the same argument as in Theorem 3.14. □

5.21 A lift of $I_{\mathbb{T}}$

Recall that $I_{\mathbb{T}}$ is the subset $\mathbf{S}(\mathbb{T})_\infty - (\mathbf{S}_\infty \cup \mathbb{T})$ defined in Theorem 5.2. We use $\tilde{I}_{\mathbb{T}}$ to denote the subset of complex embeddings of E consisting of the unique lift $\tilde{\tau}$ of every element $\tau \in I_{\mathbb{T}}$, for which $\tilde{\tau}^c \in \tilde{\mathbf{S}}(\mathbb{T})_\infty$. We describe this set explicitly as follows.

We write $I_{\mathbb{T}/\mathfrak{p}} = I_{\mathbb{T}} \cap \Sigma_{\infty/\mathfrak{p}}$ and $\tilde{I}_{\mathbb{T}/\mathfrak{p}} = \tilde{I}_{\mathbb{T}} \cap \Sigma_{E, \infty/\mathfrak{p}}$ for $\mathfrak{p} \in \Sigma_p$. They are empty sets unless \mathfrak{p} is of type α_1 or β_1 . When \mathfrak{p} is of type α_1 or β_1 , using the notation of § 5.1, $I_{\mathbb{T}/\mathfrak{p}}$ consists of $\sigma^{-m_i-1}\tau_i$ for all i such that $\#(C_i \cap \mathbb{T}/\mathfrak{p})$ is odd. In the notation of § 5.3, the set $\tilde{I}_{\mathbb{T}/\mathfrak{p}}$ consists of $\sigma^{-a_{r_i}}\tilde{\tau}_i$ for all i such that $\#(C_i \cap \mathbb{T}/\mathfrak{p})$ is odd. We remark that, in either case, for any $\tilde{\tau}$ lifting a place $\tau \in I_{\mathbb{T}}$, $\tilde{\tau} \notin \tilde{\Delta}(\mathbb{T})^+ \cup \tilde{\Delta}(\mathbb{T})^-$.

5.22 Isomorphism of $Y_{\mathbb{T}}'$ with $Z_{\mathbb{T}}'$

Let $Z_{\mathbb{T}}'$ be the moduli space over k_0 representing the functor that takes a locally noetherian k_0 -scheme S to the set of isomorphism classes of tuples $(B, \iota_B, \lambda_B, \beta_{K_{\mathbb{T}}}', J^\circ)$, where:

- (i) $(B, \iota_B, \lambda_B, \beta_{K_{\mathbb{T}}}')$ is an S -valued point of $\mathbf{Sh}_{K_{\mathbb{T}}}'(G_{\mathbf{S}(\mathbb{T})}')$;
- (ii) J° is the collection of subbundles $J_{\tilde{\tau}}^\circ \subseteq H_1^{\text{dR}}(B/S)_{\tilde{\tau}}^\circ$ locally free of rank one for each $\tilde{\tau} \in \tilde{I}_{\mathbb{T}}$.

It is clear that Z'_T is a $(\mathbb{P}^1)^{I_T}$ -bundle over $\mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(T)})$.

We define a morphism $\eta_2 : Y'_T \rightarrow Z'_T$ as follows: let S be a locally noetherian k_0 -scheme and $x = (A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_{K'_T}, C, \iota_C; \phi_A, \phi_B)$ be an S -valued point of Y'_T . We define $\eta_2(x) \in Z'_T(S)$ to be the isomorphism class of the tuple $(B, \iota_B, \lambda_B, \beta_{K'_T}, J^\circ)$, where $J^\circ_{\tilde{\tau}}$ is given by $\phi_{B,*,\tilde{\tau}}^{-1} \circ \phi_{A,*,\tilde{\tau}}(\omega_{A^\vee,\tilde{\tau}}^\circ)$ for the isomorphisms

$$H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^\circ \xrightarrow[\cong]{\phi_{A,*,\tilde{\tau}}} H_1^{\text{dR}}(C/S)_{\tilde{\tau}}^\circ \xleftarrow[\cong]{\phi_{B,*,\tilde{\tau}}} H_1^{\text{dR}}(B/S)_{\tilde{\tau}}^\circ.$$

Note that $\tilde{\tau} \notin \tilde{\Delta}(T)^\pm$ implies that both $\phi_{A,*,\tilde{\tau}}$ and $\phi_{B,*,\tilde{\tau}}$ are isomorphisms.

PROPOSITION 5.23. *The morphism $\eta_2 : Y'_T \rightarrow Z'_T$ is an isomorphism.*

We note that Theorem 5.8 follows immediately from this proposition and Proposition 5.17.

Proof. As in the proof of Proposition 5.17, it suffices to prove that η_2 induces a bijection on the closed points and on tangent spaces.

Step I. We show first that η_2 induces a bijection on closed points. Let $z = (B, \iota_B, \lambda_B, \beta_{K'_T}, J^\circ)$ be a closed point of Z'_T with values in $k = \overline{\mathbb{F}}_p$. We have to show that there exists a *unique* point $y = (A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_{K'_T}, C, \iota_C; \phi_A, \phi_B) \in Y'_T(k)$ with $\eta_2(y) = z$. To prove this, we basically reverse the construction in the proof of Lemma 5.18.

We start by reconstructing C from B and J° . We denote by $\tilde{\mathcal{D}}_B = (\tilde{\mathcal{D}}_B^\circ)^{\oplus 2}$ the covariant Dieudonné module of B and by $\tilde{\mathcal{D}}_B^\circ = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ the canonical decomposition according to the \mathcal{O}_E -action. We construct a Dieudonné submodule $M^\circ = \bigoplus_{\tau \in \Sigma_{E,\infty}} M_\tau^\circ \subseteq \tilde{\mathcal{D}}_B^\circ[1/p]$ with $\tilde{\mathcal{D}}_B^\circ \subseteq M^\circ \subseteq p^{-1}\tilde{\mathcal{D}}_B^\circ$ as follows. Let $\tilde{\tau} \in \Sigma_{E,\infty/p}$ with $\mathfrak{p} \in \Sigma_p$. If $\tilde{\tau} \notin \tilde{\Delta}(T)^-$, we put $M_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$. To define $M_{\tilde{\tau}}^\circ$ in the other case, we separate the discussion according to the type of \mathfrak{p} .

- (Case $\alpha 1$ and $\beta 1$) Recall our notation from §§ 5.1, 5.3, 5.11, and 5.21. There are two subcases according to the parity of $\#(C_i \cap T/\mathfrak{p})$, where C_i is a chain of $\mathbb{S}_{\infty/p} \cup T/\mathfrak{p}$ as in § 5.1. (It should not be confused with the abelian variety C .)

- * When $r_i = \#(C_i \cap T/\mathfrak{p})$ is odd, $\sigma^{-m_i-1}\tilde{\tau}_i \in \tilde{I}_{T/\mathfrak{p}}$, so that $J_{\sigma^{-m_i-1}\tilde{\tau}_i}^\circ$ is defined. In this case, all $\tau = \sigma^{-\ell}\tau_i$ belong to $\mathbb{S}(T)_{\infty/p}$ for $0 \leq \ell \leq m_i + 1$; so $s_{T,\sigma^{-\ell}\tilde{\tau}_i} \in \{0, 2\}$ and the essential Frobenii

$$F_{B,\text{es}}^{m_i+1-\ell} : \tilde{\mathcal{D}}_{B,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ \xrightarrow[\cong]{F_{B,\text{es}}} \tilde{\mathcal{D}}_{B,\sigma^{-m_i}\tilde{\tau}_i}^\circ \xrightarrow[\cong]{F_{B,\text{es}}} \cdots \xrightarrow[\cong]{F_{B,\text{es}}} \tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ$$

are isomorphisms for such an ℓ . If $a_j \leq \ell < a_{j+1}$ for some odd number j , we put

$$M_{\sigma^{-\ell}\tilde{\tau}_i}^\circ = p^{-1}F_{B,\text{es}}^{m_i+1-\ell}(\tilde{J}_{\sigma^{-m_i-1}\tilde{\tau}_i}^\circ),$$

where $\tilde{J}_{\sigma^{-m_i-1}\tilde{\tau}_i}^\circ$ denotes the inverse image in $\tilde{\mathcal{D}}_{B,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ$ of $J_{\sigma^{-m_i-1}\tilde{\tau}_i}^\circ \subseteq \mathcal{D}_{B,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ$ under the natural reduction map modulo p . For other ℓ , we have already defined $M_{\sigma^{-\ell}\tilde{\tau}_i}^\circ$ to be $\tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ$.

- * When $r_i = \#(C_i \cap T/\mathfrak{p})$ is even, there is no J° involved in this construction. Note that all $\tau = \sigma^{-\ell}\tau_i$ belong to $\mathbb{S}(T)_{\infty/p}$ for $0 \leq \ell \leq m_i$. So, $s_{T,\sigma^{-\ell}\tilde{\tau}_i} \in \{0, 2\}$ and in the sequence

of essential Frobenii

$$F_{B,es}^{m_i-\ell+1} : \tilde{\mathcal{D}}_{B,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ \xrightarrow{F_B} \tilde{\mathcal{D}}_{B,\sigma^{-m_i}\tilde{\tau}_i}^\circ \xrightarrow{F_{B,es} \cong} \tilde{\mathcal{D}}_{B,\sigma^{-m_i+1}\tilde{\tau}_i}^\circ \xrightarrow{F_{B,es} \cong} \cdots \xrightarrow{F_{B,es} \cong} \tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ$$

all the maps except the first one are isomorphisms. If $a_j \leq \ell < a_{j+1}$ for some odd number j , we put

$$M_{\sigma^{-\ell}\tilde{\tau}_i}^\circ = p^{-1}F_{B,es}^{m_i-\ell+1}(\tilde{\mathcal{D}}_{B,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ);$$

then we have $\dim_k(M_{\sigma^{-\ell}\tilde{\tau}_i}^\circ/\tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ) = 1$, since the cokernel of $F_B : \tilde{\mathcal{D}}_{B,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ \rightarrow \tilde{\mathcal{D}}_{B,\sigma^{-m_i}\tilde{\tau}_i}^\circ$ has dimension 1, as $s_{T,\sigma^{-m_i-1}\tilde{\tau}_i} = 1$. (For other ℓ , we have already defined $M_{\sigma^{-\ell}\tilde{\tau}_i}^\circ$ to be $\tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ$.)

- (Case $\alpha 2$) In this case, \mathfrak{p} is a prime of type α^\sharp for $\text{Sh}_{K_T}(G_{S(T)})$ and it splits into two primes \mathfrak{q} and $\bar{\mathfrak{q}}$ in E . Let $H_{\mathfrak{q}} \subseteq B[\mathfrak{q}]$ be the closed subgroup scheme given in the data β_{K_T} . Let $H_{\bar{\mathfrak{q}}}$ be its annihilator under the Weil pairing between $B[\mathfrak{q}]$ and $B[\bar{\mathfrak{q}}]$ induced by λ_B . (We collectively write $H_{\mathfrak{p}}$ for $H_{\mathfrak{q}} \times H_{\bar{\mathfrak{q}}}$.) Let $\mathcal{D}_{H_{\mathfrak{p}}}^\circ = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}} \mathcal{D}_{H_{\mathfrak{p}},\tilde{\tau}}^\circ \subseteq \mathcal{D}_B^\circ$ be the reduced covariant Dieudonné module of $H_{\mathfrak{p}} = H_{\mathfrak{q}} \times H_{\bar{\mathfrak{q}}}$. Then each $\mathcal{D}_{H_{\mathfrak{p}},\tilde{\tau}}^\circ$ is necessarily one dimensional over k for all $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$. For $\tilde{\tau} \in \tilde{\Delta}(T)_{/\mathfrak{p}}^-$, we define

$$M_{\tilde{\tau}}^\circ = p^{-1}\tilde{\mathcal{D}}_{H_{\mathfrak{p}},\tilde{\tau}}^\circ,$$

where $\tilde{\mathcal{D}}_{H_{\mathfrak{p}},\tilde{\tau}}^\circ$ denotes the inverse image in $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ of the subspace $\mathcal{D}_{H_{\mathfrak{p}},\tilde{\tau}}^\circ \subseteq \mathcal{D}_{B,\tilde{\tau}}^\circ$. (We have defined $M_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ for $\tilde{\tau} \notin \tilde{\Delta}(T)_{/\mathfrak{p}}^-$ before.)

- (Case $\beta 2$) In this case, \mathfrak{p} is a prime of type β^\sharp for $\text{Sh}_{K_T}(G_{S(T)})$. For $\tilde{\tau} \in \tilde{\Delta}(T)_{/\mathfrak{p}}^-$, let $\lambda_{B,*,\tilde{\tau}} : \mathcal{D}_{B,\tilde{\tau}}^\circ \rightarrow \mathcal{D}_{B^\vee,\tilde{\tau}}^\circ$ be the morphism induced by the polarization λ_B . By Theorem 3.14(b3), $J_{\tilde{\tau}}^\circ := \text{Ker}(\lambda_{B,*,\tilde{\tau}})$ is a k -vector space of dimension 1. We set $M_{\tilde{\tau}}^\circ = p^{-1}\tilde{J}_{\tilde{\tau}}^\circ$ for such $\tilde{\tau}$, where $\tilde{J}_{\tilde{\tau}}^\circ$ is the preimage of $J_{\tilde{\tau}}^\circ$ under the reduction map $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ \rightarrow \mathcal{D}_{B,\tilde{\tau}}^\circ$. Note that when viewing $\tilde{\mathcal{D}}_{B^\vee,\tilde{\tau}}^\circ$ as a lattice of $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ[1/p]$ using the polarization, we have $M_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{B^\vee,\tilde{\tau}}^\circ$. (We have defined $M_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ for $\tilde{\tau} \notin \tilde{\Delta}(T)_{/\mathfrak{p}}^-$ before.)

This concludes the definition of $M^\circ \subseteq p^{-1}\tilde{\mathcal{D}}_B^\circ$. One checks easily that M° is stable under F_B and V_B . Consider the quotient Dieudonné modules

$$M/\tilde{\mathcal{D}}_B = (M^\circ/\tilde{\mathcal{D}}_B^\circ)^{\oplus 2} \subseteq p^{-1}\tilde{\mathcal{D}}_B/\tilde{\mathcal{D}}_B \cong \mathcal{D}_B.$$

Then $M/\tilde{\mathcal{D}}_B$ corresponds to a closed finite group scheme $G \subseteq B[p]$ stable under the action of \mathcal{O}_D . We put $C = B/G$ with the induced \mathcal{O}_D -action and define $\phi_B : B \rightarrow C$ as the canonical \mathcal{O}_D -equivariant isogeny. Then the natural induced map $\phi_{B,*} : \tilde{\mathcal{D}}_B^\circ \rightarrow \tilde{\mathcal{D}}_C^\circ$ is identified with the inclusion $\tilde{\mathcal{D}}_B^\circ \hookrightarrow M^\circ$.

We now construct A from C . Similar to above, we first define a $W(k)$ -lattice $L^\circ = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} L_{\tilde{\tau}}^\circ \subseteq \tilde{\mathcal{D}}_C^\circ$, with $L_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ$ unless $\tilde{\tau} \in \tilde{\Delta}(T)^+$. If $\tilde{\tau} \in \tilde{\Delta}(T)^+$, then the corresponding p -adic place $\mathfrak{p} \in \Sigma_p$ cannot be of type $\beta 2$ or β^\sharp . In this case, we identify $\tilde{\mathcal{D}}_B^\circ[1/p]$ with $\tilde{\mathcal{D}}_C^\circ[1/p]$, so that $\tilde{\mathcal{D}}_B^\circ$ and $\tilde{\mathcal{D}}_C^\circ$ are both viewed as $W(k)$ -lattices in $\tilde{\mathcal{D}}_B^\circ[1/p]$. The polarization λ_B induces a perfect pairing

$$\langle \cdot, \cdot \rangle_{\lambda_B} : \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ[1/p] \times \tilde{\mathcal{D}}_{B,\tilde{\tau}^c}^\circ[1/p] \rightarrow W(k)[1/p],$$

which induces a perfect pairing between $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ and $\tilde{\mathcal{D}}_{B,\tilde{\tau}^c}^\circ$. We put (for $\tilde{\epsilon}\tilde{\Delta}(\mathbb{T})^+$)

$$L_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{C,\tilde{\tau}^c}^{\circ,\vee} := \{v \in \tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ[1/p] : \langle v, w \rangle_{\lambda_B} \in W(k) \text{ for all } w \in \tilde{\mathcal{D}}_{C,\tilde{\tau}^c}^\circ\}.$$

Note that $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+$ always implies that $\tilde{\tau}^c \in \tilde{\Delta}(\mathbb{T})^- - \tilde{\Delta}(\mathbb{T})^+$. So, $\tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ and $\tilde{\mathcal{D}}_{C,\tilde{\tau}^c}^\circ \supset \tilde{\mathcal{D}}_{B,\tilde{\tau}^c}^\circ$ with quotient isomorphic to k . This implies that $L_{\tilde{\tau}}^\circ \subseteq \tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ$ with quotient isomorphic to k .

As usual, one verifies that L° is stable under F_B and V_B (because it is equal to either $\tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ$ or $\tilde{\mathcal{D}}_{C,\tilde{\tau}^c}^{\circ,\vee}$ in various cases), and we put $L = (L^\circ)^{\oplus 2}$. The quotient Dieudonné module $L/p\tilde{\mathcal{D}}_C$ corresponds to a closed subgroup scheme $K \subseteq C[p]$ stable under the action of \mathcal{O}_D . We put $A = C/K$ equipped with the induced \mathcal{O}_D -action and define $\phi_A: A \rightarrow C$ as the canonical \mathcal{O}_D -equivariant isogeny with kernel $C[p]/K$. Then $\phi_{A,*}: \tilde{\mathcal{D}}_A \rightarrow \tilde{\mathcal{D}}_C$ is identified with the natural inclusion $L \hookrightarrow \tilde{\mathcal{D}}_C$.

We define $\lambda_A: A \rightarrow A^\vee$ to be the quasi-isogeny:

$$\lambda_A: A \xrightarrow{\phi_A} C \xleftarrow{\phi_B} B \xrightarrow{\lambda_B} B^\vee \xleftarrow{\phi_B^\vee} C^\vee \xrightarrow{\phi_A^\vee} A^\vee,$$

and we will verify that λ_A is a genuine isogeny (hence a polarization since λ_B is) satisfying condition (b) of the moduli space $\mathbf{Sh}_{K'_p}(G'_S)$ as in Theorem 3.14. We may identify $\tilde{\mathcal{D}}_A^\circ[1/p]$ and $\tilde{\mathcal{D}}_{A^\vee}^\circ[1/p]$ with $\tilde{\mathcal{D}}_B^\circ[1/p]$, and view both $\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ and $\tilde{\mathcal{D}}_{A^\vee,\tilde{\tau}}^\circ$ as lattices of $\tilde{\mathcal{D}}_B^\circ[1/p]$. It suffices to show that we have a natural inclusion

$$\tilde{\mathcal{D}}_{A^\vee,\tilde{\tau}}^\circ \subseteq (\tilde{\mathcal{D}}_{A,\tilde{\tau}^c}^\circ)^\vee = \{v \in \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ[1/p] : \langle v, w \rangle_{\lambda_B} \in W(k) \text{ for all } w \in \tilde{\mathcal{D}}_{A,\tilde{\tau}^c}^\circ\},$$

which is an isomorphism unless $\tilde{\tau}$ induces a p -adic place of type β^\sharp for $\text{Sh}_K(G_S)$, in which case it is an inclusion with quotient k .

- By the construction of A , this is clear for $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^+$ and hence for all their complex conjugates (as the duality is reciprocal).
- For all places $\tilde{\tau} \in \Sigma_{E,\infty/p}$ such that \mathfrak{p} is not of type $\beta 2$ and $\tilde{\tau} \notin \tilde{\Delta}(\mathbb{T})^\pm$, we know that $\tilde{\tau}^c \notin \tilde{\Delta}(\mathbb{T})^\pm$. So, $\tilde{\mathcal{D}}_{A,?}^\circ = \tilde{\mathcal{D}}_{B,?}^\circ$ for $? = \tilde{\tau}, \tilde{\tau}^c$ under the identification. The statement is clear. Note that this includes the case that \mathfrak{p} is a prime of type β^\sharp for $\text{Sh}_K(G_S)$.
- The only case left is when $\tilde{\tau} \in \Sigma_{E,\infty/p}$ for \mathfrak{p} of type $\beta 2$. In this case, $\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ$, which is the dual of $\tilde{\mathcal{D}}_{C,\tilde{\tau}^c}^\circ$ for all $\tilde{\tau} \in \Sigma_{E,\infty/p}$ by construction.

This concludes the verification that λ_A is an isogeny satisfying condition (b) of Theorem 3.14 for the moduli space $\mathbf{Sh}_{K'_p}(G'_S)$.

We now define the level structure $\alpha_{K'} = \alpha^p \alpha_p$ on A . For the prime-to- p level structure α^p , we define it to be the K'_Γ -orbit of the isomorphism class:

$$\alpha^p: \Lambda^{(p)} \xrightarrow[\beta^p]{\sim} T^{(p)}(B) \xrightarrow[\phi_{B,*}]{\sim} T^{(p)}(C) \xleftarrow[\phi_{A,*}]{\sim} T^{(p)}(A).$$

We take the closed subgroup scheme $\alpha_{\mathfrak{p}} \subseteq A[\mathfrak{p}]$ for each $\mathfrak{p} \in \Sigma_p$ of type α^\sharp for \mathbf{S} (and hence for $\mathbf{S}(\mathbb{T})$) to be the subgroup scheme corresponding to $\beta_{\mathfrak{p}}$ under the sequence of isomorphisms (note that $\tilde{\Delta}(\mathbb{T})^\pm_{/\mathfrak{p}} = \emptyset$ for \mathfrak{p} of type α^\sharp) of p -divisible groups

$$A[\mathfrak{p}^\infty] \xrightarrow[\cong]{\phi_A} C[\mathfrak{p}^\infty] \xleftarrow[\cong]{\phi_B} B[\mathfrak{p}^\infty].$$

It is clear that $\alpha_{K'}$ verifies condition (c) in Theorem 3.14.

This finishes the construction of all the data $y = (A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_{K'}, C, \iota_C; \phi_A, \phi_B)$. To see that y is indeed a k -point of Y'_T , we have to check that y satisfies the conditions (i)–(ix) for Y'_T in § 5.15. Conditions (ii), (iii), and (vi)–(ix) being clear from our construction, it remains to check (i), (iv) and (v). Moreover, the Kottwitz signature condition Theorem 3.14(a) follows from Lemma 5.19 immediately. So, for property (i), it remains to show that the partial Hasse invariant $h_{\tilde{\tau}}(A)$ vanishes if $\tau = \tilde{\tau}|_F \in \mathbf{T}$.

We now check these properties (i), (iv), and (v) in various cases. For this, we identify $\tilde{\mathcal{D}}_A^\circ[1/p]$, $\tilde{\mathcal{D}}_B^\circ[1/p]$, and $\tilde{\mathcal{D}}_C^\circ[1/p]$ via $\phi_{A,*}$ and $\phi_{B,*}$.

(1) Assume that \mathfrak{p} is a prime of case $\alpha 1$ or $\beta 1$. We keep the notation as before. If $\tilde{\tau}$ does not lift a place belonging to some chain C_i inside $\mathbf{S}_{\infty/\mathfrak{p}} \cup \mathbf{T}/\mathfrak{p}$, the conditions (i), (iv), and (v) trivially hold. So, we assume that $\tilde{\tau}|_F \in C_i$ for some C_i .

If $r_i = \#(\mathbf{T}/\mathfrak{p} \cap C_i)$ is odd, unwinding our earlier construction gives, for $0 \leq \ell \leq m_i + 1$,

$$\begin{aligned} \tilde{\mathcal{D}}_{A,\sigma^{-\ell}\tilde{\tau}_i}^\circ &= \tilde{\mathcal{D}}_{C,\sigma^{-\ell}\tilde{\tau}_i}^\circ = \begin{cases} p^{-1}F_{B,\text{es}}^{m_i+1-\ell}(\tilde{J}_{\sigma^{-m_i-1}\tilde{\tau}_i}^\circ) & \text{if } a_j \leq \ell < a_{j+1} \text{ for some odd } j, \\ \tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ & \text{otherwise;} \end{cases} \\ \tilde{\mathcal{D}}_{A,\sigma^{-\ell}\tilde{\tau}_i^c}^\circ &= \begin{cases} pF_{B,\text{es}}^{m_i+1-\ell}(\tilde{J}_{\sigma^{-m_i-1}\tilde{\tau}_i^c}^{\circ,\perp}) & \text{if } a_j \leq \ell < a_{j+1} \text{ for some odd } j, \\ \tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i^c}^\circ & \text{otherwise;} \text{ and} \end{cases} \\ \tilde{\mathcal{D}}_{C,\sigma^{-\ell}\tilde{\tau}_i^c}^\circ &= \tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i^c}^\circ \quad \text{for all } \ell. \end{aligned}$$

For condition (v) in § 5.15, it is trivial unless $\tilde{\tau} = \sigma^{-\ell}\tilde{\tau}_i$ for some $\ell \in [a_j, a_{j+1})$ with j odd. In the exceptional case, it is equivalent to proving that (for the n as in condition (v))

$$\tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ = F_{A,\text{es}}^n(\tilde{\mathcal{D}}_{A,\sigma^{-\ell-n}\tilde{\tau}_i}^\circ).$$

Note that $n = a_{j+1} - \ell$; it follows that $\tilde{\mathcal{D}}_{A,\sigma^{-\ell-n}\tilde{\tau}_i}^\circ = \tilde{\mathcal{D}}_{B,\sigma^{-a_{j+1}}\tilde{\tau}_i}^\circ$ by definition. As $s_{\mathbf{T},\sigma^{-a_{j+1}}\tilde{\tau}_i} = 2$ and $s_{\sigma^{-a_{j+1}}\tilde{\tau}_i} = 1$, $F_{A,\text{es}}^n(\tilde{\mathcal{D}}_{A,\sigma^{-\ell-n}\tilde{\tau}_i}^\circ)$ coincides with $F_{B,\text{es}}^n(\tilde{\mathcal{D}}_{B,\sigma^{-a_{j+1}}\tilde{\tau}_i}^\circ)$ by the definition of essential Frobenius. The desired equality follows from the fact that $F_{B,\text{es}}^n(\tilde{\mathcal{D}}_{B,\sigma^{-a_{j+1}}\tilde{\tau}_i}^\circ) = \tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ$.

Similarly, condition (iv) is trivial unless $\tilde{\tau} = \sigma^{-\ell}\tilde{\tau}_i^c$ for some $\ell \in [a_j, a_{j+1})$ with j odd. In the exceptional case, it is equivalent to the following equality (for the n as in condition (iv)):

$$p\tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i^c}^\circ = \tilde{F}_{A,\text{es}}^n(\tilde{\mathcal{D}}_{A,\sigma^{-\ell-n}\tilde{\tau}_i^c}^\circ).$$

But $n = a_{j+1} - \ell$ by definition; so, $\tilde{\mathcal{D}}_{A,\sigma^{-\ell-n}\tilde{\tau}_i^c}^\circ = \tilde{\mathcal{D}}_{B,\sigma^{-a_{j+1}}\tilde{\tau}_i^c}^\circ$. Since $s_{\mathbf{T},\sigma^{-a_{j+1}}\tilde{\tau}_i^c} = 0$ and $s_{\sigma^{-a_{j+1}}\tilde{\tau}_i^c} = 1$, the essential Frobenius of A at $\sigma^{-a_{j+1}}\tilde{\tau}_i^c$ is defined to be F_A , while that of B at $\sigma^{-a_{j+1}}\tilde{\tau}_i^c$ is defined to be V_B^{-1} . Therefore, $F_{A,\text{es},\sigma^{-\ell}\tilde{\tau}_i^c}^n$ is the same as $pF_{B,\text{es},\sigma^{-\ell}\tilde{\tau}_i^c}^n$. The equality above is now clear.

We now check the vanishing of partial Hasse invariants $h_{\sigma^{a_j}\tilde{\tau}_i}(A)$ with $1 \leq j \leq r_i - 1$. By Lemma 4.5, it suffices to show that, for any $j = 1, \dots, r_i - 1$ and setting $a_0 = -1$, the image of

$$F_{A,\text{es}}^{a_{j+1}-a_{j-1}} : \tilde{\mathcal{D}}_{A,\sigma^{-a_{j+1}}\tilde{\tau}_i}^\circ \rightarrow \tilde{\mathcal{D}}_{A,\sigma^{-a_{j-1}}\tilde{\tau}_i}^\circ$$

is contained in $p\tilde{\mathcal{D}}_{A,\sigma^{-a_{j-1}}\tilde{\tau}_i}^\circ$. First, regardless of the parity of j , we find easily that $F_{A,\text{es}}^{a_{j+1}-a_{j-1}} = pF_{B,\text{es}}^{a_{j+1}-a_{j-1}}$ as maps from $\tilde{\mathcal{D}}_{A,\sigma^{-a_{j+1}}\tilde{\tau}_i}^\circ$ to $\tilde{\mathcal{D}}_{A,\sigma^{-a_{j-1}}\tilde{\tau}_i}^\circ$ by checking carefully the dependence of the essential Frobenii on the signatures. Now, if j is odd, then

$$\tilde{\mathcal{D}}_{A,\sigma^{-\ell}\tilde{\tau}_i}^\circ = \tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ \quad \text{for } \ell = a_{j+1} \text{ and } a_{j-1}.$$

Hence, one gets $F_{A,es}^{a_{j+1}-a_{j-1}}(\tilde{\mathcal{D}}_{A,\sigma^{-a_{j+1}\tilde{\tau}_i}}^\circ) = p\tilde{\mathcal{D}}_{A,\sigma^{-a_{j-1}\tilde{\tau}_i}}^\circ$, since $F_{B,es}^{a_{j+1}-a_{j-1}}(\tilde{\mathcal{D}}_{B,\sigma^{-a_{j+1}\tilde{\tau}_i}}^\circ) = \tilde{\mathcal{D}}_{B,\sigma^{-a_{j-1}\tilde{\tau}_i}}^\circ$. If j is even, then

$$\tilde{\mathcal{D}}_{A,\sigma^{-\ell}\tilde{\tau}_i}^\circ = p^{-1}F_{B,es}^{m_i+1-\ell}(\tilde{\mathcal{J}}_{\sigma^{-m_i-1}\tilde{\tau}_i}^\circ) \quad \text{for } \ell = a_{j+1} \text{ and } a_{j-1}.$$

It is also clear that $F_{A,es}^{a_{j+1}-a_{j-1}}(\tilde{\mathcal{D}}_{A,\sigma^{-a_{j+1}\tilde{\tau}_i}}^\circ) = p\tilde{\mathcal{D}}_{A,\sigma^{-a_{j-1}\tilde{\tau}_i}}^\circ$.

If $r_i = \#(\mathbb{T}/\mathfrak{p} \cap C_i)$ is even, all conditions can be proved in exactly the same way, except replacing $F_{B,es}^{m_i+1-\ell}(\tilde{\mathcal{J}}_{\sigma^{-m_i-1}\tilde{\tau}_i}^\circ)$ by $F_{B,es}^{m_i-\ell}(F_B(\tilde{\mathcal{D}}_{B,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ))$ and the proof of the vanishing of the last Hasse invariant $h_{\sigma^{-a_{r_i}\tilde{\tau}_i}}(A)$ needs a small modification. In fact, we have

$$\tilde{\mathcal{D}}_{A,\sigma^{-\ell}\tilde{\tau}_i}^\circ = \begin{cases} \tilde{\mathcal{D}}_{B,\sigma^{-\ell}\tilde{\tau}_i}^\circ & \text{for } a_{r_i} \leq \ell \leq m_i + 1, \\ p^{-1}F_{B,es}^{m_i-\ell}(F_B(\tilde{\mathcal{D}}_{B,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ)) & \text{for } a_{r_i-1} \leq \ell < a_{r_i}. \end{cases}$$

Note that the number $n_{\sigma^{-a_{r_i}\tau_i}}$ defined in § 4.4 is equal to $m_i + 1 - a_{r_i}$, and the essential Frobenius $F_{A,es} : \tilde{\mathcal{D}}_{A,\sigma^{-a_{r_i}\tilde{\tau}_i}}^\circ \rightarrow \tilde{\mathcal{D}}_{A,\sigma^{-a_{r_i}+1}\tilde{\tau}_i}^\circ$ is simply F_A . We have

$$F_A F_{A,es}^{m_i+1-a_{r_i}}(\tilde{\mathcal{D}}_{A,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ) = F_{A,es}^{m_i+2-a_{r_i}}(\tilde{\mathcal{D}}_{A,\sigma^{-m_i-1}\tilde{\tau}_i}^\circ) = p\tilde{\mathcal{D}}_{A,\sigma^{-a_{r_i}+1}\tilde{\tau}_i}^\circ.$$

This verifies the vanishing of $h_{\sigma^{-a_{r_i}\tilde{\tau}_i}}(A)$.

(2) Assume that \mathfrak{p} is a prime of Case $\alpha 2$. We write $H_{\mathfrak{p}} = H_{\mathfrak{q}} \oplus H_{\bar{\mathfrak{q}}}$ and $\tilde{\mathcal{D}}_{H_{\mathfrak{q}},\tilde{\tau}}^\circ \subseteq \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ as before. We have

$$\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ = \begin{cases} p^{-1}\tilde{\mathcal{D}}_{H_{\mathfrak{p}},\tilde{\tau}}^\circ & \text{if } \tilde{\tau} \in \tilde{\Delta}(\mathbb{T})_{\mathfrak{p}}^-, \\ p(\tilde{\mathcal{D}}_{H_{\mathfrak{p}},\tilde{\tau}^c}^\circ)^\vee & \text{if } \tilde{\tau} \in \tilde{\Delta}(\mathbb{T})_{\mathfrak{p}}^+, \\ \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ = \begin{cases} p^{-1}\tilde{\mathcal{D}}_{H_{\mathfrak{p}},\tilde{\tau}}^\circ & \text{if } \tilde{\tau} \in \tilde{\Delta}(\mathbb{T})_{\mathfrak{p}}^-, \\ \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ & \text{otherwise.} \end{cases}$$

The same argument as in (1) allows us to check conditions (i), (iv), and (v).

(3) Assume now that \mathfrak{p} is prime of Case $\beta 2$ in § 5.1. For each $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$, let $\lambda_{B,*,\tilde{\tau}} : \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ \rightarrow \tilde{\mathcal{D}}_{B^\vee,\tilde{\tau}}^\circ$ be the map induced by the polarization λ_B . By condition (b3) of Theorem 3.14, its cokernel has dimension 1 over k . When viewing $\tilde{\mathcal{D}}_{B^\vee,\tilde{\tau}}^\circ$ as a lattice of $\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ[1/p]$ via $\lambda_{B,*,\tilde{\tau}}^{-1}$, we have

$$\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{C,\tilde{\tau}}^\circ = \begin{cases} \tilde{\mathcal{D}}_{B^\vee,\tilde{\tau}}^\circ & \text{if } \tilde{\tau} \in \tilde{\Delta}(\mathbb{T})_{\mathfrak{p}}^-, \\ \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ & \text{otherwise.} \end{cases}$$

The same argument as in (1) allows us to check conditions (i), (iv), and (v). This then concludes the proof of Step I.

Step II: Let $y = (A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_{K'_T}, C, \iota_C; \phi_A, \phi_B) \in Y'_T$ be a closed point with values in $k = \overline{\mathbb{F}}_p$, and $z = \eta_2(y) = (B, \iota_B, \lambda_B, \beta_{K'_T}, J^\circ) \in Z'_T$. We prove that $\eta_2 : Y'_T \rightarrow Z'_T$ induces a bijection of tangent spaces $\eta_{2,y} : T_{Y'_T,y} \xrightarrow{\cong} T_{Z'_T,z}$. We follow the same strategy as in Lemma 5.20.

Set $\mathbb{I} = \text{Spec}(k[\epsilon]/\epsilon^2)$. The tangent space $T_{Z'_T,z}$ is identified with the set of deformations $z_{\mathbb{I}} = (B_{\mathbb{I}}, \iota_{B,\mathbb{I}}, \lambda_{B,\mathbb{I}}, \beta_{K'_T,\mathbb{I}}, J_{\mathbb{I}}^\circ) \in Z'_T(\mathbb{I})$ of z , where $J_{\mathbb{I}}^\circ$ is the collection of subbundles $J_{\mathbb{I},\tilde{\tau}}^\circ \subseteq H_1^{\text{dR}}(B_{\mathbb{I}}/\mathbb{I})_{\tilde{\tau}}^\circ = H_1^{\text{cris}}(B/k)_{\mathbb{I},\tilde{\tau}}^\circ$ for each $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})$. We have to show that every point $z_{\mathbb{I}}$ lifts uniquely to a deformation $y_{\mathbb{I}} = (A_{\mathbb{I}}, \iota_{A,\mathbb{I}}, \lambda_{A,\mathbb{I}}, \alpha_{K',\mathbb{I}}, B_{\mathbb{I}}, \iota_{B,\mathbb{I}}, \lambda_{B,\mathbb{I}}, \beta_{K'_T,\mathbb{I}}, C_{\mathbb{I}}, \iota_{C,\mathbb{I}}; \phi_{A,\mathbb{I}}, \phi_{B,\mathbb{I}}) \in Y'_T(\mathbb{I})$ with $\eta(y_{\mathbb{I}}) = z_{\mathbb{I}}$.

We start with $C_{\mathbb{I}}$ and $\phi_{B_{\mathbb{I}}}$. For $\tilde{\tau} \in \Sigma_{E,\infty}$, denote by

$$\phi_{B,*,\tilde{\tau}}^{\text{cris}} : H_1^{\text{cris}}(B/k)_{\mathbb{I},\tilde{\tau}}^{\circ} \rightarrow H_1^{\text{cris}}(C/k)_{\mathbb{I},\tilde{\tau}}^{\circ}$$

the natural morphism induced by ϕ_B and by $\phi_{B,*,\tilde{\tau}}^{\text{dR}}$ the analogous map between the de Rham homology H_1^{dR} . The crystalline nature of H_1^{cris} implies that $\phi_{B,*,\tilde{\tau}}^{\text{cris}} = \phi_{B,*,\tilde{\tau}}^{\text{dR}} \otimes_k k[\epsilon]/\epsilon^2$. To construct $C_{\mathbb{I}}$ and $\phi_{B,\mathbb{I}}$, it suffices to specify, for each $\tilde{\tau} \in \Sigma_{E,\infty}$, a subbundle $\omega_{C^{\vee},\mathbb{I},\tilde{\tau}}^{\circ} \subseteq H_1^{\text{cris}}(C/k)_{\mathbb{I},\tilde{\tau}}^{\circ}$ which lifts $\omega_{C^{\vee},\tilde{\tau}}^{\circ}$ and satisfies

$$\phi_{B,*,\tilde{\tau}}^{\text{cris}}(\omega_{B_{\mathbb{I}}^{\vee},\tilde{\tau}}^{\circ}) \subseteq \omega_{C^{\vee},\mathbb{I},\tilde{\tau}}^{\circ}. \tag{5.23.1}$$

We distinguish a few cases.

- (i) If neither $\tilde{\tau}$ nor $\sigma\tilde{\tau}$ belongs to $\tilde{\Delta}(\mathbb{T})^-$, both $\phi_{B,*,\tilde{\tau}}^{\text{cris}}$ and $\phi_{B,*,\sigma\tilde{\tau}}^{\text{cris}}$ are isomorphisms. It follows that $\phi_{B,*,\tilde{\tau}}^{\text{dR}}(\omega_{B^{\vee},\tilde{\tau}}^{\circ}) = \omega_{C^{\vee},\tilde{\tau}}^{\circ}$. Hence, we have to take $\omega_{C^{\vee},\mathbb{I},\tilde{\tau}}^{\circ} = \phi_{B,*,\tilde{\tau}}^{\text{cris}}(\omega_{B_{\mathbb{I}}^{\vee},\tilde{\tau}}^{\circ})$.
- (ii) If both $\tilde{\tau}, \sigma\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^-$, then $\tau = \tilde{\tau}|_F \in \mathbb{S}_{\infty/\mathfrak{p}}$ by Lemma 5.14. A simple dimension count similar to Lemma 5.19 implies that $\dim_k(\omega_{C^{\vee},\tilde{\tau}}^{\circ}) = \dim_k(\omega_{B^{\vee},\tilde{\tau}}^{\circ}) \in \{0, 2\}$. We take $\omega_{C^{\vee},\mathbb{I},\tilde{\tau}}^{\circ}$ to be 0 or $H_1^{\text{cris}}(C/k)_{\mathbb{I},\tilde{\tau}}^{\circ}$ correspondingly, and (5.23.1) trivially holds.
- (iii) If $\tilde{\tau} \in \tilde{I}_{\mathbb{T}}$, the property of the morphism η_2 forces $\omega_{C^{\vee},\mathbb{I},\tilde{\tau}}^{\circ} = \phi_{B,*,\tilde{\tau}}^{\text{cris}}(J_{B_{\mathbb{I}},\tilde{\tau}}^{\circ})$.
- (iv) For all other $\tilde{\tau}$, $\tilde{\tau}|_F$ must belong to \mathbb{T} . Let n be the number associated to $\tilde{\tau}$ as in Lemma 5.13. By the vanishing of the partial Hasse invariant on A at $\tilde{\tau}$, we have $\omega_{C^{\vee},\tilde{\tau}}^{\circ} = F_{C,\text{es}}^n(H_1^{\text{dR}}(C/k)_{\sigma^{-n}\tilde{\tau}}^{\circ})$. We take

$$\omega_{C^{\vee},\mathbb{I},\tilde{\tau}}^{\circ} = F_{C,\text{es}}^n(H_1^{\text{cris}}(C^{(p^n)}/k)_{\mathbb{I},\tilde{\tau}}^{\circ}).$$

This is not a forced choice now, but it will become one when we have constructed the lift $A_{\mathbb{I}}$ and require $A_{\mathbb{I}}$ to have vanishing partial Hasse invariant.

Since $F_{B,\text{es}}^n : H_1^{\text{cris}}(B^{(p^n)}/k)_{\mathbb{I},\tilde{\tau}}^{\circ} \rightarrow H_1^{\text{cris}}(B/k)_{\mathbb{I},\tilde{\tau}}^{\circ}$ is an isomorphism, we conclude that (5.23.1) holds for $\tilde{\tau}$.

We now construct $A_{\mathbb{I}}$ and the isogeny $\phi_{A_{\mathbb{I}}} : A_{\mathbb{I}} \rightarrow C_{\mathbb{I}}$. As usual, we have to specify, for each $\tilde{\tau} \in \Sigma_{E,\infty}$, a subbundle $\omega_{A^{\vee},\mathbb{I},\tilde{\tau}}^{\circ} \subseteq H_1^{\text{cris}}(A/k)_{\mathbb{I},\tilde{\tau}}^{\circ}$ that lifts $\omega_{A^{\vee},\tilde{\tau}}^{\circ}$ and satisfies $\phi_{A,*,\tilde{\tau}}^{\text{cris}}(\omega_{A^{\vee},\mathbb{I},\tilde{\tau}}^{\circ}) \subseteq \omega_{C^{\vee},\mathbb{I},\tilde{\tau}}^{\circ}$. Let $\mathfrak{p} \in \Sigma_p$ be the prime such that $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$.

- If neither $\tilde{\tau}$ nor $\sigma\tilde{\tau}$ belongs to $\tilde{\Delta}(\mathbb{T})^+$, then $\phi_{A,*,\tilde{\tau}}^{\text{dR}}$ and hence $\phi_{A,*,\tilde{\tau}}^{\text{cris}}$ is an isomorphism. We are forced to take $\omega_{A^{\vee},\mathbb{I},\tilde{\tau}}^{\circ} = (\phi_{A,*,\tilde{\tau}}^{\text{cris}})^{-1}(\omega_{C^{\vee},\mathbb{I},\tilde{\tau}}^{\circ})$. In particular, if $\tilde{\tau} \in \tilde{I}_{\mathbb{T}}$, we have $\omega_{A^{\vee},\mathbb{I},\tilde{\tau}}^{\circ} = (\phi_{A,*,\tilde{\tau}}^{\text{cris}})^{-1} \phi_{B,*,\tilde{\tau}}^{\text{cris}}(J_{B_{\mathbb{I}},\tilde{\tau}}^{\circ})$.
- In all other cases, we must have $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$ for \mathfrak{p} not of type $\beta 2$ or β^{\sharp} . Then we have to take $\omega_{A^{\vee},\mathbb{I},\tilde{\tau}}^{\circ}$ to be the orthogonal complement of $\omega_{A^{\vee},\mathbb{I},\tilde{\tau}^c}^{\circ}$ (which is already defined in the previous case) under the perfect pairing

$$\langle \cdot, \cdot \rangle_{\lambda_A} : H_1^{\text{cris}}(A/k)_{\mathbb{I},\tilde{\tau}}^{\circ} \times H_1^{\text{cris}}(A/k)_{\mathbb{I},\tilde{\tau}^c}^{\circ} \rightarrow k[\epsilon]/\epsilon^2$$

induced by the polarization λ_A . It is clear that $\omega_{A^{\vee},\mathbb{I},\tilde{\tau}}^{\circ}$ is a lift of $\omega_{A^{\vee},\tilde{\tau}}^{\circ}$. It remains to show

that $\phi_{A,*,\tilde{\tau}}^{\text{cris}}(\omega_{A^\vee,\mathbb{I},\tilde{\tau}}^\circ) \subseteq \omega_{C^\vee,\mathbb{I},\tilde{\tau}}^\circ$. We consider the following commutative diagram:

$$\begin{array}{ccc}
 H_1^{\text{cris}}(A/k)_{\mathbb{I},\tilde{\tau}}^\circ & \times & H_1^{\text{cris}}(A/k)_{\mathbb{I},\tilde{\tau}^c}^\circ & \xrightarrow{\langle \cdot, \cdot \rangle_{\lambda_A}} & \mathbb{I} \\
 \downarrow \phi_{A,*,\tilde{\tau}}^{\text{cris}} & & \cong \downarrow \phi_{A,*,\tilde{\tau}^c}^{\text{cris}} & & \parallel \\
 H_1^{\text{cris}}(C/k)_{\mathbb{I},\tilde{\tau}}^\circ & \times & H_1^{\text{cris}}(C/k)_{\mathbb{I},\tilde{\tau}^c}^\circ & & \\
 \cong \uparrow \phi_{B,*,\tilde{\tau}}^{\text{cris}} & & \uparrow \phi_{B,*,\tilde{\tau}^c}^{\text{cris}} & & \\
 H_1^{\text{cris}}(B/k)_{\mathbb{I},\tilde{\tau}}^\circ & \times & H_1^{\text{cris}}(B/k)_{\mathbb{I},\tilde{\tau}^c}^\circ & \xrightarrow{\langle \cdot, \cdot \rangle_{\lambda_B}} & \mathbb{I}
 \end{array} \tag{5.23.2}$$

where both duality pairings are perfect. By our choice of $\tilde{\tau}$, we have $\tilde{\tau}, \sigma\tilde{\tau} \notin \tilde{\Delta}(\mathbb{T})^-$ and $\tilde{\tau}^c, \sigma\tilde{\tau}^c \notin \tilde{\Delta}(\mathbb{T})^+$; so both $\phi_{A,*,\tilde{\tau}^c}^{\text{cris}}$ and $\phi_{B,*,\tilde{\tau}}^{\text{cris}}$ in (5.23.2) are isomorphisms and they induce isomorphisms on the reduced differentials. Using the diagram (5.23.2) of perfect duality, the inclusion $\phi_{A,*,\tilde{\tau}}^{\text{cris}}(\omega_{A^\vee,\mathbb{I},\tilde{\tau}}^\circ) \subseteq \omega_{C^\vee,\mathbb{I},\tilde{\tau}}^\circ$ is equivalent to the inclusion $\phi_{B,*,\tilde{\tau}^c}^{\text{cris}}(\omega_{B^\vee,\mathbb{I},\tilde{\tau}^c}^\circ) \subseteq \omega_{C^\vee,\mathbb{I},\tilde{\tau}^c}^\circ$, which was already checked.

By construction, the $\tilde{\tau}$ -partial Hasse invariant of $A_{\mathbb{I}}$ vanishes if $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^-$ and $\tilde{\tau}|_F \in \mathbb{T}$. The duality guarantees the vanishing of Hasse invariants at their conjugate places. This condition conversely forces the uniqueness of our choice of $C_{\mathbb{I}}$ and $A_{\mathbb{I}}$, as indicated earlier. From the construction, $\omega_{A^\vee,\mathbb{I}}^\circ = \bigoplus_{\tilde{\tau} \in \Sigma_{E,\infty}} \omega_{A^\vee,\mathbb{I},\tilde{\tau}}^\circ$ is isotropic under the pairing on $H_1^{\text{cris}}(A/k)_{\mathbb{I}}^\circ$ induced by λ_A . This concludes checking condition (1) of § 5.15.

The lift of the level structure $\alpha_{K',\mathbb{I}}$ is automatic for the tame part, and can be done in a unique way as in the proof of Theorem 3.14.

It then remains to check that $\phi_{A_{\mathbb{I}}}$ and $\phi_{B_{\mathbb{I}}}$ satisfy conditions (iv) and (v) of § 5.15. For condition (v), it is obvious except when $\tilde{\tau} \in \tilde{\Delta}(\mathbb{T})^-$, in which case

$$\text{Im}(\phi_{B,*,\tilde{\tau}}^{\text{cris}}) = \text{Im}(\phi_{B,*,\tilde{\tau}}^{\text{dR}}) \otimes_k \mathbb{I} \quad \text{and} \quad \phi_{A,*,\tilde{\tau}}^{\text{cris}}(\text{Im}(F_{\text{es},A_{\mathbb{I}},\tilde{\tau}}^n)) = \phi_{A,*,\tilde{\tau}}^{\text{dR}}(\text{Im}(F_{\text{es},A,\tilde{\tau}}^n)) \otimes_k \mathbb{I},$$

where $n > 1$ is the number determined in Lemma 5.13. So, condition (v) for the lift follows from that for $\phi_A : A \rightarrow C$. (Note that $n \geq 1$ implies that the image of the essential image is determined by the reduction.) Exactly the same argument proves condition (iv).

This concludes Step II of the proof of Proposition 5.23. □

5.24 End of proof of Theorem 5.8

Statement (1) of Theorem 5.8 follows from Propositions 5.17 and 5.23. Statements (2) and (3) are clear from the proof of (1). It remains to prove statement (4), namely the compatibility of the twisted partial Frobenius. We use $X'_{\tilde{S},\mathbb{T}}, Y'_{\tilde{S},\mathbb{T}}$, and $Z'_{\tilde{S},\mathbb{T}}$ to denote the original $X'_{\mathbb{T}}, Y'_{\mathbb{T}}$, and $Z'_{\mathbb{T}}$ in § 5.15 to indicate their dependence on \tilde{S} . We will define a twisted partial Frobenius

$$\mathfrak{F}'_{\mathfrak{p}^2,\tilde{S}} : Y'_{\tilde{S},\mathbb{T}} \rightarrow Y'_{\sigma_{\tilde{S}}^2\tilde{S},\sigma_{\tilde{S}}^2\mathbb{T}}$$

compatible via η_1 with the $\mathfrak{F}'_{\mathfrak{p}^2,\tilde{S}}$ on $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0}$ defined in § 3.22. For an S -valued point $x = (A, \iota_A, \lambda_A, \alpha_{K'}, B, \iota_B, \lambda_B, \beta_p, C, \iota_C; \phi_A, \phi_B)$ of $Y'_{\tilde{S},\mathbb{T}}$, its image

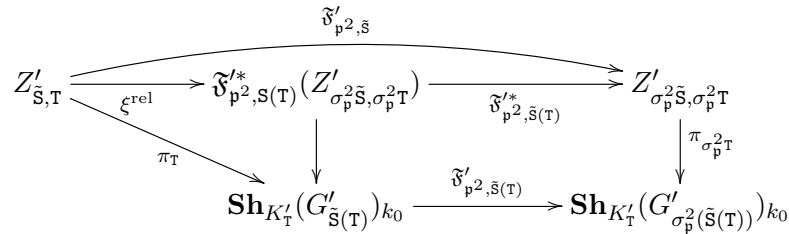
$$\mathfrak{F}'_{\mathfrak{p}^2,\tilde{S}}(x) = (A', \iota_{A'}, \lambda_{A'}, \alpha'_{K'}, B', \iota_{B'}, \lambda_{B'}, \beta'_p, C', \iota_{C'}; \phi_{A'}, \phi_{B'})$$

is given as follows. Here, for $G = A, B, C$, we put $G' = (G/\text{Ker}_{G,\mathfrak{p}^2}) \otimes_{\mathcal{O}_F} \mathfrak{p}$, where $\text{Ker}_{G',\mathfrak{p}^2}$ is the \mathfrak{p} -component of the \mathfrak{p}^2 -Frobenius of G . The induced structures $(\iota_{A'}, \lambda_{A'}, \alpha'_{K'}, \iota_{B'}, \lambda_{B'}, \beta'_p)$

are defined in the same way as in § 3.22. The isogenies $\phi_{A'} : A' \rightarrow C'$ and $\phi_{B'} : B' \rightarrow C'$ are constructed from ϕ_A and ϕ_B by the functoriality of the p^2 -Frobenius. We have to prove that the induced map on de Rham homologies $\phi_{A',*,\tilde{\tau}}$ and $\phi_{B',*,\tilde{\tau}}$ satisfies the required conditions in (v) and (vi) of § 5.15. If $\tilde{\tau} \in \Sigma_{E,\infty/p'}$ with $p' \neq p$, this is clear, because the p -divisible groups $G'[p'^\infty]$ are canonically identified with $G[p'^\infty]$ for $G = A, B, C$. Now consider the case $p' = p$. As in the proof of Lemma 4.12, for $G = A, B, C$, the p -divisible group $G'[p^\infty]$ is isomorphic to the base change of $G[p^\infty]$ via the p^2 -Frobenius on S . One thus deduces isomorphisms of de Rham homology groups

$$H_1^{\text{dR}}(G'/S)_{\tilde{\tau}}^\circ = (H_1^{\text{dR}}(G/S)_{\sigma^{-2}\tilde{\tau}}^\circ)^{(p^2)}, \tag{5.24.1}$$

which are compatible with F and V as $\tilde{\tau} \in \Sigma_{E,\infty/p}$ varies, and compatible with $\phi_{A',*,\tilde{\tau}}$ and $\phi_{B',*,\tilde{\tau}}$ by functoriality. Hence, the required properties on $\phi_{A',*,\tilde{\tau}}$ and $\phi_{B',*,\tilde{\tau}}$ follow from those on $\phi_{A,*,\sigma^{-2}\tilde{\tau}}$ and $\phi_{B,*,\sigma^{-2}\tilde{\tau}}$. This finishes the construction of \mathfrak{F}'_{p^2} on $Y'_{\tilde{S},T}$. Via the isomorphism $\eta_2 : Y'_{\tilde{S},T} \xrightarrow{\sim} Z'_{\tilde{S},T}$ proved in Proposition 5.23, $\mathfrak{F}'_{p^2,S}$ induces a map $\mathfrak{F}'_{p^2,S} : Z'_{\tilde{S},T} \rightarrow Z'_{\sigma_p^2\tilde{S},\sigma_p^2T}$. Let $z = (B, \iota_B, \lambda_B, \beta_{K'_T}, J^\circ)$ be an S -valued point of $Z'_{\tilde{S},T}$ as described in Proposition 5.22. Then its image $\mathfrak{F}'_{p^2,\tilde{S}}(z)$ is given by $(B', \iota_{B'}, \lambda_{B'}, \beta'_{K'_T}, J^{\circ'}) \in Z'_{\sigma_p^2\tilde{S},\sigma_p^2T}$, where $(B', \iota_{B'}, \lambda_{B'}, \beta'_{K'_T})$ are defined as in § 3.22, and $J^{\circ'}$ is the collection of line bundles $J_{\tilde{\tau}}^{\circ'} \subseteq H_1^{\text{dR}}(B'/S)_{\tilde{\tau}}^\circ$ for each $\tilde{\tau} \in \bigcup_{p' \in \Sigma_p} \sigma_p^2(\tilde{I}_{T,p'})$ given as follows. For $\tilde{\tau} \in \tilde{I}_{T,p'}$ with $p' \neq p$, we have $J_{\tilde{\tau}}^{\circ'} = J_{\tilde{\tau}}^\circ$ since $H_1^{\text{dR}}(B'/S)_{\tilde{\tau}}^\circ$ is canonically identified with $H_1^{\text{dR}}(B/S)_{\tilde{\tau}}^\circ$. For $\tilde{\tau} \in \sigma_p^2(\tilde{I}_{T,p})$, we have $J_{\tilde{\tau}}^{\circ'} = (J_{\sigma^{-2}\tilde{\tau}}^\circ)^{(p^2)}$, which makes sense thanks to the isomorphism (5.24.1) for $G = B'$. After identifying $\mathbf{Sh}_{K'_T}(G'_{\tilde{S}})_{k_0,T} = X'_{\tilde{S},T}$ with $Z'_{\tilde{S},T}$, the projection $\pi_T : Z'_{\tilde{S},T} \rightarrow \mathbf{Sh}_{K'_T}(G'_{\tilde{S}(T)})_{k_0}$ is given by $(B, \iota_B, \lambda_B, \beta_{K'_T}, J^\circ) \mapsto (B, \iota_B, \lambda_B, \beta_{K'_T})$. It is clear that we have the following commutative diagram:



where ξ^{rel} is given by $(B, \iota_B, \lambda_B, \beta_{K'_T}, J^\circ) \mapsto (B, \iota_B, \lambda_B, \beta_{K'_T}, J^{\circ'})$ with $J^{\circ'}$ defined above. This proves statement (4) immediately.

6. Ampleness of modular line bundles

In this section, we suppose that $F \neq \mathbb{Q}$. We will apply Theorem 5.8 to prove some necessary conditions for the ampleness of certain modular line bundles on quaternionic/unitary Shimura varieties. In this section, let $X' = \mathbf{Sh}_{K'}(G'_S)_{k_0}$ be a unitary Shimura variety over k_0 considered in § 4.1. This is a smooth and quasi-projective variety over k_0 , and projective if $S_\infty \neq \emptyset$. Let $(\mathbf{A}', \iota, \lambda, \alpha_{K'})$ be the universal abelian scheme over X' . For each $\tilde{\tau} \in \Sigma_{E,\infty}$, the $\mathcal{O}_{X'}$ -module $\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ$ is locally free of rank $2 - s_{\tilde{\tau}}$. It is a line bundle if $\tilde{\tau}|_F$ belongs to $\Sigma_\infty - S_\infty$.

6.1 Rational Picard group

For a variety Y over k_0 , we write $\text{Pic}(Y)_{\mathbb{Q}}$ for $\text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$. For a line bundle \mathcal{L} on Y , we denote by $[\mathcal{L}]$ its class in $\text{Pic}(Y)_{\mathbb{Q}}$.

LEMMA 6.2. (1) For any $\tilde{\tau} \in \Sigma_{E,\infty}$ lifting a place $\tau \in \Sigma_\infty - \mathbf{S}_\infty$, we have equalities

$$[\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ] = [\omega_{\mathbf{A}'/X',\tilde{\tau}^c}^\circ] = [\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ] = [\omega_{\mathbf{A}'/X',\tilde{\tau}^c}^\circ].$$

(2) For any $\tilde{\tau} \in \Sigma_{E,\infty}$, we have $[\wedge_{\mathcal{O}_{X'}}^2 H_1^{\text{dR}}(\mathbf{A}'/X')_{\tilde{\tau}}^\circ] = 0$.

(3) Let X'^* denote the minimal compactification of X' (which is just X' if $\mathbf{S}_\infty \neq \emptyset$). Then the natural morphism $j : \text{Pic}(X'^*) \rightarrow \text{Pic}(X')$ is injective. Moreover, for each $\tilde{\tau} \in \Sigma_{E,\infty}$ lifting a place $\tau \in \Sigma_\infty - \mathbf{S}_\infty$, $[\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ]$ belongs to the image of $j_{\mathbb{Q}} : \text{Pic}(X'^*)_{\mathbb{Q}} \rightarrow \text{Pic}(X')_{\mathbb{Q}}$.

Proof. (1) Suppose that $\tau \in \Sigma_{\infty/\mathfrak{p}} - \mathbf{S}_{\infty/\mathfrak{p}}$ for $\mathfrak{p} \in \Sigma_{\mathfrak{p}}$. Clearly, \mathfrak{p} is not of type α^\sharp or β^\sharp . The equality $[\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ] = [\omega_{\mathbf{A}'/X',\tilde{\tau}^c}^\circ]$ follows from the isomorphism $\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ \cong \omega_{\mathbf{A}'/X',\tilde{\tau}^c}^\circ$ thanks to the polarization λ on \mathbf{A}' . To prove the equality $[\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ] = [\omega_{\mathbf{A}'/X',\tilde{\tau}^c}^\circ]$, we consider the partial Hasse invariants $h_{\tilde{\tau}} \in \Gamma(X', (\omega_{\mathbf{A}'/X',\sigma^{-n\tau}\tilde{\tau}}^\circ)^{\otimes p^{n\tau}} \otimes \omega_{\mathbf{A}'/X',\tilde{\tau}}^{\circ,(-1)})$, and $h_{\tilde{\tau}^c}$ defined similarly with $\tilde{\tau}$ replaced by $\tilde{\tau}^c$. By Lemma 4.5 and Proposition 4.7, the vanishing of $h_{\tilde{\tau}}$ and $h_{\tilde{\tau}^c}$ defines the same divisor: $X'_\tau \subseteq X'$. Hence, for each $\tilde{\tau} \in \Sigma_{E,\infty}$ lifting some $\tau \in \Sigma_{\infty/\mathfrak{p}} - \mathbf{S}_{\infty/\mathfrak{p}}$, we have an equality

$$p^{n\tau}[\omega_{\mathbf{A}'/X',\sigma^{-n\tau}\tilde{\tau}}^\circ] - [\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ] = p^{n\tau}[\omega_{\mathbf{A}'/X',\sigma^{-n\tau}\tilde{\tau}^c}^\circ] - [\omega_{\mathbf{A}'/X',\tilde{\tau}^c}^\circ]. \tag{6.2.1}$$

Let C be the square matrix with coefficients in \mathbb{Q} , whose rows and columns are labeled by those places $\tilde{\tau} \in \Sigma_{E,\infty}$ lifting a place $\tau \in \Sigma_{\infty/\mathfrak{p}} - \mathbf{S}_{\infty/\mathfrak{p}}$, and whose $(\tilde{\tau}_1, \tilde{\tau}_2)$ -entry is

$$c_{\tilde{\tau}_1, \tilde{\tau}_2} = \begin{cases} -1 & \text{if } \tilde{\tau}_1 = \tilde{\tau}_2, \\ p^{n\tau_2} & \text{if } \tilde{\tau}_1 = \sigma^{-n\tau_2}\tilde{\tau}_2, \\ 0 & \text{otherwise.} \end{cases}$$

One checks easily that C is invertible and hence it follows from (6.2.1) that $[\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ] = [\omega_{\mathbf{A}'/X',\tilde{\tau}^c}^\circ]$.

(2) Assume first that $\tilde{\tau} \in \Sigma_{E,\infty}$ lifts some $\tau \in \Sigma_{\infty/\mathfrak{p}} - \mathbf{S}_\infty$. From the Hodge filtration $0 \rightarrow \omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ \rightarrow H_1^{\text{dR}}(\mathbf{A}'/X')_{\tilde{\tau}}^\circ \rightarrow \text{Lie}(\mathbf{A}'/X')_{\tilde{\tau}}^\circ \rightarrow 0$, one deduces that

$$[\wedge_{\mathcal{O}_{X'}}^2 H_1^{\text{dR}}(\mathbf{A}'/X')_{\tilde{\tau}}^\circ] = [\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ] + [\text{Lie}(\mathbf{A}'/X')_{\tilde{\tau}}^\circ].$$

Then statement (2) follows from (1) and the fact that $[\text{Lie}(\mathbf{A}'/X')_{\tilde{\tau}}^\circ] = -[\omega_{\mathbf{A}'/X',\tilde{\tau}}^\circ] = -[\omega_{\mathbf{A}'/X',\tilde{\tau}^c}^\circ]$. Consider now the case when $\tilde{\tau} \in \Sigma_{E,\infty}$ lifts some $\tau \in \mathbf{S}_{\infty/\mathfrak{p}}$ for a place \mathfrak{p} of type α or β . Then there is an integer $m \geq 1$ such that $\sigma^m\tau \in \Sigma_\infty - \mathbf{S}_\infty$ and $\sigma^i\tau \in \mathbf{S}_\infty$ for all $0 \leq i \leq m-1$ and we have a sequence of isomorphisms

$$H_1^{\text{dR}}(\mathbf{A}'/X')_{\tilde{\tau}}^{\circ,(p^m)} \xrightarrow[\cong]{F_{\mathbf{A}',\text{es}}} H_1^{\text{dR}}(\mathbf{A}'/X')_{\sigma\tilde{\tau}}^{\circ,(p^{m-1})} \xrightarrow[\cong]{F_{\mathbf{A}',\text{es}}} \dots \xrightarrow[\cong]{F_{\mathbf{A}',\text{es}}} H_1^{\text{dR}}(\mathbf{A}'/X')_{\sigma^m\tilde{\tau}}^\circ.$$

From this, one gets

$$p^m[\wedge_{\mathcal{O}_{X'}}^2 H_1^{\text{dR}}(\mathbf{A}'/X')_{\tilde{\tau}}^\circ] = [\wedge_{\mathcal{O}_{X'}}^2 H_1^{\text{dR}}(\mathbf{A}'/X')_{\sigma^m\tilde{\tau}}^\circ] = 0.$$

Finally, if $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{p}}$ for a place \mathfrak{p} of type α^\sharp or β^\sharp and if m is the inertia degree of \mathfrak{p} over p , then the sequence of isomorphisms

$$H_1^{\text{dR}}(\mathbf{A}'/X')_{\tilde{\tau}}^{\circ,(p^{2m})} \xrightarrow[\cong]{F_{\mathbf{A}',\text{es}}} H_1^{\text{dR}}(\mathbf{A}'/X')_{\sigma\tilde{\tau}}^{\circ,(p^{2m-1})} \xrightarrow[\cong]{F_{\mathbf{A}',\text{es}}} \dots \xrightarrow[\cong]{F_{\mathbf{A}',\text{es}}} H_1^{\text{dR}}(\mathbf{A}'/X')_{\sigma^{2m}\tilde{\tau}}^\circ$$

gives rise to an equality

$$p^{2m}[\wedge^2_{\mathcal{O}_{X'}} H_1^{\text{dR}}(\mathbf{A}'/X')^\circ_{\tilde{\tau}}] = [\wedge^2_{\mathcal{O}_{X'}} H_1^{\text{dR}}(\mathbf{A}'/X')^\circ_{\sigma^{2m}\tilde{\tau}}] = [\wedge^2_{\mathcal{O}_{X'}} H_1^{\text{dR}}(\mathbf{A}'/X')^\circ_{\tilde{\tau}}]$$

as $\sigma^{2m}\tilde{\tau} = \tilde{\tau}$. This forces $[\wedge^2_{\mathcal{O}_{X'}} H_1^{\text{dR}}(\mathbf{A}'/X')^\circ_{\tilde{\tau}}] = 0$.

(3) If X' is a Shimura curve, then $X'^* = X'$ as $F \neq \mathbb{Q}$. Assume now that X' has dimension at least 2. The injectivity of $j : \text{Pic}(X'^*) \rightarrow \text{Pic}(X')$ follows from the facts that X'^* is normal [Lan13, Proposition 7.2.4.3] and that the complement $X'^* - X'$ has codimension ≥ 2 . Recall from (1) that the partial Hasse invariant defines the class as described in (6.2.1). Note that the inverse matrix of C has all entries positive. It follows that each $[\omega_{\mathbf{A}'\vee/X', \tilde{\tau}_0}^\circ]$ for $\tilde{\tau}_0$ lifting $\tau_0 \in \Sigma_\infty - \mathbf{S}_\infty$ is a positive linear combination of the $\mathcal{O}_{X'}(X'_\tau)$. Let $X'^{n\text{-ord}} = \bigcup_{\tau \in \Sigma_\infty - \mathbf{S}_\infty} X'_\tau \subseteq X'$ be the union of the Goren–Oort strata of codimension 1. Since $X'^{n\text{-ord}}$ is closed in X'^* and is disjoint from the cusps, each line bundle $\mathcal{O}_{X'}(X'_\tau)$ extends to a line bundle $\mathcal{O}_{X'^*}(X'_\tau)$. By linear combination, each $[\omega_{\mathbf{A}'\vee/X', \tilde{\tau}_0}^\circ]$ extends to a class in $\text{Pic}(X'^*)_{\mathbb{Q}}$. \square

Notation 6.3. For any $\tau \in \Sigma_\infty - \mathbf{S}_\infty$, we put $[\omega_\tau] = [\omega_{\mathbf{A}'\vee/X', \tilde{\tau}}^\circ]$ for simplicity, where $\tilde{\tau}$ is a lift of τ . This is a well-defined element in $\text{Pic}(X'^*)_{\mathbb{Q}}$ by Lemma 6.2.

PROPOSITION 6.4. *Let \mathfrak{p} be a p -adic place such that $\#(\Sigma_{\infty/\mathfrak{p}} - \mathbf{S}_{\infty/\mathfrak{p}}) > 1$. When \mathbf{T} consists of a single element $\tau \in \Sigma_{\infty/\mathfrak{p}}$ with \mathfrak{p} not of type β_2 , let $n_\tau = n_{\tau, \mathbf{S}}$ be as in § 4.4. Let $N_{X'_\mathbf{T}}(X')$ denote the normal bundle of the embedding $X'_\mathbf{T} \hookrightarrow X'$. Then the equality $[N_{X'_\mathbf{T}}(X')] = [\mathcal{O}(-2p^{n_\tau})]$ holds in $\text{Pic}(X'_\mathbf{T})_{\mathbb{Q}}$, where $\mathcal{O}(1)$ is the canonical quotient bundle of the \mathbb{P}^1 -bundle $\pi_\tau : X'_\mathbf{T} = \mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \mathbf{T}} \rightarrow \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\mathbf{T})})_{k_0}$ and $\mathcal{O}(-2p^{n_\tau})$ is the dual of $\mathcal{O}(1)^{\otimes 2p^{n_\tau}}$.*

Proof. By the construction in § 5.22, the set $\tilde{I}_\tau = \{\sigma^{-n_\tau}\tilde{\tau}\}$ for a specific lift $\tilde{\tau}$ of τ . We have

$$J_{\sigma^{-n_\tau}\tilde{\tau}}^\circ = \phi_{B, *, \sigma^{-n_\tau}\tilde{\tau}}^{-1} \circ \phi_{A, *, \sigma^{-n_\tau}\tilde{\tau}}(\omega_{\mathbf{A}'\vee, \sigma^{-n_\tau}\tilde{\tau}}^\circ)$$

in terms of the moduli description of $Y'_\mathbf{T} \cong X'_\mathbf{T}$. So, the restriction of $[\omega_{\sigma^{-n_\tau}\tilde{\tau}}]$ to $X'_\mathbf{T}$ is $[\mathcal{O}(-1)]$. The Goren–Oort stratum $X'_\mathbf{T}$ is defined as the zero locus of

$$h_{\tilde{\tau}} : \omega_{\mathbf{A}'\vee/X', \tilde{\tau}}^\circ \rightarrow (\omega_{\mathbf{A}'\vee/X', \sigma^{-n_\tau}\tilde{\tau}}^\circ)^{\otimes p^{n_\tau}}.$$

So, firstly, the class of $N_{X'_\mathbf{T}}(X')$ in $\text{Pic}(X'_\mathbf{T})_{\mathbb{Q}}$ is given by the restriction of $p^{n_\tau}[\omega_{\sigma^{-n_\tau}\tilde{\tau}}] - [\omega_\tau]$ to $X'_\mathbf{T}$ and, secondly, on $X'_\mathbf{T}$ we have an isomorphism

$$\omega_{\mathbf{A}'\vee/X', \tilde{\tau}}^\circ \xrightarrow{\cong} H_1^{\text{dR}}(\mathbf{A}'/X')^\circ_{\tilde{\tau}}(p^{n_\tau}) / (\omega_{\mathbf{A}'\vee/X', \sigma^{-n_\tau}\tilde{\tau}}^\circ)^{\otimes p^{n_\tau}}.$$

This implies that $[\omega_\tau]$ equals $-p^{n_\tau}[\omega_{\sigma^{-n_\tau}\tilde{\tau}}]$ in $\text{Pic}(X'_\mathbf{T})_{\mathbb{Q}}$.

To sum up, we have equalities in $\text{Pic}(X'_\mathbf{T})_{\mathbb{Q}}$:

$$\begin{aligned} [N_{X'_\mathbf{T}}(X')] &= p^{n_\tau}[\omega_{\sigma^{-n_\tau}\tilde{\tau}}] - [\omega_\tau] \\ &= 2p^{n_\tau}[\omega_{\sigma^{-n_\tau}\tilde{\tau}}] = [\mathcal{O}(-2p^{n_\tau})]. \end{aligned} \quad \square$$

THEOREM 6.5. *Let $\underline{t} = (t_\tau) \in \mathbb{Q}^{\Sigma_\infty - \mathbf{S}_\infty}$. If the element $[\omega^{\underline{t}}] = \sum_{\tau \in \Sigma_\infty - \mathbf{S}_\infty} t_\tau [\omega_\tau]$ of $\text{Pic}(X')_{\mathbb{Q}}$ is ample, then*

$$p^{n_\tau} t_\tau > t_{\sigma^{-n_\tau}\tilde{\tau}} \quad (\text{and } t_\tau > 0) \text{ for all } \tau. \tag{6.5.1}$$

Here we put the second condition in parentheses, because it follows from the first one.

Proof. Assume that $[\omega^\ell]$ is ample. Let $\tau \in \Sigma_\infty - \mathbf{S}_\infty$ and $\mathfrak{p} \in \Sigma_p$ be the prime of F such that $\tau \in \Sigma_{\infty/\mathfrak{p}}$. We distinguish two cases.

- $\Sigma_{\infty/\mathfrak{p}} - \mathbf{S}_{\infty/\mathfrak{p}} = \{\tau\}$. Condition (6.5.1) for τ is simply $t_\tau > 0$. We consider the GO-stratum X'_{T_τ} with $T_\tau = \Sigma_\infty - (\mathbf{S}_\infty \cup \{\tau\})$. Then X'_{T_τ} is isomorphic to a Shimura curve by Theorem 5.8 and we let $i_\tau : X'_{T_\tau} \rightarrow X'$ denote the canonical embedding. For any $\tilde{\tau}' \in \Sigma_{E,\infty} - \mathbf{S}_{E,\infty}$ with restriction $\tau' = \tilde{\tau}'|_F \neq \tau$, let

$$F_{\mathbf{A}', \text{es}, \tilde{\tau}'}^{n_{\tau'}} : H_1^{\text{dR}}(\mathbf{A}'^{(p^{n_{\tau'}})}/X')_{\tilde{\tau}'}^\circ \rightarrow H_1^{\text{dR}}(\mathbf{A}'/X')_{\tilde{\tau}'}^\circ \tag{6.5.2}$$

be the $n_{\tau'}$ th iteration of the essential Frobenius in §4.4. We always have $\text{Ker}(F_{\mathbf{A}', \text{es}, \tilde{\tau}'}^{n_{\tau'}}) = (\omega_{\mathbf{A}'/X', \sigma^{-n_{\tau'} \tilde{\tau}'}}^\circ)^{(p^{n_{\tau'}})}$. The vanishing of $h_{\tilde{\tau}'}$ on X'_{T_τ} is equivalent to

$$\text{Im}(F_{\mathbf{A}', \text{es}, \tilde{\tau}'}^{n_{\tau'}})|_{X'_{T_\tau}} = (\omega_{\mathbf{A}'/X', \tilde{\tau}'}^\circ)|_{X'_{T_\tau}}.$$

Therefore, one deduces an equality in $\text{Pic}(X'_{T_\tau})_{\mathbb{Q}}$:

$$p^{n_{\tau'}} i_\tau^*[\omega_{\sigma^{-n_{\tau'} \tau}'}] + i_\tau^*[\omega_{\tau'}] = 0.$$

Letting τ' run through the set $\Sigma_{\infty/\mathfrak{q}} - \mathbf{S}_{\infty/\mathfrak{q}}$ with $\mathfrak{q} \neq \mathfrak{p}$, one obtains that $i_\tau^*[\omega_{\tau'}] = 0$. Therefore, we have $i_\tau^*[\omega^\ell] = t_\tau i_\tau^*[\omega_\tau]$, which is ample on X'_{T_τ} since so is $[\omega^\ell]$ on X'_{T_τ} by assumption. By the ampleness of $\det(\omega) = \bigotimes_{\tilde{\tau}' \in \Sigma_{E,\infty}} \omega_{\mathbf{A}', \tilde{\tau}'}^\circ$ on X' and hence on X'_{T_τ} , we see that $i_\tau^*[\omega_\tau]$ is ample on X'_{T_τ} . It follows that $t_\tau > 0$.

- $\Sigma_{\infty/\mathfrak{p}} - \mathbf{S}_{\infty/\mathfrak{p}} \neq \{\tau\}$. Consider the GO-stratum X'_T given by $T = \{\tau\}$; then $\mathbf{S}(T) = \mathbf{S} \cup \{\tau, \sigma^{-n_\tau \tau}\}$ in the notation of §5.1. By Propositions 5.17 and 5.23, X'_T is isomorphic to a \mathbb{P}^1 -bundle over $\mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(T)})_{k_0}$. Let $\pi : X'_T \rightarrow \mathbf{Sh}_{K'_T}(G'_{\mathbb{S}(T)})_{k_0}$ denote the natural projection. The ampleness of $[\omega^\ell]$ on X' implies the ampleness of its restriction to each closed fiber \mathbb{P}^1_s of π . By the proof of Proposition 6.4, we have

$$\omega_{\mathbf{A}'/X', \tilde{\tau}'}^\circ|_{\mathbb{P}^1_s} \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1_s}(-1) & \text{if } \tau' = \sigma^{-n_\tau \tau}, \\ \mathcal{O}_{\mathbb{P}^1_s}(p^{n_\tau}) & \text{if } \tau' = \tau, \\ \mathcal{O}_{\mathbb{P}^1_s} & \text{otherwise.} \end{cases}$$

The relation (6.5.1) follows immediately. □

Since the Hilbert modular varieties and the unitary Shimura varieties have the same neutral geometric connected components, the following is an immediate corollary of Theorem 6.5.

COROLLARY 6.6. *Let X denote the special fiber of the Hilbert modular variety and X^* its minimal compactification. Then, for each $\tau \in \Sigma_\infty$, the class $[\omega_\tau] \in \text{Pic}(X)_{\mathbb{Q}}$ uniquely extends to a class $[\omega_\tau] \in \text{Pic}(X^*)_{\mathbb{Q}}$. Moreover, $[\omega^\ell] = \sum_{\tau \in \Sigma_\infty} t_\tau [\omega_\tau]$ is ample only when $t_\tau > 0$ and $pt_\tau > t_{\sigma^{-1}\tau}$ for all $\tau \in \Sigma_\infty$.*

For the converse to Theorem 6.5, we have the following conjecture.

CONJECTURE 6.7. The conditions in Theorem 6.5 and Corollary 6.6 are also sufficient for $[\omega^\ell]$ to be ample.

Remark 6.8. In the case of a Hilbert modular surface, Corollary 6.6 and the sufficiency of the condition were proved by Andreatta and Goren [AG04, Theorem 8.1.1], and relies heavily on some intersection theory on surfaces. It seems difficult to generalize their method.

Using our global geometric description, it seems possible to prove, for small inertia degrees (at least when all inertia degrees are ≤ 5), Conjecture 6.7 using variants of the Nakai–Moishezon criterion. The combinatorics becomes complicated when the inertia degree is large.

7. Link morphisms

We will introduce certain generalizations of the partial Frobenius morphisms, called the *link morphisms*, on unitary Shimura varieties associated to quaternionic ones. These morphisms appear naturally when considering the restriction of the projection maps π_T in Theorem 5.8 to other Goren–Oort strata. The explicit descriptions of these morphisms are essential for the application considered in the forthcoming paper [TX14]. For simplicity, we will assume that p is inert of degree g in the totally real field F . Denote by \mathfrak{p} the unique prime of F above p . Let E be a CM extension of F . If \mathfrak{p} splits in E , fix a prime \mathfrak{q} of E above \mathfrak{p} and denote the other prime by $\bar{\mathfrak{q}}$; if \mathfrak{p} is inert in E , we denote by \mathfrak{q} the unique prime of E above \mathfrak{p} .

7.1 Links

We introduce some combinatorial objects. Let $n \geq 1$ be an integer. Put n points aligned equidistantly on a horizontal section of a vertical cylinder. We label the n points by the elements of $\mathbb{Z}/n\mathbb{Z}$ so that the $(i + 1)$ st point is next to the i th point on the right. Let S be a subset of the n points above. To such an S , we associate a graph as follows: we start left to right with the plot labeled $0 \in \mathbb{Z}/n\mathbb{Z}$ and draw a *plus sign* if the element is in S and a *node* if it is in $\mathbb{Z}/n\mathbb{Z} - S$. We call such a picture a *band of length n* associated to S . For instance, if $n = 5$ and $S = \{1, 3\}$, then the band is $\bullet + \bullet + \bullet$.

Let S' be another subset of $\mathbb{Z}/n\mathbb{Z}$ of the same cardinality as S . Then a *link* $\eta : S \rightarrow S'$ is a graph of the following kind: put the band attached to S on the top of the band for S' in the same cylinder; draw non-intersecting curves from each of the nodes from the top band to a node on the bottom band. We say that a curving is turning to the *left* (respectively to the *right*) if it is moving from the top band to the bottom band. If a curve travels m -numbers of points (of both plus signs and nodes) to the *right* (respectively *left*), we say that the displacement of this curve is m (respectively $-m$). When both S and S' are equal to $\mathbb{Z}/n\mathbb{Z}$ (so that there are no nodes at all), then we say that $\eta : S \rightarrow S'$ is the trivial link. We define the *total displacement* of a link η as the sum of the displacements of all curves in η . For example, if $n = 5$, $S = \{1, 3\}$, and $S' = \{1, 4\}$, then

$$\eta = \begin{array}{c} \bullet + \bullet + \bullet \\ \text{---} \text{---} \text{---} \\ \bullet + \bullet + \bullet \end{array} \tag{7.1.1}$$

is a link from S to S' , and its total displacement is $v(\eta) = 3 + 3 + 3 = 9$.

For a link $\eta : S \rightarrow S'$, we denote by $\eta^{-1} : S' \rightarrow S$ the link obtained by flipping the picture about the equator of the cylinder. For two links $\eta : S \rightarrow S'$ and $\eta' : S' \rightarrow S''$, we define the composition of $\eta' \circ \eta : S \rightarrow S''$ by putting the picture of η on top of the picture of η' and identify the nodes corresponding to η' . It is obvious that $v(\eta^{-1}) = -v(\eta)$ and $v(\eta' \circ \eta) = v(\eta') + v(\eta)$.

7.2 Links for a subset of places of F or E

We return to the setup of Notation 1.12 and recall that p is inert in F . We fix an isomorphism $\Sigma_\infty \cong \mathbb{Z}/g\mathbb{Z}$, so that $i \mapsto i + 1$ corresponds to the action of Frobenius on Σ_∞ .

For an even subset \mathbf{S} of places in F , we have the *band* for \mathbf{S} when applying § 7.1 to the subset \mathbf{S}_∞ of Σ_∞ . Let \mathbf{S}' be another even subset of places of F such that $\#\mathbf{S}_\infty = \#\mathbf{S}'_\infty$ and \mathbf{S}' contains the same finite places of F as \mathbf{S} does. A link η from the band for \mathbf{S} to that for \mathbf{S}' is denoted by $\eta : \mathbf{S} \rightarrow \mathbf{S}'$. When $\mathbf{S}' = \mathbf{S}$ and $\mathbf{S}_\infty = \Sigma_\infty$, $\eta : \mathbf{S} \rightarrow \mathbf{S}'$ is necessarily the trivial link (so that there are no curves at all).

The Frobenius action on Σ_∞ defines a link $\sigma : \mathbf{S} \rightarrow \sigma(\mathbf{S})$, in which all curves turn to the right with displacement 1; the total displacement of this link σ is $v(\sigma) = g - \#\mathbf{S}_\infty$. Here $\sigma(\mathbf{S})$ denotes

the subset of places of F whose finite part is the same as \mathbf{S} and whose infinite part is the image of Frobenius on \mathbf{S}_∞ .

Notation 7.3. Recall the definition of n_τ for $\tau \in \Sigma_\infty - \mathbf{S}_\infty$ from Notation 4.3. For simplicity, we write τ^- for $\sigma^{-n_\tau}\tau$ and we use τ^+ to denote the unique place in $\Sigma_\infty - \mathbf{S}_\infty$ such that $\tau = (\tau^+)^- = \sigma^{-n_\tau+\tau^+}$. When there are several \mathbf{S} involved, we will write $n_\tau(\mathbf{S})$ for n_τ to emphasize its dependence on \mathbf{S} .

7.4 Link morphisms

Let $\eta : \mathbf{S} \rightarrow \mathbf{S}'$ be a link of two even subsets of places of F . If $\mathbf{S}_\infty \neq \Sigma_\infty$, we denote by $m(\tau)$ the displacement of the curve at τ in the link η for each $\tau \in \Sigma_\infty - \mathbf{S}_\infty$, and put $m(\tau) = 0$ for $\tau \in \mathbf{S}_\infty$.

Let $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}})_{k_0}$ and $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}'}})_{k_0}$ denote the special fibers of some unitary Shimura varieties of type considered in §3.7. (There is no restriction on the signatures, i.e. the sets $\tilde{\mathbf{S}}_\infty$ and $\tilde{\mathbf{S}'}_\infty$ that lift \mathbf{S}_∞ and \mathbf{S}'_∞ ; but we fix them.) Here we have fixed (compatible) isomorphisms $\mathcal{O}_D := \mathcal{O}_{D_{\tilde{\mathbf{S}}}} \cong \mathcal{O}_{D_{\tilde{\mathbf{S}'}}} \cong M_{2 \times 2}(\mathcal{O}_E)$ and $G'_{\tilde{\mathbf{S}}}(\mathbb{A}^\infty) \cong G'_{\tilde{\mathbf{S}'}}(\mathbb{A}^\infty)$, and regard K' as an open compact subgroup of both of the groups; this is possible because \mathbf{S} and \mathbf{S}' have the same finite part, and the argument in Lemma 5.4 applies verbatim in this situation. Note that K'_p is assumed to be hyperspecial as in §3.7. Let $\mathbf{A}'_{\tilde{\mathbf{S}},k_0}$ be the universal abelian scheme over $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}})_{k_0}$. For a point x of $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}})_{k_0}$ with values in a perfect field $k(x)$, we denote by $\mathbf{A}'_{\tilde{\mathbf{S}},x}$ the base change of $\mathbf{A}'_{\tilde{\mathbf{S}}}$ to x , and $\tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}},x})^\circ$ the reduced part of the covariant Dieudonné module of $\mathbf{A}'_{\tilde{\mathbf{S}},x}$ (cf. §4.1 and the proof of Lemma 5.18). For each $\tilde{\tau} \in \Sigma_{E,\infty} - \mathbf{S}_{E,\infty}$, we have the essential Frobenius map defined in Notation 4.2:

$$F_{\mathbf{A}',\text{es}} : \tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}},x})^\circ_{\sigma^{-1}\tilde{\tau}} \rightarrow \tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}},x})^\circ_{\tilde{\tau}}$$

Finally, recall that a p -quasi-isogeny of abelian varieties means a quasi-isogeny of the form $f_1 \circ f_2^{-1}$, where f_1 and f_2 are isogenies of p -power order. The degree of the quasi-isogeny is $\text{deg } f_1 / \text{deg } f_2$.

DEFINITION 7.5. Assume that $m(\tau) \geq 0$ for each $\tau \in \Sigma_\infty$, i.e. all curves (if any) in η are either straight lines or all turning to the right. Let n be an integer. If \mathfrak{p} is inert in E , we assume that $n = 0$. A link morphism of indentation degree n associated to η on $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}})_{k_0}$ (if it exists) is a morphism of varieties

$$\eta'_{(n),\#} : \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}'}})_{k_0}$$

together with a p -quasi-isogeny of abelian varieties

$$\eta''_{(n)} : \mathbf{A}'_{\tilde{\mathbf{S}},k_0} \rightarrow \eta'^*_{(n),\#}(\mathbf{A}'_{\tilde{\mathbf{S}'},k_0}),$$

such that the following conditions are satisfied:

- (1) $\eta'_{(n),\#}$ induces a bijection on geometric points;
- (2) the quasi-isogeny $\eta''_{(n)}$ is compatible with the actions of \mathcal{O}_D , level structures, and the polarizations on both abelian varieties;
- (3) there exists, for each $\tilde{\tau} \in \Sigma_{E,\infty} - \mathbf{S}_{E,\infty}$, some $t_{\tilde{\tau}} \in \mathbb{Z}$, such that, for every $\overline{\mathbb{F}}_p$ -point x of $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}})_{k_0}$ with image $x' = \eta'_{(n),\#}(x)$,

$$\eta''_{(n),*} (F_{\text{es},\mathbf{A}'_{\tilde{\mathbf{S}},x}}^{m(\tau)} (\tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}},x})^\circ_{\tilde{\tau}})) = p^{t_{\tilde{\tau}}} \tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}'},x'})^\circ_{\sigma^{m(\tau)}\tilde{\tau}},$$

where $\tau \in \Sigma_\infty$ is the image of $\tilde{\tau}$;

(4) the quasi-isogeny of the \mathfrak{q} -divisible group

$$\eta_{(n),\mathfrak{q}}^{\#} : \mathbf{A}'_{\tilde{\mathfrak{S}},k_0}[\mathfrak{q}^\infty] \rightarrow \eta_{(n),\#}^{\prime*} \mathbf{A}'_{\tilde{\mathfrak{S}'},k_0}[\mathfrak{q}^\infty]$$

has degree p^{2n} . Here our convention for \mathfrak{q} is as at the beginning of this section. In particular, if \mathfrak{p} splits in E , then the quasi-isogeny on the $\bar{\mathfrak{q}}^\infty$ -divisible groups

$$\eta_{(n),\bar{\mathfrak{q}}}^{\#} : \mathbf{A}'_{\tilde{\mathfrak{S}},k_0}[\bar{\mathfrak{q}}^\infty] \rightarrow \eta_{(n),\#}^{\prime*} \mathbf{A}'_{\tilde{\mathfrak{S}'},k_0}[\bar{\mathfrak{q}}^\infty]$$

has necessarily degree p^{-2n} . Here the exponent $2n$ is due to the fact that $\mathbf{A}'_{k_0}[\mathfrak{q}^\infty]$ is two copies of its reduced part $\mathbf{A}'_{k_0}[\mathfrak{q}^\infty]^\circ$.

Let $\eta_i : \mathbf{S}_i \rightarrow \mathbf{S}_{i+1}$ for $i = 1, 2$ be two links with all curves turning to the right, and let $(\eta'_{i,\#}, \eta_i^{\#})$ for $i = 1, 2$ be a link morphism of indentation degree n_i on $\mathbf{Sh}_{K'_p}(G'_{\tilde{\mathfrak{S}}_i})_{k_0}$ attached to η_i . The composition of $(\eta'_{2,\#}, \eta_2^{\#})$ with $(\eta'_{1,\#}, \eta_1^{\#})$ defined by

$$\eta'_{12,\#} : \mathbf{Sh}_{K'_p}(G'_{\tilde{\mathfrak{S}}_1})_{k_0} \xrightarrow{\eta'_{1,\#}} \mathbf{Sh}_{K'_p}(G'_{\tilde{\mathfrak{S}}_2})_{k_0} \xrightarrow{\eta'_{2,\#}} \mathbf{Sh}_{K'_p}(G'_{\tilde{\mathfrak{S}}_3})_{k_0}$$

and

$$\eta_{12}^{\#} : \mathbf{A}'_{\tilde{\mathfrak{S}}_1,k_0} \xrightarrow{\eta_1^{\#}} \eta_{1,\#}^{\prime*}(\mathbf{A}'_{\tilde{\mathfrak{S}}_2,k_0}) \xrightarrow{\eta_{1,\#}^{\prime*}(\eta_2^{\#})} \eta_{12,\#}^{\prime*}(\mathbf{A}'_{\tilde{\mathfrak{S}}_3,k_0})$$

is a link morphism attached to the composed link $\eta_{12} := \eta_2 \circ \eta_1$ with indentation degree $n_1 + n_2$.

7.6 Variants

The formulation of link morphisms on $\mathbf{Sh}_{K'}(G'_{\tilde{\mathfrak{S}}})_{k_0}$ is compatible with changing the tame level K'^p . By taking the inverse limit of K'^p , one can define a link morphism on $\mathbf{Sh}_{K'_p}(G'_{\tilde{\mathfrak{S}}})_{k_0}$ associated to η in the obvious way.

One can define similarly a link morphism of indentation degree n on $\mathbf{Sh}_{K''_p}(G''_{\tilde{\mathfrak{S}}})_{k_0}$ as a pair $(\eta''_{(n),\#}, \eta''_{(n),\#})$, where

$$\eta''_{(n),\#} : \mathbf{Sh}_{K''_p}(G''_{\tilde{\mathfrak{S}}})_{k_0} \rightarrow \mathbf{Sh}_{K''_p}(G''_{\tilde{\mathfrak{S}'}})_{k_0}$$

is a morphism of varieties and

$$\eta''_{(n)} : \mathbf{A}''_{\tilde{\mathfrak{S}},k_0} \rightarrow \eta''_{(n),\#}(\mathbf{A}''_{\tilde{\mathfrak{S}'},k_0})$$

is a p -quasi-isogeny of abelian schemes such that the same conditions (1)–(4) in Definition 7.5 are satisfied (except the primes are replaced by double primes). Here $\mathbf{A}''_{\tilde{\mathfrak{S}},k_0}$ is the family of abelian varieties constructed in § 3.20.

Example 7.7. (1) Consider the second iteration of the Frobenius link $\sigma^2 = \sigma_{\mathfrak{p}}^2 : \mathbf{S} \rightarrow \sigma^2(\mathbf{S})$. The twisted (partial) Frobenius map (3.22.3)

$$\mathfrak{F}'_{\mathfrak{p}^2} : \mathbf{Sh}_{K'}(G'_{\tilde{\mathfrak{S}}})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\sigma^2 \tilde{\mathfrak{S}}})_{k_0}$$

together with the isogeny $\eta'_{\mathfrak{p}^2}$ defined in (3.22.4) is a link morphism associated to σ^2 ; the indentation degree is 0 if \mathfrak{p} is inert in E/F , and is $2\#\tilde{\mathfrak{S}}_{\infty/\bar{\mathfrak{q}}} - 2\#\tilde{\mathfrak{S}}_{\infty/\mathfrak{q}}$ if \mathfrak{p} splits in E/F .

(2) Assume that $\mathbf{S}_\infty = \Sigma_\infty$ and $\mathfrak{p} \notin \mathfrak{S}$ (so that \mathfrak{p} splits in E by our choice of E). The Shimura variety $\mathbf{Sh}_{K'}(G'_{\tilde{\mathfrak{S}}})_{k_0}$ is just a finite union of closed points. Let $\tau_0 \in \Sigma_\infty$ and $\tilde{\tau}_0 \in \tilde{\Sigma}_\infty$ be the lift of τ_0 with signature $s_{\tilde{\tau}_0} = 0$. We assume that $\sigma^{-1}\tilde{\tau}_0 \notin \tilde{\Sigma}_\infty$ (so that $\sigma^{-1}\tilde{\tau}_0^c \in \tilde{\Sigma}_\infty$).

Let \tilde{S}' denote the subset of places of F containing the same finite places as \tilde{S} and such that $\tilde{S}'_\infty = \tilde{S}_\infty \cup \{\tilde{\tau}_0^c, \sigma^{-1}\tilde{\tau}_0\} \setminus \{\tilde{\tau}_0, \sigma^{-1}\tilde{\tau}_0^c\}$. Let S' be the subset of places of F defined by the restriction of \tilde{S}'_∞ . Then there exists a link morphism $(\delta'_{\tau_0, \#}, \delta'^{\#}_{\tau_0})$ from $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0}$ to $\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0}$ associated to the trivial link $S \rightarrow S'$ defined as follows; its indentation is 0 if \mathfrak{p} is inert in E/F , is 2 if \mathfrak{p} splits in E/F and $\tilde{\tau}$ induces the p -adic place \mathfrak{q} , and is -2 if \mathfrak{p} splits in E/F and $\tilde{\tau}$ induces the p -adic place $\bar{\mathfrak{q}}$.

It suffices to define $\delta'_{\tau_0, \#}$ on the geometric closed points, as both Shimura varieties are zero dimensional. For each $\overline{\mathbb{F}}_p$ -point $x = (A, \iota, \lambda_A, \alpha_{K'}) \in \mathbf{Sh}_{K'}(G'_{\tilde{S}})(\overline{\mathbb{F}}_p)$, let $\tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ$ denote the $\tilde{\tau}$ -component of the reduced covariant Dieudonné module of A for each $\tilde{\tau} \in \Sigma_{E, \infty}$. We put $M_{\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ$ for $\tilde{\tau} \neq \tilde{\tau}_0, \tilde{\tau}_0^c$ and

$$M_{\tilde{\tau}_0^c}^\circ = p\tilde{\mathcal{D}}_{A, \tilde{\tau}_0^c}^\circ, \quad M_{\tilde{\tau}_0}^\circ = \frac{1}{p}\tilde{\mathcal{D}}_{A, \tilde{\tau}_0}^\circ \subseteq \tilde{\mathcal{D}}_{A, \tilde{\tau}_0}^\circ \left[\frac{1}{p} \right].$$

It is straightforward to check that the signature condition implies that the $M_{\tilde{\tau}}$ are stable under the actions of Frobenius and Verschiebung of $\tilde{\mathcal{D}}_{A, \tilde{\tau}_0}^\circ[1/p]$. As in the proof of Proposition 5.17, this gives rise to an abelian variety B of dimension $4g$ with an action of \mathcal{O}_D and an \mathcal{O}_D -quasi-isogeny $\phi : B \rightarrow A$ such that the induced morphism on Dieudonné modules $\phi_* : \tilde{\mathcal{D}}_{B, \tilde{\tau}}^\circ \rightarrow \tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ[1/p]$ is identified with the natural inclusion $M_{\tilde{\tau}}^\circ \hookrightarrow \tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ[1/p]$ for all $\tilde{\tau} \in \Sigma_{E, \infty}$. The polarization λ_A induces naturally a polarization λ_B on B such that $\lambda_B = \phi^\vee \circ \lambda_A \circ \phi$, since $M_{\tilde{\tau}}^\circ$ is the dual lattice of $M_{\tilde{\tau}^c}^\circ$ for every $\tilde{\tau} \in \Sigma_{E, \infty}$. When \mathfrak{p} is of type α_2 , then K'_p is the Iwahoric subgroup and the level structure at \mathfrak{p} is equivalent to the data of a collection of submodules $L_{\tilde{\tau}}^\circ \subset \tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ$ for $\tilde{\tau} \in \Sigma_{E, \infty/\mathfrak{q}}$ which are stable under the action of Frobenius and Verschiebung morphisms and such that $\tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ/L_{\tilde{\tau}}^\circ$ is a one-dimensional vector space over $\overline{\mathbb{F}}_p$ for each $\tilde{\tau}$. This then gives rise to a level structure at \mathfrak{p} for B_x by taking $L_{\tilde{\tau}}^\circ = L_{\tilde{\tau}}^\circ$ if $\tilde{\tau} \neq \tilde{\tau}_0$ and $L_{\tilde{\tau}_0}^\circ = p^{-1}L_{\tilde{\tau}_0}^\circ$. It is clear that other level structures of A transfer to those of B automatically. This then defines a morphism

$$\delta'_{\tau_0, \#} : \mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0}.$$

One checks easily that one can reverse the construction to recover A from B . So, $\delta'_{\tau_0, \#}$ is an isomorphism and there exists a p -quasi-isogeny

$$\delta'^{\#}_{\tau_0} : \mathbf{A}'_{\tilde{S}, k_0} \rightarrow (\delta'_{\tau_0, \#})^* \mathbf{A}'_{\tilde{S}', k_0},$$

whose base change to x is $\phi^{-1} : A \rightarrow B$ constructed above. It is evident by construction that $(\delta'_{\tau_0, \#}, \delta'^{\#}_{\tau_0})$ is a link morphism of the prescribed indentation associated to the trivial link $S \rightarrow S'$.

The following proposition will play a crucial role in our application in [TX14].

PROPOSITION 7.8. *For a given link $\eta : S \rightarrow S'$ with all curves (if any) turning to the right and an integer $n \in \mathbb{Z}$ (with $n = 0$ if \mathfrak{p} is inert in E), there exists at most one link morphism of indentation degree n from $\mathbf{Sh}_{K'_p}(G'_{\tilde{S}})_{k_0}$ to $\mathbf{Sh}_{K'_p}(G'_{\tilde{S}'})_{k_0}$ (or from $\mathbf{Sh}_{K''_p}(G''_{\tilde{S}})_{k_0}$ to $\mathbf{Sh}_{K''_p}(G''_{\tilde{S}'})_{k_0}$) associated to η .*

Proof. Since $\mathbf{Sh}_{K'_p}(G'_{\tilde{S}})_{k_0}$ and $\mathbf{Sh}_{K''_p}(G''_{\tilde{S}})_{k_0}$ have a canonically isomorphic neutral connected component (and the restrictions of $\mathbf{A}'_{\tilde{S}, k_0}$ and $\mathbf{A}''_{\tilde{S}, k_0}$ to this neutral connected component are also canonically isomorphic), it suffices to treat the case of $\mathbf{Sh}_{K'_p}(G'_{\tilde{S}})_{k_0}$. Let $(\eta'_{i, \#}, \eta'^{\#}_i)$ for $i = 1, 2$

be two link morphisms of indentation degree n associated to η . By the moduli property of $\mathbf{Sh}_{K'_p}(G'_{\mathbb{S}'})_{k_0}$, it suffices to show that the p -quasi-isogeny of abelian varieties

$$\phi : \eta_{1,\sharp}^{t*}(\mathbf{A}'_{\mathbb{S}',k_0}) \xleftarrow{\eta_1^\sharp} \mathbf{A}'_{\mathbb{S},k_0} \xrightarrow{\eta_2^\sharp} \eta_{2,\sharp}^{t*}(\mathbf{A}'_{\mathbb{S}',k_0})$$

is an isomorphism. By [RZ96, Proposition 2.9], the locus where ϕ is an isomorphism is a closed subscheme of $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$. As $\mathbf{Sh}_{K'_p}(G'_{\mathbb{S}})_{k_0}$ is a reduced variety, ϕ is an isomorphism if and only if it is so after base changing to every $\overline{\mathbb{F}}_p$ -point of $\mathbf{Sh}_{K'_p}(G'_{\mathbb{S}})_{k_0}$.

Let x be an $\overline{\mathbb{F}}_p$ -point of $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$ and put $x_i = \eta_{i,\sharp}^t(x)$ for $i = 1, 2$. Consider first the case $\mathbf{S}_\infty \neq \Sigma_\infty$. By condition 7.5(3), there exists an integer $u_{\tilde{\tau}}$ for each $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{q}} - \mathbf{S}'_{E,\infty/\mathfrak{q}}$ such that

$$\phi_{x,*}(\tilde{\mathcal{D}}(\mathbf{A}_{\mathbb{S}',x_1})_{\tilde{\tau}}^\circ) = p^{u_{\tilde{\tau}}} \tilde{\mathcal{D}}(\mathbf{A}_{\mathbb{S}',x_2})_{\tilde{\tau}}^\circ.$$

We claim that $u_{\tilde{\tau}}$ must be 0 for all $\tilde{\tau}$. Note that the cokernel of

$$F_{\text{es}, \mathbf{A}_{\mathbb{S}',x_i}}^{n_{\tilde{\tau}}} : \tilde{\mathcal{D}}(\mathbf{A}_{\mathbb{S}',x_i})_{\sigma^{-n_{\tilde{\tau}}\tilde{\tau}}}^\circ \rightarrow \tilde{\mathcal{D}}(\mathbf{A}_{\mathbb{S}',x_i})_{\tilde{\tau}}^\circ$$

has dimension 1 over $k(x_i)$ for $i = 1, 2$. Since $\phi_{x,*}$ commutes with $F_{\text{es}}^{n_{\tilde{\tau}}}$, we see that $u_{\tilde{\tau}} = u_{\sigma^{-n_{\tilde{\tau}}\tilde{\tau}}}$. Consequently, for all $\tilde{\tau} \in \Sigma_{E,\infty/\mathfrak{q}}$, $u_{\tilde{\tau}}$ takes the same value, which we denote by u . However, both $\eta_{1,\mathfrak{q}}^\sharp$ and $\eta_{2,\mathfrak{q}}^\sharp$ have degree p^{2n} by condition 7.5(4). It follows that

$$\phi_{x,\mathfrak{q}} : \eta_{1,\sharp}^{t*}(\mathbf{A}'_{\mathbb{S}',x_1})[\mathfrak{q}^\infty] \rightarrow \eta_{2,\sharp}^{t*}(\mathbf{A}'_{\mathbb{S}',x_2})[\mathfrak{q}^\infty]$$

is a quasi-isogeny of degree 0, which forces u to be 0. Hence, ϕ_* is an isomorphism.

When $\mathbf{S}_\infty = \Sigma_\infty$, we have similarly an integer $u_{\tilde{\tau}}$ for all $\tilde{\tau} \in \Sigma_{E,\infty}$ such that $\phi_*(\tilde{\mathcal{D}}(\mathbf{A}_{\mathbb{S}',x_1})_{\tilde{\tau}}^\circ) = p^{u_{\tilde{\tau}}} \tilde{\mathcal{D}}(\mathbf{A}_{\mathbb{S}',x_2})_{\tilde{\tau}}^\circ$. Since $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$ and $\mathbf{Sh}_{K'_p}(G'_{\mathbb{S}'})_{k_0}$ are both zero dimensional and

$$F_{\text{es}, \mathbf{A}'_{\mathbb{S},x_i}} : \tilde{\mathcal{D}}(\mathbf{A}_{\mathbb{S}',x_i})_{\sigma^{-1\tilde{\tau}}}^\circ \rightarrow \tilde{\mathcal{D}}(\mathbf{A}_{\mathbb{S}',x_i})_{\tilde{\tau}}^\circ$$

is an isomorphism for all $\tilde{\tau} \in \Sigma_{E,\infty}$ and $i = 1, 2$, the commutativity of ϕ_* with essential Frobenii shows that $u_{\tilde{\tau}} = u_{\sigma^{-1\tilde{\tau}}}$. The same arguments as above show that ϕ_* is an isomorphism. \square

Remark 7.9. This proposition does not guarantee the existence of the link morphism associated to a given link. However, we do expect the link morphisms to exist in general (for links with all curves turning to the right).

7.10 Link morphisms and Hecke operators

Assume that $\mathbf{S}_\infty = \Sigma_\infty$, so that \mathfrak{p} is of type α or α^\sharp and the band associated to \mathbf{S} consists of only plus signs. Let \mathfrak{q} and $\bar{\mathfrak{q}}$ be the two primes of E above \mathfrak{p} . We will focus on the compatibility of link morphisms with the Hecke operators at \mathfrak{q} , whose definition we recall now.

We have the following description:

$$G''_{\mathbb{S}}(\mathbb{Q}_p) \cong \text{GL}_2(F_{\mathfrak{p}}) \times_{F_{\mathfrak{p}}^\times} (E_{\mathfrak{q}}^\times \times E_{\bar{\mathfrak{q}}}^\times) \xrightarrow{\sim} \text{GL}_2(E_{\mathfrak{q}}) \times F_{\mathfrak{p}}^\times,$$

where the last isomorphism is given by $(g, (\lambda_1, \lambda_2)) \mapsto (g\lambda_1, \det(g)\lambda_1\lambda_2)$ for $g \in \text{GL}_2(F_{\mathfrak{p}})$, $\lambda_1 \in E_{\mathfrak{q}}^\times$, and $\lambda_2 \in E_{\bar{\mathfrak{q}}}^\times$. Then $G'_{\mathbb{S}}(\mathbb{Q}_p)$ is the subgroup $\text{GL}_2(E_{\mathfrak{q}}) \times \mathbb{Q}_p^\times$ of $G''_{\mathbb{S}}(\mathbb{Q}_p)$. Let $\gamma_{\mathfrak{q}}$ (respectively $\xi_{\mathfrak{q}}$) be the element of $G'_{\mathbb{S}}(\mathbb{A}^\infty)$ which is equal to

$$\left(\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix}, 1 \right) \in \text{GL}_2(E_{\mathfrak{q}}) \times \mathbb{Q}_p^\times \quad \left(\text{respectively} \quad \left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \in \text{GL}_2(E_{\mathfrak{q}}) \times \mathbb{Q}_p^\times \right)$$

at p and is equal to 1 at other places. Assume that $K' \subseteq G'_S(\mathbb{A}^\infty)$ is hyperspecial at p , i.e. $K'_p = \mathrm{GL}_2(\mathcal{O}_{E_q}) \times \mathbb{Z}_p^\times$. We use S_q and T_q to denote the Hecke correspondences on $\mathbf{Sh}_{K'}(G'_S)$ defined by $K'\gamma_q K'$ and $K'\xi_q K'$, respectively. Explicitly, if $\mathrm{Iw}'_p = \mathrm{Iw}_q \times \mathbb{Z}_p^\times \subseteq G'_S(\mathbb{Q}_p)$ with $\mathrm{Iw}_q \subseteq \mathrm{GL}_2(\mathcal{O}_{E_q})$ the standard Iwahoric subgroup reducing to upper triangular matrices when modulo p , then the Hecke correspondence T_q is given by the following diagram:

$$\begin{array}{ccc}
 & \mathbf{Sh}_{K'p\mathrm{Iw}'_p}(G'_S) & \\
 \mathrm{pr}_1 \swarrow & & \searrow \mathrm{pr}_2 \\
 \mathbf{Sh}_{K'}(G'_S) & & \mathbf{Sh}_{K'}(G'_S)
 \end{array} \tag{7.10.1}$$

where pr_1 is the natural projection and pr_2 is induced by the right multiplication by ξ_q . Note that S_q is an automorphism of $\mathbf{Sh}_{K'}(G'_S)$ and there is a natural p -quasi-isogeny of universal abelian schemes

$$\Phi_{S_q} : \mathbf{A}'_S \rightarrow S_q^* \mathbf{A}'_S$$

compatible with all structures such that the induced quasi-isogeny of p -divisible groups $\Phi_{S_q}[q^\infty] : \mathbf{A}'_S[q^\infty] \rightarrow (S_q^* \mathbf{A}'_S)[q^\infty]$ is the canonical isogeny with kernel $\mathbf{A}'_S[q]$.

Similarly, the elements γ_q and ξ_q induce Hecke correspondences on $\mathbf{Sh}_{K''}(G''_S)$, which we denote still by S_q and T_q , respectively.

PROPOSITION 7.11. *Assume that $S_\infty = \Sigma_\infty$. Let \tilde{S}_∞ and \tilde{S}'_∞ be two different choices of signatures in § 3.4. Suppose that there exists a link morphism $(\eta'_\#, \eta''_\#)$ from $\mathbf{Sh}_{K'}(G'_S)_{k_0}$ to $\mathbf{Sh}_{K''}(G''_S)_{k_0}$ (of some indentation) associated to the trivial link $S \rightarrow S'$, where $K'_p = \mathrm{GL}_2(\mathcal{O}_{E_q}) \times \mathbb{Z}_p^\times \subseteq G'(\mathbb{Q}_p)$ is hyperspecial. Then $(\eta'_\#, \eta''_\#)$ lifts uniquely to a link morphism $(\eta'_{\#, \mathrm{Iw}}, \eta''_{\#, \mathrm{Iw}})$ on $\mathbf{Sh}_{K'p\mathrm{Iw}'_p}(G'_S)_{k_0}$ such that the following commutative diagrams are Cartesian:*

$$\begin{array}{ccc}
 \mathbf{Sh}_{K'}(G'_S)_{k_0} & \xleftarrow{\mathrm{pr}_1} \mathbf{Sh}_{K'p\mathrm{Iw}'_p}(G'_S)_{k_0} \xrightarrow{\mathrm{pr}_2} \mathbf{Sh}_{K'}(G'_S)_{k_0} & \xrightarrow{S_q} \mathbf{Sh}_{K'}(G'_S)_{k_0} \\
 \downarrow \eta'_\# & \downarrow \eta'_{\#, \mathrm{Iw}} & \downarrow \eta'_\# \\
 \mathbf{Sh}_{K'}(G'_S)_{k_0} & \xleftarrow{\mathrm{pr}_1} \mathbf{Sh}_{K'p\mathrm{Iw}'_p}(G'_S)_{k_0} \xrightarrow{\mathrm{pr}_2} \mathbf{Sh}_{K'}(G'_S)_{k_0} & \xrightarrow{S_q} \mathbf{Sh}_{K'}(G'_S)_{k_0} \\
 & & \downarrow \eta''_\# \\
 & & \mathbf{Sh}_{K''}(G''_S)_{k_0}
 \end{array}$$

where the top and the bottom lines of the left-hand diagram are the Hecke correspondences T_q defined above. The same holds for the link morphism $(\eta''_\#, \eta'''_\#) : \mathbf{Sh}_{K''}(G''_S)_{k_0} \rightarrow \mathbf{Sh}_{K''}(G''_S)_{k_0}$.

Proof. Note that S_q is in fact an isomorphism of Shimura varieties; so the compatibility with S_q -action follows from the uniqueness of the link morphism by Proposition 7.8.

We prove now the existence of the lift $(\eta'_{\#, \mathrm{Iw}}, \eta''_{\#, \mathrm{Iw}})$, whose uniqueness is proved in Proposition 7.8. Let $x = (A, \iota, \lambda, \alpha_{K'^p})$ be a point of $\mathbf{Sh}_{K'}(G'_S)(\overline{\mathbb{F}}_p)$. Put $x' = \eta'_\#(x) = (A', \iota', \lambda', \alpha'_{K''^p})$. By Definition 7.5(3), for any $\tilde{\tau} \in \Sigma_{E, \infty}$, there exists $t_{\tilde{\tau}} \in \mathbb{Z}$ independent of x such that $\eta''_\#(\tilde{D}_{A, \tilde{\tau}}^\circ) = p^{t_{\tilde{\tau}}} \tilde{D}_{A', \tilde{\tau}}^\circ$. Fix a $\tilde{\tau}_0 \in \Sigma_{E, \infty/q}$. Giving a point y of $\mathbf{Sh}_{K'p\mathrm{Iw}'_p}(G'_S)_{k_0}$ with $\mathrm{pr}_1(y) = x$ is equivalent to giving a $W(\overline{\mathbb{F}}_p)$ -submodule $\tilde{H}_{\tilde{\tau}_0}^\circ \subseteq \tilde{D}_{A, \tilde{\tau}_0}^\circ$ such that $F_{\mathrm{es}, A}^g(\tilde{H}_{\tilde{\tau}_0}^\circ) = \tilde{H}_{\tilde{\tau}_0}^\circ$ and $\tilde{D}_{A, \tilde{\tau}_0}^\circ / \tilde{H}_{\tilde{\tau}_0}^\circ$ is one dimensional over $\overline{\mathbb{F}}_p$. We put $\tilde{H}_{\tilde{\tau}_0}^{\circ} = p^{-t_{\tilde{\tau}_0}} \eta''_\#(\tilde{H}_{\tilde{\tau}_0}^\circ) \subseteq \tilde{D}_{A', \tilde{\tau}_0}^\circ$. Then one sees easily that the quotient $\tilde{D}_{A', \tilde{\tau}_0}^\circ / \tilde{H}_{\tilde{\tau}_0}^{\circ}$ is one dimensional over $\overline{\mathbb{F}}_p$ and $\tilde{H}_{\tilde{\tau}_0}^{\circ}$ is fixed by $F_{\mathrm{es}, A'}^g$. This gives rise to a point y' of $\mathbf{Sh}_{K'p\mathrm{Iw}'_p}(G'_S)_{k_0}$ with $\mathrm{pr}_1(y') = x'$. One thus defines $\eta'_{\#, \mathrm{Iw}}(y) = y'$, and the quasi-isogeny $\eta''_{\#, \mathrm{Iw}}$ as the pullback of $\eta''_\#$ via pr_1 . It is clear by construction that $\eta''_{\#, \mathrm{Iw}} \circ \mathrm{pr}_1 = \mathrm{pr}_1 \circ \eta'_{\#, \mathrm{Iw}}$.

It remains to prove that $\eta'_\# \circ \text{pr}_2 = \text{pr}_2 \circ \eta'_{\#, \text{Iw}}$. Let $y = (A, \iota, \lambda, \alpha_{K'^p}, \tilde{H}_{\tilde{\tau}_0}^\circ) \in \mathbf{Sh}_{K'^p \text{Iw}_p}(G'_\mathfrak{S})(\overline{\mathbb{F}}_p)$ be a point above x as above. We put $\tilde{\mathcal{D}}_{A, \mathfrak{q}}^\circ := \bigoplus_{\tilde{\tau} \in \Sigma_{E, \infty/\mathfrak{q}}} \tilde{\mathcal{D}}_{A, \tilde{\tau}}^\circ$ and we define $\tilde{\mathcal{D}}_{A, \bar{\mathfrak{q}}}^\circ$ similarly with \mathfrak{q} replaced by $\bar{\mathfrak{q}}$. Then $\tilde{H}_\mathfrak{q}^\circ := 1/p \bigoplus_{i=0}^{g-1} F_{\text{es}, A}^i(\tilde{H}_{\tilde{\tau}_0}^\circ)$ is a $W(\overline{\mathbb{F}}_p)$ -lattice of $\tilde{\mathcal{D}}_{A, \mathfrak{q}}^\circ[1/p]$ stable under the action of F and V . Let $\tilde{H}_\mathfrak{q}^{\circ, \vee} = \tilde{H}_\mathfrak{q}^{\circ, \vee} \subseteq \tilde{\mathcal{D}}_{A, \mathfrak{q}}^\circ[1/p]$ denote the dual lattice of $\tilde{H}_\mathfrak{q}^\circ$ under the perfect pairing between $\tilde{\mathcal{D}}_{A, \mathfrak{q}}^\circ[1/p]$ and $\tilde{\mathcal{D}}_{A, \bar{\mathfrak{q}}}^\circ[1/p]$ induced by λ . By the theory of Dieudonné modules, there exists a unique abelian variety B equipped with \mathcal{O}_D -action ι_B together with an \mathcal{O}_D -linear p -quasi-isogeny $\phi: B \rightarrow A$ such that $\phi_*(\tilde{\mathcal{D}}_B^\circ)$ is identified with the lattice $\tilde{H}_\mathfrak{q}^\circ \oplus \tilde{H}_\bar{\mathfrak{q}}^\circ$ of $\tilde{\mathcal{D}}_A^\circ[1/p]$. Note that B satisfies the signature condition of $\mathbf{Sh}_{K'}(G'_\mathfrak{S})_{k_0}$. Since $\tilde{H}_\mathfrak{q}^\circ$ and $\tilde{H}_\bar{\mathfrak{q}}^\circ$ are dual to each other, the quasi-isogeny $\lambda_B = \phi^\vee \circ \lambda \circ \phi: B \rightarrow B^\vee$ is a prime-to- p polarization λ_B on B . We equip moreover B with the K'^p -level structure $\beta_{K'^p}$ such that $\alpha_{K'^p} = \phi \circ \beta_{K'^p}$. Thus, $z := (B, \iota_B, \lambda_B, \beta_{K'^p})$ gives rise to an $\overline{\mathbb{F}}_p$ -point of $\mathbf{Sh}_{K'}(G'_\mathfrak{S})_{k_0}$ and we have $z = \text{pr}_2(y)$ by the moduli interpretation of pr_2 .

Let $(B', \iota_{B'}, \lambda_{B'}, \beta'_{K'^p})$ denote the image $\eta'_\#(z)$. Then $\tilde{\mathcal{D}}_{B', \tilde{\tau}_0}^\circ$ is identified via $(\eta'_\#)^{-1}$ with the lattice $p^{-t_{\tilde{\tau}_0}} \tilde{\mathcal{D}}_{B, \tilde{\tau}_0}^\circ$ of $\tilde{\mathcal{D}}_{B, \tilde{\tau}_0}^\circ[1/p]$ and hence with the lattice $p^{-t_{\tilde{\tau}_0}-1} \tilde{H}_{\tilde{\tau}_0}$ of $\tilde{\mathcal{D}}_{A, \tilde{\tau}_0}^\circ[1/p]$. By our construction of $y' = \eta'_{\#, \text{Iw}}(y)$, it is easy to see that, if $\text{pr}_2(y') = (B'', \iota_{B''}, \lambda_{B''}, \beta''_{K'^p})$, then $\tilde{\mathcal{D}}_{B'', \tilde{\tau}_0}^\circ$ can be canonically identified with $\tilde{\mathcal{D}}_{B', \tilde{\tau}_0}^\circ$ as lattices of $\tilde{\mathcal{D}}_{A, \tilde{\tau}_0}^\circ[1/p]$. Since other components $\tilde{\mathcal{D}}_{B', \tilde{\tau}}^\circ$ or $\tilde{\mathcal{D}}_{B'', \tilde{\tau}}^\circ$ for $\tilde{\tau} \in \Sigma_{E, \infty}$ are determined from $\tilde{\mathcal{D}}_{B', \tilde{\tau}_0}^\circ$ and $\tilde{\mathcal{D}}_{B'', \tilde{\tau}_0}^\circ$ by the same rules (i.e. stability under the essential Frobenius and the duality), we see that B' is canonically isomorphic to B'' , compatible with all structures. This concludes the proof of $\text{pr}_2 \circ \eta'_{\#, \text{Iw}} = \eta'_\# \circ \text{pr}_2$. \square

For the rest of this paper, we discuss two topics, whose proofs are nested together. One topic is to understand the behavior of the description of the Goren–Oort strata under the link morphisms; the other is to understand the restriction of the \mathbb{P}^1 -bundle description of the Goren–Oort strata to other Goren–Oort strata.

PROPOSITION 7.12. *Let $\tau \in \Sigma_\infty - \mathfrak{S}_\infty$ be a place such that $\tau^- \neq \tau$ and let $\pi_\tau: \mathbf{Sh}_{K'}(G'_\mathfrak{S})_{k_0, \tau} \rightarrow \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0}$ be the \mathbb{P}^1 -bundle fibration given by Theorem 5.8 for the Goren–Oort stratum defined by the vanishing of the partial Hasse invariant at τ . Let \mathfrak{T} be a subset of $\Sigma_\infty - \mathfrak{S}_\infty$ containing τ .*

- (1) *If $\tau^+ \notin \mathfrak{T}$, then we put $\mathfrak{T}_\tau = \mathfrak{T} \setminus \{\tau, \tau^-\}$ and we have a commutative diagram.*

$$\begin{array}{ccc}
 \mathbf{Sh}_{K'}(G'_\mathfrak{S})_{k_0, \mathfrak{T}} & \xrightarrow{\quad} & \mathbf{Sh}_{K'}(G'_\mathfrak{S})_{k_0, \tau} \\
 \downarrow & & \downarrow \pi_\tau \\
 \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0, \mathfrak{T}_\tau} & \xrightarrow{\quad} & \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0}
 \end{array} \tag{7.12.1}$$

If $\tau^- \in \mathfrak{T}$, the left vertical arrow is an isomorphism. If $\tau^- \notin \mathfrak{T}$, this diagram is Cartesian.

- (2) *If $\tau, \tau^- \in \mathfrak{T}$ and $\tau^+ \neq \tau^-$, then we put $\mathfrak{T}_\tau = \mathfrak{T} \setminus \{\tau, \tau^-\}$, and π_τ induces a natural isomorphism*

$$\pi_\tau: \mathbf{Sh}_{K'}(G'_\mathfrak{S})_{k_0, \mathfrak{T}} \rightarrow \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0, \mathfrak{T}_\tau}. \tag{7.12.2}$$

Moreover, all descriptions above are compatible with the natural quasi-isogenies on universal abelian varieties, and analogous results hold for $\mathbf{Sh}_{K''}(G''_\mathfrak{S})_{k_0}$.

Proof. The statements for $\mathbf{Sh}_{K''}(G''_\mathfrak{S})_{k_0}$ follow from those analogs for $\mathbf{Sh}_{K'}(G'_\mathfrak{S})_{k_0}$ by § 2.12 (or in this case more explicitly by § 3.20). Thus, we will just prove the proposition for $\mathbf{Sh}_{K'}(G'_\mathfrak{S})_{k_0}$.

(1) If $\tau^+ \notin T$, the prime \mathfrak{p} must be of type α_1 or β_1 . By the proof of Proposition 5.17, the natural quasi-isogeny $\phi : \pi_\tau^*(\mathbf{A}'_{\mathfrak{S}(\tau), k_0}) \rightarrow \mathbf{A}'_{\mathfrak{S}, k_0} |_{\mathbf{sh}_{K'}(G'_{\mathfrak{S}})_{k_0, \tau}}$ induces an isomorphism on the (reduced) differential forms at $\tilde{\tau}'$ for all $\tilde{\tau}' \in \Sigma_{E, \infty}$ not lifting $\tau, \sigma^{-1}\tau, \dots, \sigma^{-n_\tau}\tau = \tau^-$. So, π_τ induces a Cartesian square

$$\begin{CD} \mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, T_\tau \cup \{\tau\}} @>>> \mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, \tau} \\ @VV \pi_\tau V @VV \pi_\tau V \\ \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0, T_\tau} @>>> \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0} \end{CD}$$

This already proves (1) in case $\tau^- \notin T$. Suppose now that $\tau^- \in T$. By Proposition 5.23, $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, T_\tau \cup \{\tau\}}$ is the moduli space of tuples $(B, \iota_B, \lambda_B, \beta_{K'_T}; J_{\tilde{\tau}^-}^\circ)$, where:

- $(B, \iota_B, \lambda_B, \beta_{K'_T})$ is a point of $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0, T_\tau}$ with values in a scheme S over k_0 ;
- $J_{\tilde{\tau}^-}^\circ$ is a subbundle of $H_1^{\text{dR}}(B/S)_{\tilde{\tau}^-}^\circ$ of rank one (here $\tilde{\tau}^- \in \Sigma_{E, \infty}$ is the specific lift of τ^- defined in § 5.21).

Then the closed subscheme $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, T}$ of $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, T_\tau \cup \{\tau\}}$ is defined by the condition:

$$J_{\tilde{\tau}^-}^\circ = F_{B, \text{es}}^{n_{\tau^-}}(H_1^{\text{dR}}(B^{(p^{n_{\tau^-}})})/S)_{\sigma^{-n_{\tau^-}}\tilde{\tau}^-}^\circ.$$

This shows that the restriction of $\pi_\tau : \mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, T_\tau \cup \{\tau\}} \rightarrow \mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, T_\tau}$ to $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, T}$ is an isomorphism.

(2) When $\tau^+ \notin T$, this was proved in (1). Assume now that $\tau^+ \in T$. To complete the proof, it suffices to prove that, for a k_0 -scheme S , the τ^+ th partial Hasse invariant vanishes at an S -point $x = (A, \iota_A, \lambda_A, \alpha_{K'}) \in \mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, \{\tau\}}$ if and only if it vanishes at $\pi_\tau(x) = (B, \iota_B, \lambda_B, \beta_{K'}) \in \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0}$. Let $\tilde{\tau}$ be the lift of τ contained in $\tilde{\Delta}(\tau)^+$ (see § 5.11 for the notation $\tilde{\Delta}(\tau)^+$). Put $\tilde{\tau}^+ = \sigma^{n_{\tau^+}}\tilde{\tau}$ and $\tilde{\tau}^- = \sigma^{-n_\tau}\tilde{\tau}$. By Lemma 4.5, it suffices to show that

$$\begin{aligned} F_{A, \text{es}}^{n_{\tau^+}}(H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ, (p^{n_{\tau^+}})}) &= \omega_{A^\vee/S, \tilde{\tau}^+}^\circ \\ \Leftrightarrow F_{B, \text{es}}^{n_{\tau^+} + n_\tau + n_{\tau^-}}(H_1^{\text{dR}}(B/S)_{\sigma^{-n_{\tau^-}}\tilde{\tau}^-}^{\circ, (p^{n_{\tau^+} + n_\tau + n_{\tau^-}})}) &= \omega_{B^\vee/S, \tilde{\tau}^+}^\circ. \end{aligned} \tag{7.12.3}$$

But this follows from the following three facts.

- (a) By the definition of essential Frobenius in Notation 4.2, one deduces a commutative diagram.

$$\begin{CD} H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^{\circ, (p^{n_{\tau^+}})} @> F_{A, \text{es}}^{n_{\tau^+}} >> H_1^{\text{dR}}(A/S)_{\tilde{\tau}^+}^\circ \\ @V \phi_{*, \tilde{\tau}} VV @V \cong \phi_{*, \tilde{\tau}^+} VV \\ H_1^{\text{dR}}(B/S)_{\tilde{\tau}}^{\circ, (p^{n_{\tau^+}})} @> F_{B, \text{es}}^{n_{\tau^+}} >> H_1^{\text{dR}}(B/S)_{\tilde{\tau}^+}^\circ \end{CD}$$

- (b) It follows from condition (v) of the moduli description in § 5.15 that

$$\phi_{*, \tilde{\tau}}(H_1^{\text{dR}}(A/S)_{\tilde{\tau}}^\circ) = F_{B, \text{es}}^{n_{\tau^-} + n_\tau}(H_1^{\text{dR}}(B/S)_{\sigma^{-n_{\tau^-}}\tilde{\tau}^-}^{\circ, (p^{n_{\tau^-} + n_\tau})}).$$

- (c) The condition $\tau^- \neq \tau^+$ implies that the quasi-isogeny $\phi : A \rightarrow B$ induces an isomorphism $\phi_{*, \tilde{\tau}^+} : H_1^{\text{dR}}(A/S)_{\tilde{\tau}^+}^\circ \cong H_1^{\text{dR}}(B/S)_{\tilde{\tau}^+}^\circ$ preserving the Hodge filtrations, in particular identifying the submodules $\phi_{*, \tilde{\tau}}(\omega_{A^\vee/S, \tilde{\tau}^+}^\circ) = \omega_{B^\vee/S, \tilde{\tau}^+}^\circ$. □

7.13 Compatibility of link morphisms and the description of Goren–Oort strata

We first recall that, although the subset $\mathbf{S}(\tau)$ is completely determined by \mathbf{S} and τ as in §5.1, the lift $\tilde{\mathbf{S}}(\tau)_\infty$, which consists of all $\tilde{\tau}' \in \Sigma_{E,\infty}$ with signature $s_{\tilde{\tau}'} = 0$ (see §3.4), depends on an auxiliary choice in §5.3: a lift $\tilde{\tau}$ of τ to be contained in $\tilde{\mathbf{S}}(\tau)_\infty$. We assume that $\#(\Sigma_\infty - \mathbf{S}_\infty) \geq 2$. If \mathfrak{p} splits as $\mathfrak{q}\bar{\mathfrak{q}}$ in E for a fixed place \mathfrak{q} , then the $\tilde{\tau}$ contained in $\tilde{\mathbf{S}}(\tau)_\infty$ is always chosen to be the one in $\Sigma_{E,\infty/\mathfrak{q}}$. If \mathfrak{p} is inert in E , then there are two possible choices: $\tilde{\tau}$ and its conjugate $\tilde{\tau}^c$ for a fixed lift $\tilde{\tau}$ of τ . In the latter case, we denote by $\tilde{\tau}$ the lift of τ contained in $\tilde{\mathbf{S}}(\tau)_\infty$, by $\tilde{\mathbf{S}}(\tau)' = (\mathbf{S}(\tau), \mathbf{S}(\tau)'_\infty)$ the lift of $\mathbf{S}(\tau)$ such that $\tilde{\tau}^c \in \tilde{\mathbf{S}}(\tau)'_\infty$, and let $\pi'_\tau: \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\tau} \rightarrow \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)'})_{k_0}$ be the corresponding \mathbb{P}^1 -bundle. The following proposition says that π_τ and π'_τ differ from each other by a link isomorphism.

PROPOSITION 7.14. *Assume that \mathfrak{p} is inert in E . Then there exists a link isomorphism*

$$(\eta'_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)',\sharp}, \eta''_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)'}) : \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0} \xrightarrow{\sim} \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)'})_{k_0}$$

(of indentation degree 0) associated to the identity link $\eta : \mathbf{S}(\tau) \rightarrow \mathbf{S}(\tau)$ such that the diagram

$$\begin{array}{ccc} & \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\tau} & \\ \pi_\tau \swarrow & & \searrow \pi'_\tau \\ \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0} & \xrightarrow[\eta_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)',\sharp}]{\cong} & \mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)'})_{k_0} \end{array}$$

commutes. Similar statements hold for $\mathbf{Sh}_{K''}(G''_{\tilde{\mathbf{S}}(\tau)})_{k_0}$ and $\mathbf{Sh}_{K''}(G''_{\tilde{\mathbf{S}}(\tau)'})_{k_0}$.

Proof. Consider the closed subvariety $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\{\tau,\tau^-\}}$ of $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\tau}$. Then, by Proposition 7.12(2) (note that \mathfrak{p} being inert in E implies that $\tau^+ \neq \tau^-$), $\pi|_{\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\{\tau,\tau^-\}}}$ is an isomorphism. We put

$$\eta'_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)',\sharp} := \pi'_\tau \circ (\pi|_{\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\{\tau,\tau^-\}}})^{-1}$$

and define $\eta''_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)'}$ as the pullback via $(\pi_\tau|_{\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\{\tau^-, \tau\}}})^{-1}$ of the quasi-isogeny

$$\pi_\tau^* \mathbf{A}'_{\tilde{\mathbf{S}}(\tau),k_0} \rightarrow \mathbf{A}'_{\tilde{\mathbf{S}}(\tau),k_0}|_{\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\{\tau,\tau^-\}}} \rightarrow \pi_\tau^{*} \mathbf{A}_{\tilde{\mathbf{S}}(\tau)',k_0},$$

where the two quasi-isogenies are given by Theorem 5.8(2). By Proposition 7.12(2) again, $\pi'_\tau|_{\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0,\{\tau,\tau^-\}}}$ is an isomorphism and hence so is $\eta'_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)',\sharp}$. It remains to show that $(\eta'_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)',\sharp}, \eta''_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)'})$ is a link morphism associated to the identity link on \mathbf{S} . Let x be a geometric point of $\mathbf{Sh}_{K'}(G'_{\tilde{\mathbf{S}}(\tau)})_{k_0}$, and $x' = \eta'_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)',\sharp}(x)$. By construction, it is easy to see that the quasi-isogeny $\eta''_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)'}$ induces an isomorphism $\tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}}(\tau),k_0,x})_{\tilde{\tau}'}^\circ \xrightarrow{\sim} \tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}}(\tau)',k_0,x'})_{\tilde{\tau}'}^\circ$ for $\tilde{\tau}' \neq \sigma^a \tilde{\tau}, \sigma^a \tilde{\tau}^c$ with $a = 0, \dots, n_\tau - 1$. In the exceptional cases, we have

$$\eta''_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)' }(\tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}}(\tau),k_0,x})_{\tilde{\tau}'}^\circ) = \begin{cases} p \tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}}(\tau)',k_0,x'})_{\tilde{\tau}'}^\circ & \text{for } \tilde{\tau}' = \sigma^a \tilde{\tau} \text{ for } a = 0, \dots, n_\tau - 1, \\ \frac{1}{p} \tilde{\mathcal{D}}(\mathbf{A}'_{\tilde{\mathbf{S}}(\tau)',k_0,x'})_{\tilde{\tau}'}^\circ & \text{for } \tilde{\tau}' = \sigma^a \tilde{\tau}^c \text{ for } a = 0, \dots, n_\tau - 1. \end{cases}$$

Hence, $(\eta'_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)',\sharp}, \eta''_{\tilde{\mathbf{S}}(\tau),\tilde{\mathbf{S}}(\tau)'})$ verifies Definition 7.5. □

The following lemma will be needed in the proof of the main result of this section.

LEMMA 7.15. Assume that $S_\infty \neq \emptyset$. Let $\chi(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{F}_p}) := \sum_{i=0}^{+\infty} (-1)^i \dim H_{\text{et}}^i(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{F}_p}, \mathbb{Q}_\ell)$ denote the Euler–Poincaré characteristic of $\mathbf{Sh}_{K'}(G'_S)_{\mathbb{F}_p}$ for some fixed prime $\ell \neq p$. Then we have $\chi(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{F}_p}) \neq 0$.

Proof. The assumption that $S_\infty \neq \emptyset$ implies that the Shimura variety $\mathbf{Sh}_{K'}(G'_S)$ is proper. Consider the integral model $\mathbf{Sh}_{K'}(G'_S)$ over $\mathcal{O}_{\tilde{\rho}}$ and choose an embedding $\mathcal{O}_{\tilde{\rho}} \hookrightarrow \mathbb{C}$. By the proper base change theorem and the standard comparison theorems, we have

$$\chi(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{F}_p}) = \chi(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}) := \sum_{i=0}^{\infty} (-1)^i \dim H_{\text{sing}}^i(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}, \mathbb{C}),$$

where $H_{\text{sing}}^i(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}, \mathbb{C})$ denotes the singular cohomology of $\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}$ for the usual complex topology. For each $\tilde{\tau}$ lifting an element of $\Sigma_\infty - S_\infty$, put $\omega_{\tilde{\tau}}^\circ = \omega_{\mathbf{A}'_{S,\mathbb{C}}, \tilde{\tau}}^\circ$ and $\mathfrak{t}_{\tilde{\tau}}^\circ = \text{Lie}(\mathbf{A}'_{S,\mathbb{C}})_{\tilde{\tau}}^\circ = \omega_{\tilde{\tau}}^{\circ, \vee}$ to simplify the notation. They are line bundles over $\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}$. We have a Hodge filtration

$$0 \rightarrow \omega_{\tilde{\tau}c}^\circ \rightarrow H_1^{\text{dR}}(\mathbf{A}'_{S,\mathbb{C}}/\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}})_{\tilde{\tau}}^\circ \rightarrow \mathfrak{t}_{\tilde{\tau}}^\circ \rightarrow 0.$$

Note that $H_1^{\text{dR}}(\mathbf{A}'_{S,\mathbb{C}}/\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}})_{\tilde{\tau}}^\circ$ is equipped with the integrable Gauss–Manin connection, so that its Chern classes are trivial by classical Chern–Weil theory. One thus obtains

$$\begin{cases} c_1(\omega_{\tilde{\tau}c}^\circ)c_1(\mathfrak{t}_{\tilde{\tau}}^\circ) = c_2(H_1^{\text{dR}}(\mathbf{A}'_{S,\mathbb{C}}/\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}})_{\tilde{\tau}}^\circ) = 0, \\ c_1(\omega_{\tilde{\tau}c}^\circ) + c_1(\mathfrak{t}_{\tilde{\tau}}^\circ) = c_1(H_1^{\text{dR}}(\mathbf{A}'_{S,\mathbb{C}}/\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}})_{\tilde{\tau}}^\circ) = 0 \end{cases} \implies \begin{cases} c_1(\omega_{\tilde{\tau}}^\circ)^2 = 0, \\ c_1(\omega_{\tilde{\tau}}^\circ) = c_1(\omega_{\tilde{\tau}c}^\circ), \end{cases}$$

where $c_i(\mathcal{E}) \in H_{\text{sing}}^{2i}(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}, \mathbb{C})$ denotes the i th Chern class of a vector bundle \mathcal{E} . Let \mathcal{T} denote the tangent bundle of $\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}$ and put $\det(\omega) := \bigotimes_{\tilde{\tau} \in \Sigma_{E,\infty} - S_{E,\infty}} \omega_{\tilde{\tau}}^\circ$. By Corollary 3.17, we get

$$c_d(\mathcal{T}) = \prod_{\tau \in \Sigma_\infty - S_\infty} (-2c_1(\omega_{\tilde{\tau}}^\circ)),$$

where $\tilde{\tau} \in \Sigma_{E,\infty}$ is an arbitrary lift of τ and $d = \#\Sigma_\infty - \#S_\infty$ is the dimension of $\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}$. Note that $c_1(\omega_{\tilde{\tau}}^\circ)^2 = 0$ and $c_1(\omega_{\tilde{\tau}}^\circ) = c_1(\omega_{\tilde{\tau}c}^\circ)$ imply that

$$(c_1(\det(\omega)))^d = 2^d d! \prod_{\tau \in \Sigma_\infty - S_\infty} (c_1(\omega_{\tilde{\tau}}^\circ)) = (-1)^d d! c_d(\mathcal{T}).$$

It is well known that $\det(\omega)$ is ample (see [Lan13], for instance) and hence it follows that $c_d(\mathcal{T}) \neq 0$. On the other hand, there exists a canonical isomorphism

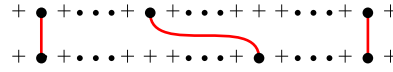
$$\text{Tr}: H_{\text{sing}}^{2d}(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}}, \mathbb{C}) \xrightarrow{\sim} \mathbb{C},$$

which sends the cycle class of a point to 1. The lemma follows immediately from the non-vanishing of $c_d(\mathcal{T})$ and the well-known fact that $\text{Tr}(c_d(\mathcal{T})) = \chi(\mathbf{Sh}_{K'}(G'_S)_{\mathbb{C}})$. \square

We state now the main result of this section, which will play a crucial role in our application to Tate cycles in [TX14].

THEOREM 7.16. Keep the same notation as in Proposition 7.12, that is, let $\tau \in \Sigma_\infty - S_\infty$ be a place such that τ^- is different from τ and let \mathbf{T} be a subset of $\Sigma_\infty - S_\infty$ containing τ .

- (1) If $\tau^- \notin \mathbf{T}$ and $\tau, \tau^+ \in \mathbf{T}$, we put $\mathbf{T}_{\tau^+} = \mathbf{T} \setminus \{\tau, \tau^+\}$. Let $\eta = \eta_{\tau^- \rightarrow \tau^+} : \mathbf{S}(\tau^+) \rightarrow \mathbf{S}(\tau)$ be the link given by straight lines except sending τ^- to τ^+ (to the right) with displacement $v(\eta) = n_\tau + n_{\tau^+}$:



(a) We consider the composition

$$\eta_{\sharp}^{\prime} : \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau^+)}) \xleftarrow[\cong]{\pi_{\tau^+}} \mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \{\tau, \tau^+\}} \xrightarrow{i_{\tau^+}} \mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \tau} \xrightarrow{\pi_{\tau}} \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau)})_{k_0},$$

and let η^{\sharp} denote the natural quasi-isogeny of abelian varieties on $\mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau^+)})_{k_0}$ given by (the pullback via $(\pi_{\tau^+}|_{\mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \{\tau, \tau^+\}}})^{-1,*}$ of)

$$(\pi_{\tau^+}|_{\mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \{\tau, \tau^+\}}})^* \mathbf{A}'_{\mathbf{S}(\tau^+), k_0} \leftarrow i_{\tau^+}^*(\mathbf{A}'_{\mathbf{S}, k_0}|_{\mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \tau}}) \rightarrow i_{\tau^+}^* \pi_{\tau}^*(\mathbf{A}'_{\mathbf{S}(\tau), k_0}).$$

Then $(\eta_{\sharp}^{\prime}, \eta^{\sharp})$ is the link morphism associated to the link η of indentation degree $n = n_{\tau^+} - n_{\tau}$ if \mathfrak{p} splits in E/F and $n = 0$ if \mathfrak{p} is inert in E/F . Moreover, the following diagram:

$$\begin{array}{ccccc} \mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \mathbf{T}} & \xrightarrow{\quad} & \mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \{\tau, \tau^+\}} & \xrightarrow{i_{\tau^+}} & \mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \tau} \\ \cong \downarrow & & \cong \downarrow \pi_{\tau^+}|_{\mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \{\tau, \tau^+\}}} & & \downarrow \pi_{\tau} \\ \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau^+)})_{k_0, \mathbf{T}_{\tau^+}} & \xrightarrow{\quad} & \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau^+)})_{k_0} & \xrightarrow{\eta_{\sharp}^{\prime}} & \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau)})_{k_0} \end{array} \tag{7.16.1}$$

is commutative, where the two vertical isomorphisms are given by Proposition 7.12(2).

- (b) For $\tilde{\tau}' \in \Sigma_{E, \infty} \setminus \mathbf{S}_{E, \infty}$, the quasi-isogeny $\eta_{\sharp}^{\prime} : \mathbf{A}'_{\mathbf{S}(\tau^+), k_0} \rightarrow \eta_{\sharp}^{\prime*} \mathbf{A}'_{\mathbf{S}(\tau), k_0}$ induces a canonical isomorphism

$$\eta_{\sharp}^{\prime*} \text{Lie}(\mathbf{A}'_{\mathbf{S}(\tau), k_0})_{\tilde{\tau}'}^{\circ} \cong \begin{cases} \text{Lie}(\mathbf{A}'_{\mathbf{S}(\tau^+), k_0})_{\sigma^{-v(\eta)} \tilde{\tau}'}^{\circ, (p^{v(\eta)})} & \text{if } \tilde{\tau}' \text{ is a lifting of } \tau^+, \\ \text{Lie}(\mathbf{A}'_{\mathbf{S}(\tau^+), k_0})_{\tilde{\tau}'}^{\circ} & \text{otherwise.} \end{cases}$$

(c) The morphism η_{\sharp}^{\prime} is finite flat of degree $p^{v(\eta)} = p^{n_{\tau} + n_{\tau^+}}$.

- (2) Assume that $\Sigma_{\infty} - \mathbf{S}_{\infty} = \{\tau, \tau^-\} = \mathbf{T}$ (so that $\tau^+ = \tau^-$ and \mathfrak{p} is of type α_2 for \mathbf{T}). Then there exists a link morphism $(\eta_{\sharp}, \eta^{\sharp}) : \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau^-)})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau)})_{k_0}$ of indentation degree $2(g - n_{\tau}) = 2n_{\tau^-}$ associated with the trivial link $\eta : \mathbf{S}(\tau^-) \rightarrow \mathbf{S}(\tau)$ such that the diagram

$$\begin{array}{ccccc} & & \mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \{\tau, \tau^-\}} & & \\ & \swarrow \pi_{\tau} & & \searrow \pi_{\tau^-} & \\ \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau)})_{k_0} & & & & \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau^-)})_{k_0} \xrightarrow{\eta_{\sharp}} \mathbf{Sh}_{K'}(G'_{\mathbf{S}(\tau)})_{k_0} \end{array}$$

coincides with the Hecke correspondence $T_{\mathfrak{q}}$ (7.10.1) if we identify $\mathbf{Sh}_{K'}(G'_{\mathbf{S}})_{k_0, \{\tau, \tau^-\}}$ with $\mathbf{Sh}_{K'^p \text{Iw}'_p}(G'_{\mathbf{S}(\tau)})_{k_0}$ via the isomorphism given by Theorem 5.8.

All descriptions above are compatible with the natural tame Hecke operator actions and similar results apply to $\mathbf{Sh}_{K''}(G''_{\mathbb{S}})_{k_0}$.

Proof. The statements for $\mathbf{Sh}_{K''}(G''_{\mathbb{S}})_{k_0}$ follow from those analogs for $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$ by § 2.12 (or in this case more explicitly by § 3.20). Thus, we will just prove the theorem for $\mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0}$.

(1)(a) The commutativity of the left square of (7.16.1) is tautological and the commutativity of the right square was proved in Proposition 7.12(1). It remains to show that $\pi_{\tau} \circ (\pi_{\tau+} |_{\mathbf{sh}_{K'}(G'_{\mathbb{S}})_{k_0, \{\tau, \tau+\}}})^{-1}$ is the link morphism η'_{\sharp} on $\mathbf{Sh}_{K'}(G'_{\mathbb{S}(\tau+)})_{k_0}$ associated to the link $\eta = \eta_{\tau \rightarrow \tau+}$.

Let $\tilde{\tau} \in \Sigma_{E, \infty}$ (respectively $\tilde{\tau}^+$) denote the lift of τ (respectively τ^+) contained in $\tilde{\mathbb{S}}(\tau)_{\infty}$ (respectively $\tilde{\mathbb{S}}(\tau^+)_{\infty}$). By § 5.3, we have $\tilde{\tau} = \sigma^{-n_{\tau}} \tilde{\tau}^+$ if \mathfrak{p} splits in E . If \mathfrak{p} is inert in E , it is also harmless to assume that $\tilde{\tau} = \sigma^{-n_{\tau}} \tilde{\tau}^+$ in view of Propositions 7.14 and 7.17. Put $\tilde{\tau}^- = \sigma^{-n_{\tau}} \tilde{\tau}$. Let $y = (B, \iota_B, \lambda_B, \beta'_{K'})$ be an S -point of $\mathbf{Sh}_{K'}(G'_{\mathbb{S}(\tau+)})_{k_0}$ for a locally noetherian k_0 -scheme S . Then the preimage of y under $\pi_{\tau+} |_{\mathbf{sh}_{K'}(G'_{\mathbb{S}})_{k_0, \{\tau, \tau+\}}}$ is given by $x = (A, \iota_A, \lambda_A, \alpha_{K'}) \in \mathbf{Sh}_{K'}(G'_{\mathbb{S}})_{k_0, \{\tau, \tau+\}}$, for which there exists a quasi-isogeny $\phi : B \rightarrow A$ such that $\phi_{*, \tilde{\tau}'} : H_1^{\text{dR}}(B/S)_{\tilde{\tau}'}^{\circ} \rightarrow H_1^{\text{dR}}(A/S)_{\tilde{\tau}'}^{\circ}$ is a well-defined isomorphism for all $\tilde{\tau}' \in \Sigma_{E, \infty}$ except for $\tilde{\tau}' = \sigma^a \tilde{\tau}$ or $\sigma^a \tilde{\tau}^c$ with $a = 1, \dots, n_{\tau+}$. Note that $\sigma^a \tilde{\tau} \in \tilde{\Delta}(\tau^+)^-$ and $\sigma^a \tilde{\tau}^c \in \tilde{\Delta}(\tau^+)^+$. So, in the exceptional cases, $\phi_{*, \sigma^a \tilde{\tau}}$ and $(\phi^{-1})_{*, \sigma^a \tilde{\tau}^c}$ are well defined, where $\phi^{-1} : A \rightarrow B$ denotes the quasi-isogeny inverse to ϕ , and we have (by the proof of Proposition 5.23)

$$\text{Ker}(\phi_{*, \sigma^a \tilde{\tau}}) = F_{B, \text{es}}^{n_{\tau+} + a} (H_1^{\text{dR}}(B/S)_{\tilde{\tau}^-}^{\circ, (p^{n_{\tau+} + a})}) \cong \text{Lie}(B/S)_{\tilde{\tau}^-}^{\circ, (p^{n_{\tau+} + a})} \tag{7.16.2}$$

$$\text{Im}((\phi^{-1})_{*, \sigma^a \tilde{\tau}^c}) = F_{B, \text{es}}^{n_{\tau+} + a} (H_1^{\text{dR}}(B/S)_{\tilde{\tau}^-, c}^{\circ, (p^{n_{\tau+} + a})}) \cong \text{Lie}(B/S)_{\tilde{\tau}^-, c}^{\circ, (p^{n_{\tau+} + a})}. \tag{7.16.3}$$

If π_{τ} sends $x = (A, \iota_A, \lambda_A, \alpha_{K'})$ to $z = (B', \iota_{B'}, \lambda_{B'}, \beta'_{K'}) \in \mathbf{Sh}_{K'}(G'_{\mathbb{S}(\tau)})(S)$, there is a quasi-isogeny $\psi : A \rightarrow B'$ such that $\psi_{*, \tilde{\tau}'}$ is an isomorphism for all $\tilde{\tau}' \in \Sigma_{E, \infty}$ except for $\tilde{\tau}' = \sigma^b \tilde{\tau}^-$ or $\sigma^b \tilde{\tau}^{-, c}$ with $b = 1, \dots, n_{\tau}$. In the exceptional cases, $\psi_{*, \sigma^b \tilde{\tau}^{-, c}}$ and $(\psi^{-1})_{*, \sigma^b \tilde{\tau}^-}$ are well defined, and we have (by the proof of Proposition 5.17 or rather the moduli description in § 5.15)

$$\begin{aligned} \text{Ker}(\psi_{*, \sigma^b \tilde{\tau}^{-, c}}) &= F_{A, \text{es}}^b (H_1^{\text{dR}}(A/S)_{\tilde{\tau}^{-, c}}^{\circ, (p^b)}) \\ \text{Im}((\psi^{-1})_{*, \sigma^b \tilde{\tau}^-}) &= F_{A, \text{es}}^b (H_1^{\text{dR}}(A/S)_{\tilde{\tau}^-}^{\circ, (p^b)}). \end{aligned}$$

By definition, we have $\eta'_{\sharp}(y) = z$ and the composed quasi-isogeny $\psi \circ \phi : B \rightarrow B'$ is nothing but the base change of η'_{\sharp} to S . For later reference, we remark that ψ and ϕ induce isomorphisms

$$\text{Lie}(B/S)_{\tilde{\tau}'}^{\circ} \cong \text{Lie}(A/S)_{\tilde{\tau}'}^{\circ} \cong \text{Lie}(B'/S)_{\tilde{\tau}'}^{\circ} \tag{7.16.4}$$

for all $\tilde{\tau}'$ with restriction $\tau' \in \Sigma_{\infty} - \mathbf{S}_{\infty}$ different from τ^-, τ, τ^+ , and

$$\text{Lie}(B/S)_{\tilde{\tau}^-}^{\circ, (p^{n_{\tau} + n_{\tau+}})} \xrightarrow[\cong]{(7.16.2)} \text{Ker}(\phi_{*, \tilde{\tau}^+}) \cong \text{Coker}(\phi_{*, \tilde{\tau}^+}) \cong \text{Lie}(A/S)_{\tilde{\tau}^+}^{\circ} \xrightarrow[\cong]{\psi_{*, \tilde{\tau}^+}} \text{Lie}(B'/S)_{\tilde{\tau}^+}^{\circ}, \tag{7.16.5}$$

$$\text{Lie}(B/S)_{\tilde{\tau}^-, c}^{\circ, (p^{n_{\tau} + n_{\tau+}})} \xrightarrow[\cong]{(7.16.3)} \text{Im}((\phi^{-1})_{*, \tilde{\tau}^+, c}) \cong \text{Lie}(A/S)_{\tilde{\tau}^+, c}^{\circ} \xrightarrow[\cong]{\psi_{*, \tilde{\tau}^+, c}} \text{Lie}(B'/S)_{\tilde{\tau}^+, c}^{\circ}. \tag{7.16.6}$$

Consider the case when $S = \text{Spec}(k)$ with k a perfect field containing k_0 . Denote by $\tilde{\mathcal{D}}_{B, \tilde{\tau}'}$ the $\tilde{\tau}'$ -component of the reduced covariant Dieudonné module of B . From the discussion above, one

sees easily that $\tilde{\mathcal{D}}_{B',\tilde{\tau}}^\circ = \tilde{\mathcal{D}}_{B,\tilde{\tau}'}^\circ$ for all $\tilde{\tau}' \in \Sigma_{E,\infty}$ except for $\tilde{\tau}' \in \{\sigma^a \tilde{\tau}, \sigma^a \tilde{\tau}^c \mid 1 \leq a \leq n_{\tau^+}\} \cup \{\sigma^b \tilde{\tau}^-, \sigma^b \tilde{\tau}^{-,c} \mid 1 \leq b \leq n_\tau\}$. In the exceptional cases, we have

$$(\psi \circ \phi)_*^{-1} \tilde{\mathcal{D}}_{B',\tilde{\tau}'}^\circ = \begin{cases} p^{-1} F_{B,\text{es}}^{n_\tau+a}(\tilde{\mathcal{D}}_{B,\tilde{\tau}^-}^\circ) & \text{if } \tilde{\tau}' = \sigma^a \tilde{\tau}, \\ F_{B,\text{es}}^{n_\tau+a}(\tilde{\mathcal{D}}_{B,\tilde{\tau}^-}^\circ) & \text{if } \tilde{\tau}' = \sigma^a \tilde{\tau}^c, \\ p^{-1} F_{B,\text{es}}^b(\tilde{\mathcal{D}}_{B,\tilde{\tau}^-}^\circ) & \text{if } \tilde{\tau}' = \sigma^b \tilde{\tau}^{-,c}, \\ F_{B,\text{es}}^b(\tilde{\mathcal{D}}_{B,\tilde{\tau}^-}^\circ) & \text{if } \tilde{\tau}' = \sigma^b \tilde{\tau}^-. \end{cases}$$

Since the essential Frobenius $F_{B,\text{es}}$ is bijective after inverting p , one sees easily that $\tilde{\mathcal{D}}_B^\circ$ can be recovered from $\tilde{\mathcal{D}}_{B'}^\circ$. This implies immediately that $\eta'_\sharp : \mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau^+)})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau)})_{k_0}$ induces a bijection on k -valued points, i.e. η'_\sharp verifies condition (1) in Definition 7.5. By the discussion above, it is also obvious that conditions (2) and (3) of Definition 7.5 are also verified for $(\eta'_\sharp, \eta^\sharp)$. Finally, from the formulas for $\tilde{\mathcal{D}}_{B',\tilde{\tau}'}^\circ$, one sees easily that the degree of the quasi-isogeny

$$(\psi \circ \phi)_q : B[\mathfrak{q}^\infty] \rightarrow B'[\mathfrak{q}^\infty]$$

is $2(n_{\tau^+} - n_\tau)$ if \mathfrak{p} splits in E , and is 0 if \mathfrak{p} is inert in E . This shows that $(\eta'_\sharp, \eta^\sharp)$ is the link morphism associated to η with the said indentation degree.

Statement (1)(b) follows from the isomorphisms (7.16.4)–(7.16.6) applied to the case when B is the universal abelian scheme $\mathbf{A}'_{\tilde{\mathcal{S}}(\tau^+),k_0}$. It remains to prove (1)(c). The morphism η'_\sharp is clearly quasi-finite and hence finite because $\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau^+)})_{k_0}$ is proper by Proposition 4.7. Since both $\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau)})_{k_0}$ and $\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau^+)})_{k_0}$ are regular, we conclude by [Mat86, Theorem 23.1] that η'_\sharp is flat at every point of $\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau)})_{k_0}$. It remains to see that the degree of η'_\sharp is $p^{v(\eta)}$. For this, one might be able to argue geometrically, but we found it more convenient to compare the top Chern classes of the tangent bundle.

Let \mathcal{T}_τ and \mathcal{T}_{τ^+} denote respectively the tangent bundles of $\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau)})_{k_0}$ and $\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau^+)})_{k_0}$, and let $d = \#\Sigma_\infty - \#\mathcal{S}(\tau)_\infty$ be the common dimension of these Shimura varieties. Fix a prime $\ell \neq p$. For a vector bundle \mathcal{E} over a proper and smooth k_0 -variety X , we denote by $c_i(\mathcal{E}) \in H_{\text{et}}^{2i}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)(i)$ the i th Chern class of \mathcal{E} . By Proposition 4.7, we have

$$c_d(\mathcal{T}_\tau) = \prod_{\tau' \in \Sigma_\infty - \mathcal{S}(\tau)_\infty} c_1(\text{Lie}(\mathbf{A}'_{\tilde{\mathcal{S}}(\tau),k_0})_{\tilde{\tau}'}^\circ \otimes \text{Lie}(\mathbf{A}'_{\tilde{\mathcal{S}}(\tau),k_0})_{\tilde{\tau}'^c}^\circ),$$

where $\tilde{\tau}', \tilde{\tau}'^c \in \Sigma_{E,\infty}$ denote the two liftings of τ' . A similar formula for $c_d(\mathcal{T}_{\tau^+})$ holds with τ replaced by τ^+ . By (1)(b), we have

$$\eta'^*_\sharp c_d(\mathcal{T}_\tau) = c_d(\eta'^*_\sharp \mathcal{T}_\tau) = p^{v(\eta)} c_d(\mathcal{T}_{\tau^+}).$$

Let

$$\text{Tr}_? : H_{\text{et}}^{2d}(\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(?)})_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)(d) \xrightarrow{\sim} \mathbb{Q}_\ell \quad \text{for } ? = \tau, \tau^+$$

be the ℓ -adic trace map. Then we have

$$\text{deg}(\eta'_\sharp) \text{Tr}_\tau(c_d(\mathcal{T}_\tau)) = \text{Tr}_\tau(\eta'^*_\sharp c_d(\mathcal{T}_\tau)) = p^{v(\eta)} \text{Tr}_{\tau^+}(c_d(\mathcal{T}_{\tau^+})).$$

It is well known that $\text{Tr}_\tau(c_d(\mathcal{T}_\tau)) = \chi(\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau)})_{\overline{\mathbb{F}}_p})$ (see [SGA5, Exposé VII, Corollaire 4.9]), where

$$\chi(\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau)})_{\overline{\mathbb{F}}_p}) := \sum_{i=0}^{2d} (-1)^i \dim H_{\text{et}}^i(\mathbf{Sh}_{K'}(G'_{\tilde{\mathcal{S}}(\tau)})_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)$$

is the (ℓ -adic) Euler–Poincaré characteristic. Hence, one obtains

$$\deg(\eta'_\sharp) \cdot \chi(\mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{\overline{\mathbb{F}}_p}) = p^{v(\eta)} \cdot \chi(\mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau^+)})_{\overline{\mathbb{F}}_p}).$$

Since η'_\sharp is purely inseparable, we have $\chi(\mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{\overline{\mathbb{F}}_p}) = \chi(\mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau^+)})_{\overline{\mathbb{F}}_p})$. By Lemma 7.15, we have $\chi(\mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})) \neq 0$ and hence $\deg(\eta'_\sharp) = p^{v(\eta)}$.

(2) Note that \mathfrak{p} splits in E and fix a prime \mathfrak{q} of E dividing \mathfrak{p} . As before, we denote by $\tilde{\tau}$ and $\tilde{\tau}^-$ the liftings of τ and τ^- in $\Sigma_{E,\infty/\mathfrak{q}}$, respectively.

We define first a link morphism $(\eta'_\sharp, \eta^\sharp) : \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau^-)})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0}$ of indentation degree $p^{2(g-n_\tau)}$ as follows. Let $y = (B', \iota_{B'}, \lambda_{B'}, \beta_{K'})$ be an $\overline{\mathbb{F}}_p$ -point of $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau^-)})_{k_0}$, so that $\dim \omega_{B'^\vee/k_0, \tilde{\tau}}^\circ = 0$ and $\dim \omega_{B'^\vee/k_0, \tilde{\tau}^c}^\circ = 2$. We define first a lattice $M_{\tilde{\tau}'}$ of $\tilde{\mathcal{D}}_{B', \tilde{\tau}'}^\circ[1/p]$ for each $\tilde{\tau}' \in \Sigma_{E,\infty}$ as follows: we put

$$M_{\sigma^i \tilde{\tau}}^\circ = \begin{cases} \frac{1}{p} \tilde{\mathcal{D}}_{B', \sigma^i \tilde{\tau}}^\circ & \text{for } i = 1, \dots, n_{\tau^-} = g - n_\tau, \\ \tilde{\mathcal{D}}_{B', \sigma^i \tilde{\tau}}^\circ & \text{for } i = n_{\tau^-} + 1, \dots, g \end{cases}$$

and $M_{\sigma^i \tilde{\tau}^c}^\circ = M_{\sigma^i \tilde{\tau}}^{\circ, \vee}$. One checks easily that $M = \bigoplus_{\tilde{\tau}' \in \Sigma_{E,\infty}} M_{\tilde{\tau}'}^{\circ, \oplus 2}$ is stable under the action of Frobenius and Verschiebung homomorphisms and hence a Dieudonné submodule of $\tilde{\mathcal{D}}_{B'}[1/p]$. As in the proof of Proposition 5.17, this gives rise to an abelian variety B'' equipped with an action by \mathcal{O}_D and an \mathcal{O}_D -equivariant quasi-isogeny $\phi : B' \rightarrow B''$ such that the induced morphism on the Dieudonné module $\phi^{-1} : \tilde{\mathcal{D}}_{B''} \rightarrow \tilde{\mathcal{D}}_{B'}[1/p]$ is identified with the inclusion $M \subset \tilde{\mathcal{D}}_{B'}[1/p]$. Since the lattice $M \subseteq \tilde{\mathcal{D}}_{B'}[1/p]$ is self-dual by construction, the polarization $\lambda_{B'}$ induces a prime-to- p polarization $\lambda_{B''}$ on B'' such that $\lambda_{B'} = \phi^\vee \circ \lambda_{B''} \circ \phi$. Finally, the K'^p -level structure $\beta_{K'}$ on B' induces naturally a K'^p -level structure $\beta''_{K'}$ on B'' . Moreover, an easy computation shows that $\dim \omega_{B''^\vee/k_0, \tilde{\tau}}^\circ = 2$ and $\dim \omega_{B''^\vee/k_0, \tilde{\tau}^c}^\circ = 0$, and $\dim \omega_{B''^\vee/k_0, \tilde{\tau}'}^\circ = \dim \omega_{B'^\vee/k_0, \tilde{\tau}'}$ at other $\tilde{\tau}'$. Thus, $(B'', \iota_{B''}, \lambda_{B''}, \beta''_{K'})$ is a point of $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0}$. Let

$$\eta_\sharp : \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau^-)})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\mathfrak{S}(\tau)})_{k_0}$$

be the map sending $y = (B', \iota_{B'}, \lambda_{B'}, \beta_{K'}) \mapsto (B'', \iota_{B''}, \lambda_{B''}, \beta''_{K'})$ and let

$$\eta^\sharp : \mathbf{A}'_{\mathfrak{S}(\tau^-), k_0} \rightarrow \eta_\sharp^* \mathbf{A}'_{\mathfrak{S}(\tau), k_0}$$

be the p -quasi-isogeny whose base change to each y is $\phi : B' \rightarrow B''$ constructed above. Then it is clear by construction that $(\eta_\sharp, \eta^\sharp)$ is the link morphism of indentation degree $2(g - n_\tau)$ associated to the trivial link from $\mathfrak{S}(\tau^-)$ to $\mathfrak{S}(\tau)$.

Denote by $\pi_{\{\tau, \tau^-\}} : \mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, \{\tau, \tau^-\}} \xrightarrow{\sim} \mathbf{Sh}_{K'^p \text{Iw}'_p}(G'_{\mathfrak{S}(\tau)})_{k_0}$ the isomorphism given by Theorem 5.8. Let $x = (A, \iota_A, \lambda_A, \alpha_{K'})$ be an $\overline{\mathbb{F}}_p$ -point of $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, \{\tau, \tau^-\}}$. Then its image $(B, \iota_B, \lambda_B, \beta_{K\mathfrak{p}}, \beta_{\mathfrak{p}})$ under $\pi_{\{\tau, \tau^-\}}$ is characterized as follows.

- (a) There exists an \mathcal{O}_D -equivariant p -quasi-isogeny $\phi : B \rightarrow A$ such that ϕ induces an isomorphism $\phi_* : \tilde{\mathcal{D}}_{B, \tilde{\tau}'}^\circ \xrightarrow{\sim} \tilde{\mathcal{D}}_{A, \tilde{\tau}'}^\circ$ for $\tilde{\tau}'$ different from $\sigma^i \tilde{\tau}^-$ with $i = 1, \dots, n_\tau$ and their complex conjugates. In the exceptional cases, we have

$$\begin{aligned} \phi_* (\tilde{\mathcal{D}}_{B, \sigma^i \tilde{\tau}^-}^\circ) &= \text{Im}(F_{A, \text{es}}^i : \tilde{\mathcal{D}}_{A, \tilde{\tau}^-}^\circ \rightarrow \tilde{\mathcal{D}}_{A, \sigma^i \tilde{\tau}^-}^\circ) \\ \phi_* (\tilde{\mathcal{D}}_{B, \sigma^i \tilde{\tau}^-, c}^\circ) &= \frac{1}{p} \text{Im}(F_{A, \text{es}}^i : \tilde{\mathcal{D}}_{A, \tilde{\tau}^-, c}^\circ \rightarrow \tilde{\mathcal{D}}_{A, \sigma^i \tilde{\tau}^-, c}^\circ). \end{aligned}$$

- (b) We have $\lambda_B = \phi^\vee \circ \lambda_A \circ \phi$ and $\beta_{K'^p} = \phi \circ \alpha_{K'^p}$.
- (c) Let $M_\tau^\circ \subseteq \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ/p\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ be the one-dimensional subspace given by the image of $p\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ via ϕ_*^{-1} . Then M_τ° is stable under $F_{B,\text{es}}^g$ and $M := \bigoplus_{i=0}^{g-1} F_{B,\text{es}}^i(M_\tau^\circ)^{\oplus 2}$ is a Dieudonné submodule of $\mathcal{D}_{B[\mathfrak{q}]}\circ = \bigoplus_{i=0}^{g-1} \tilde{\mathcal{D}}_{B,\sigma^i\tilde{\tau}}/p\tilde{\mathcal{D}}_{B,\sigma^i\tilde{\tau}}$. Let $H_{\mathfrak{q}}$ be the subgroup scheme of $B[\mathfrak{q}]$ associated to M . Then the Iwahoric level structure of B at p is given by $\beta_{\mathfrak{p}} = H_{\mathfrak{q}} \oplus H_{\bar{\mathfrak{q}}}$, where $H_{\bar{\mathfrak{q}}} \subseteq B[\bar{\mathfrak{q}}]$ is the orthogonal complement of $H_{\mathfrak{q}}$ under the natural duality between $B[\mathfrak{q}]$ and $B[\bar{\mathfrak{q}}]$.

It is clear that the image of x under π_τ is $(B, \iota_B, \lambda_B, \beta_{K'^p})$ by forgetting the Iwahoric level structure at p of $\pi_{\{\tau, \tau^-\}}(x)$. This shows that, via the isomorphism $\pi_{\{\tau, \tau^-\}}$, the map $\pi_\tau|_{\mathbf{Sh}_{K'}(G'_S)_{k_0, \{\tau, \tau^-\}}}$ coincides with the projection pr_1 in (7.10.1).

To finish the proof of (2), it remains to show that the composition

$$\eta_\# \circ \pi_{\tau^-} \circ \pi_{\{\tau, \tau^-\}}^{-1} : \mathbf{Sh}_{K'^p I_w^p}(G'_{\tilde{S}(\tau)})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\tilde{S}(\tau)})_{k_0}$$

is the second projection pr_2 in (7.10.1). Let $x = (A, \iota_A, \lambda_A, \alpha_{K'})$ be a point of $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0, \{\tau, \tau^-\}}$ with image $\pi_{\{\tau, \tau^-\}}(x) = (B, \iota_B, \lambda_B, \beta_{K'^p}, \beta_{\mathfrak{p}})$, together with the p -quasi-isogeny $\phi: B \rightarrow A$ as described above. The image of $(A, \iota_A, \lambda_A, \alpha_{K'})$ under π_{τ^-} is given by $(B', \iota_{B'}, \lambda_{B'}, \beta_{K'}) \in \mathbf{Sh}_{K'}(G'_{\tilde{S}(\tau^-)})_{k_0}$, which admits a quasi-isogeny $\psi: B' \rightarrow A$ compatible with all structures such that $\psi_*: \tilde{\mathcal{D}}_{B',\tilde{\tau}'}^\circ \cong \tilde{\mathcal{D}}_{A,\tilde{\tau}'}^\circ$ for all $\tilde{\tau}'$ except for those $\tilde{\tau}'$ lifting $\sigma^i\tau$ for any $i = 1, \dots, g - n_\tau = n_{\tau^-}$. In the exceptional cases, we have

$$\psi_*(\tilde{\mathcal{D}}_{B',\sigma^i\tilde{\tau}'}^\circ) = \text{Im}(F_{A,\text{es}}^i: \tilde{\mathcal{D}}_{A,\tilde{\tau}'}^\circ \rightarrow \tilde{\mathcal{D}}_{A,\sigma^i\tilde{\tau}'}^\circ) \quad \text{and} \quad \psi_*(\tilde{\mathcal{D}}_{B',\sigma^i\tilde{\tau}'}^\circ) = \frac{1}{p} \text{Im}(F_{A,\text{es}}^i: \tilde{\mathcal{D}}_{A,\tilde{\tau}'}^\circ \rightarrow \tilde{\mathcal{D}}_{A,\sigma^i\tilde{\tau}'}^\circ).$$

Consider the composed isogeny $\phi^{-1} \circ \psi: B' \rightarrow B$. Let $\tilde{M}_\tau^\circ \subseteq \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ be the inverse image of the one-dimensional subspace $M_\tau^\circ \subseteq \tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ/p\tilde{\mathcal{D}}_{B,\tilde{\tau}}^\circ$ given by the image of $p\tilde{\mathcal{D}}_{A,\tilde{\tau}}^\circ$ as in (c) above. Then we have

$$(\phi^{-1} \circ \psi)_*(\tilde{\mathcal{D}}_{B',\sigma^i\tilde{\tau}'}^\circ) = \begin{cases} F_{B,\text{es}}^i(\tilde{M}_\tau^\circ) & \text{for } i = 1, \dots, n_{\tau^-} = g - n_\tau, \\ \frac{1}{p} F_{B,\text{es}}^i(\tilde{M}_\tau^\circ) & \text{for } i = n_{\tau^-} + 1, \dots, g; \end{cases}$$

and $(\phi^{-1} \circ \psi)_*(\tilde{\mathcal{D}}_{B',\sigma^i\tilde{\tau}'}^\circ)$ is the orthogonal complement of $(\phi^{-1} \circ \psi)_*(\tilde{\mathcal{D}}_{B',\sigma^i\tilde{\tau}'}^\circ)$.

Let $(B'', \iota_{B''}, \lambda_{B''}, \beta''_{K'})$ be the image of $(B', \iota_{B'}, \lambda_{B'}, \beta_{K'}) \in \mathbf{Sh}_{K'}(G'_{\tilde{S}(\tau^-)})_{k_0}$ under $\eta_\#$. Then the composed p -quasi-isogeny $B \xrightarrow{\psi^{-1} \circ \phi} B' \xrightarrow{\eta_\#} B''$ identifies $\tilde{\mathcal{D}}_{B'',\sigma^i\tilde{\tau}'}^\circ$ with the lattice

$$\frac{1}{p} F_{B,\text{es}}^i(\tilde{M}_\tau^\circ) \subseteq \tilde{\mathcal{D}}_{B,\sigma^i\tilde{\tau}'}^\circ[1/p] \quad \text{for all } i = 1, \dots, g.$$

Thus, one sees immediately that the map $(B, \iota_B, \lambda_B, \beta_{K'^p}, \beta_{\mathfrak{p}}) \mapsto (B'', \iota_{B''}, \lambda_{B''}, \beta''_{K'})$ is nothing but the second projection in (7.10.1). This finishes the proof of (2). \square

Our last proposition explains the compatibility of the description of the GO-divisors as in Theorem 5.8 with respect to the link morphism, especially to the link morphism appearing in Theorem 7.16(1).

PROPOSITION 7.17. *Assume that $\#\Sigma_\infty - \#\mathbf{S}_\infty \geq 2$. Let $\tau_0 \in \Sigma_\infty - \mathbf{S}_\infty$ and $\eta: \mathbf{S} \rightarrow \mathbf{S}'$ be a link such that all curves are straight lines except for (possibly) one curve turning to the right, linking $\tau_0 \in \Sigma_\infty - \mathbf{S}_\infty$ with $\tau'_0 = \eta(\tau_0) = \sigma^{m(\tau_0)}\tau_0$ for some integer $m(\tau_0) \geq 0$. Assume that the link*

morphism $(\eta'_\sharp, \eta'^\sharp) : \mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0}$ of (some) indentation degree $n \in \mathbb{Z}$ associated to η exists. The setup automatically implies that $\tau_0^+ = \tau_0'^+$ and $\tau_0^- = \tau_0'^-$. Let $\tau \in \Sigma_\infty - \mathbf{S}_\infty$. Then the following statements hold.

- (1) The link morphism η'_\sharp sends the GO-divisor $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0, \tau}$ into the GO-divisor $\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0, \eta(\tau)}$.
- (2) Let $\eta_\tau : \mathbf{S}(\tau) \rightarrow \mathbf{S}'(\eta(\tau))$ denote the link given by removing from η the two curves attached to τ and τ^- . Then there exists a link morphism $(\eta'_{\tau, \sharp}, \eta'^\sharp_\tau)$ (of some indentation degree m) from $\mathbf{Sh}_{K'}(G'_{\tilde{S}(\tau)})_{k_0}$ to $\mathbf{Sh}_{K'}(G'_{\tilde{S}'(\eta(\tau))})_{k_0}$, associated to the link η_τ , such that we have the following commutative diagram of Shimura varieties:

$$\begin{array}{ccc}
 \mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0, \tau} & \xrightarrow{\pi_\tau} & \mathbf{Sh}_{K'}(G'_{\tilde{S}(\tau)})_{k_0} \\
 \downarrow \eta'_\sharp & & \downarrow \eta'_{\tau, \sharp} \\
 \mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0, \eta(\tau)} & \xrightarrow{\pi_{\eta(\tau)}} & \mathbf{Sh}_{K'}(G'_{\tilde{S}'(\eta(\tau))})_{k_0}
 \end{array} \tag{7.17.1}$$

and a similar commutative diagram of quasi-isogenies of universal abelian varieties. Moreover, the indentation degree of the link morphism $\eta'_{\tau, \sharp}$ is given by

$$m = n + n_\tau - n_{\eta(\tau)}(\mathbf{S}') = \begin{cases} 0 & \text{if } \mathfrak{p} \text{ is inert in } E/F, \\ n & \text{if } \mathfrak{p} \text{ splits in } E/F \text{ and } \tau \neq \tau_0, \tau_0^+, \\ n - m(\tau_0) & \text{if } \mathfrak{p} \text{ splits in } E/F \text{ and } \tau = \tau_0, \\ n + m(\tau_0) & \text{if } \mathfrak{p} \text{ splits in } E/F \text{ and } \tau = \tau_0^+. \end{cases}$$

- (3) Suppose moreover that the link η and the link morphism $(\eta'_\sharp, \eta'^\sharp)$ are those appearing in Theorem 7.16(1) (so our \tilde{S} being $\tilde{S}(\tau^+)$, \mathbf{S}' being $\tilde{S}(\tau)$, and τ_0 being τ^- therein, respectively). If $\mathcal{O}_\tau(1)$ (respectively $\mathcal{O}_{\eta(\tau)}(1)$) denotes the tautological quotient line bundle on $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0, \tau}$ (respectively on $\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0, \eta(\tau)}$) for the \mathbb{P}^1 -fibration π_τ (respectively $\pi_{\eta(\tau)}$), then we have a canonical isomorphism

$$\eta'^*_\sharp(\mathcal{O}_{\eta(\tau)}(1)) \cong \begin{cases} \mathcal{O}_\tau(1) & \text{if } \tau \neq \tau_0^+, \\ \mathcal{O}_\tau(p^{m(\tau_0)}) & \text{if } \tau = \tau_0^+. \end{cases} \tag{7.17.2}$$

Moreover, the induced link morphism $\eta'_{\tau, \sharp}$ is finite flat of degree $p^{v(\eta_\tau)}$, i.e. it is an isomorphism if $\tau \in \{\tau_0^+, \tau_0\}$, and it is finite flat of degree $p^{m(\tau_0)}$ if $\tau \notin \{\tau_0, \tau_0^+\}$.

The analogous results hold for link morphisms for the $\mathbf{Sh}_{K''}(G''_{\tilde{S}})_{k_0}$.

Proof. (1) Since $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0, \tau}$ is reduced, it suffices to prove that η'_\sharp sends every $\overline{\mathbb{F}}_p$ -point of $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0, \tau}$ to $\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0, \eta(\tau)}$. Let $x = (A, \iota, \lambda, \alpha_{K'^p})$ be an $\overline{\mathbb{F}}_p$ -point of $\mathbf{Sh}_{K'}(G'_{\tilde{S}})_{k_0, \tau}$ and $\eta'_\sharp(x) = (A', \iota', \lambda', \alpha'_{K'^p})$ be its image. Let $\tilde{\tau} \in \Sigma_{E, \infty}$ be a place above τ and put $\tilde{\tau}^- = \sigma^{-n_\tau} \tilde{\tau}$ and $\tilde{\tau}^+ = \sigma^{n_{\tau^+}} \tilde{\tau}$. By Lemma 4.5, the condition $h_\tau(A) = 0$ is equivalent to $F_{\text{es}, A}^{n_\tau}(\tilde{\mathcal{D}}_{A, \tilde{\tau}^-}^\circ) = \tilde{\omega}_{A^\vee, \tilde{\tau}^-}^\circ$, where $\tilde{\omega}_{A^\vee, \tilde{\tau}^-}^\circ \subseteq \tilde{\mathcal{D}}_{A, \tilde{\tau}^-}^\circ$ denotes the inverse image of $\omega_{A^\vee, \tilde{\tau}^-}^\circ \subseteq \mathcal{D}_{A, \tilde{\tau}^-}^\circ$. The latter condition is in turn equivalent to $F_{\text{es}, A}^{n_{\tau^+}} \circ F_{\text{es}, A}^{n_\tau}(\tilde{\mathcal{D}}_{A, \tilde{\tau}^-}^\circ) = p\tilde{\mathcal{D}}_{A, \tilde{\tau}^+}^\circ$. We set $\eta(\tilde{\tau}^-) = \sigma^{m(\tau^-)} \tilde{\tau}^-$, where $m(\tau^-)$ is the displacement of the curve in η connecting τ^- and $\eta(\tau^-)$ (which equals 0 except when $\tau^- = \tau_0$); similarly, we put $\eta(\tilde{\tau}^+) = \sigma^{m(\tau^+)} \tilde{\tau}^+$. Since $\eta'_\sharp : \tilde{\mathcal{D}}_A[1/p] \rightarrow \tilde{\mathcal{D}}_{A'}[1/p]$ commutes with Frobenius

and Verschiebung homomorphisms, one sees easily from condition (3) in Definition 7.5 that $F_{\text{es},A'}^{n_{\eta(\tau^+)}} \circ F_{\text{es},A'}^{n_{\eta(\tau)}}(\tilde{\mathcal{D}}_{A',\eta(\tilde{\tau}^-)}^\circ) = p^u \tilde{\mathcal{D}}_{A',\eta(\tilde{\tau}^+)}^\circ$ for some integer $u \in \mathbb{Z}$. Here $n_{\eta(\tau)}(S')$ is the integer defined in Notation 4.3 associated to τ for the set S' . But $F_{\text{es},A'}^{n_{\eta(\tau^+)}} \circ F_{\text{es},A'}^{n_{\eta(\tau)}}(\tilde{\mathcal{D}}_{A',\eta(\tilde{\tau}^-)}^\circ)$ is a $W(\overline{\mathbb{F}}_p)$ -sublattice of $\tilde{\mathcal{D}}_{A,\eta(\tilde{\tau}^+)}^\circ$ with quotient of length 2 over $W(\overline{\mathbb{F}}_p)$. Hence, the integer u has to be 1. By the same reasoning using Lemma 4.5, this is equivalent to saying that $h_{\eta(\tau)}(A') = 0$ or equivalently $\eta'_{\sharp}(x) \in \mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0,\eta(\tau)}$.

(2) Assume first that $\#\Sigma_\infty - \#\mathbf{S}_\infty > 2$. By Proposition 7.12(2), $\pi_\tau|_{\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0,\{\tau,\tau^-\}}}$ is an isomorphism. We define

$$\eta'_{\tau,\sharp} := \pi_{\eta(\tau)} \circ \eta'_{\sharp} \circ (\pi_\tau|_{\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0,\{\tau,\tau^-\}}})^{-1}$$

and η'^{\sharp} as the pullback via $(\pi_\tau|_{\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0,\{\tau^-, \tau\}}})^{-1}$ of the quasi-isogeny

$$\pi_\tau^* \mathbf{A}'_{\tilde{S}'(\tau),k_0} \xrightarrow{\phi_\tau^{-1}} (\mathbf{A}'_{\tilde{S}',k_0}|_{\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0,\tau}}) \xrightarrow{\eta'^{\sharp}} \eta'^{*}_{\sharp}(\mathbf{A}'_{\tilde{S}',k_0}|_{\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0,\eta(\tau)}}) \xrightarrow{\eta'^{*}_{\sharp}(\phi_{\eta(\tau)})} \eta'^{*}_{\sharp} \pi_{\eta(\tau)}^*(\mathbf{A}'_{\tilde{S}'(\eta(\tau)),k_0})$$

of abelian schemes on $\mathbf{Sh}_{K'}(G'_{\tilde{S}'})_{k_0,\tau}$. Here the first and the third quasi-isogenies are given by Theorem 5.8(2). It is clear that the diagram (7.17.1) is commutative. It remains to show that $(\eta'_{\tau,\sharp}, \eta'^{\sharp}_{\tau})$ defines a link morphism. Conditions (1) and (2) of Definition 7.5 being clear, condition (3) can be verified by a tedious but straightforward computation. To see condition (4) on the indentation degree, we need only to discuss the case when \mathfrak{p} splits in E/F . In this case, $\phi_\tau^{-1}[\mathfrak{q}^\infty]$ has degree p^{2n_τ} , $\eta'^{\sharp}[\mathfrak{q}^\infty]$ has degree p^{2n} , and $\phi_{\eta(\tau)}[\mathfrak{q}^\infty]$ has degree $p^{-2n_{\eta(\tau)}(S')}$. So, the total degree of the quasi-isogeny of η'^{\sharp}_{τ} is $p^{2m} = p^{2n+2n_\tau-2n_{\eta(\tau)}(S')}$. A case-by-case discussion proves condition (4) of Definition 7.5 on indentation degrees.

Assume now that $\#\Sigma_\infty - \#\mathbf{S}_\infty = 2$, so that $\Sigma_\infty - \mathbf{S}_\infty = \{\tau_0, \tau_0^+ = \tau_0^-\}$. This implies that \mathfrak{p} splits as $\mathfrak{q}\bar{\mathfrak{q}}$ in E . Since the Shimura variety $\mathbf{Sh}_{K'}(G'_{\tilde{S}'(\tau)})_{k_0}$ is zero dimensional, we just need to define the desired link morphism $(\eta'_{\tau,\sharp}, \eta'^{\sharp}_{\tau})$ on $\overline{\mathbb{F}}_p$ -points. For each $\tilde{\tau}' \in \Sigma_{E,\infty}$ with restriction $\tau' \in \{\tau_0, \tau_0^-\}$, let $t_{\tilde{\tau}'} \in \mathbb{Z}$ denote the integer as in Definition 7.5(3) attached to $\tilde{\tau}'$ for the link morphism $(\eta'_{\sharp}, \eta'^{\sharp})$. Let $y = (B, \iota_B, \lambda_B, \beta_{K'^p})$ be an $\overline{\mathbb{F}}_p$ -point of $\mathbf{Sh}_{K'}(G'_{\tilde{S}'(\tau)})_{k_0}$. We now distinguish two cases.

- (a) Consider first the case $\tau = \tau_0$. Let $\tilde{\tau}_0^-$ be the lift of τ_0 in $\Sigma_{E,\infty/\mathfrak{q}}$. We define $M_{\tilde{\tau}_0^-}^\circ = p^{-t_{\tilde{\tau}_0^-}} \tilde{\mathcal{D}}_{B,\tilde{\tau}_0^-}^\circ$ and $M_{\sigma^i \tilde{\tau}_0^-}^\circ = p^{-\delta_i} F^i(M_{\tilde{\tau}_0^-}^\circ)$ for each integer i with $1 \leq i \leq g-1$, where δ_i denotes the number of integers j with $1 \leq j \leq i$ such that $\sigma^j \tilde{\tau}_0^- \in \tilde{S}'(\eta(\tau))_\infty$. Put $M_{\mathfrak{q}}^\circ = \bigoplus_{0 \leq i \leq g-1} M_{\sigma^i \tilde{\tau}_0^-}^\circ$ and let $M_{\bar{\mathfrak{q}}}^\circ \subseteq \tilde{\mathcal{D}}_{B,\bar{\mathfrak{q}}}^\circ[1/p]$ be the dual lattice of $M_{\mathfrak{q}}^\circ$ with respect to the pairing induced by λ_B . Then $M^\circ := M_{\mathfrak{q}}^\circ \oplus M_{\bar{\mathfrak{q}}}^\circ$ is a Dieudonné submodule of $\tilde{\mathcal{D}}_B^\circ[1/p]$. By the same argument as in the proof of Proposition 5.17, there exists a unique abelian variety B' equipped with an \mathcal{O}_D -action $\iota_{B'}$ together with a p -quasi-isogeny $\phi : B \rightarrow B'$ such that the induced map $\phi_*^{-1} : \tilde{\mathcal{D}}_{B'}^\circ \rightarrow \tilde{\mathcal{D}}_B^\circ[1/p]$ is identified with the natural inclusion $M^\circ \hookrightarrow \tilde{\mathcal{D}}_B^\circ[1/p]$. As usual, since M is a self-dual lattice, λ_B induces a prime-to- p polarization $\lambda_{B'}$ such that $\phi^\vee \circ \lambda_{B'} \circ \phi = \lambda_B$. We equip B' with the K'^p -level structure $\beta'_{K'^p} = \phi \circ \beta_{K'^p}$. By the construction, one also sees easily that B' satisfies the necessary signature condition, so that $y' = (B', \iota_{B'}, \lambda_{B'}, \beta'_{K'^p})$ is a point of $\mathbf{Sh}_{K'}(G'_{\tilde{S}'(\eta(\tau))})_{k_0}$. We define

$$\eta'_{\tau,\sharp} : \mathbf{Sh}_{K'}(G'_{\tilde{S}'(\tau)})_{k_0} \rightarrow \mathbf{Sh}_{K'}(G'_{\tilde{S}'(\eta(\tau))})_{k_0} \quad \text{and} \quad \eta'^{\sharp}_{\tau} : \mathbf{A}'_{\tilde{S}'(\tau),k_0} \rightarrow \mathbf{A}'_{\tilde{S}'(\eta(\tau)),k_0}$$

by $\eta'_{\tau, \#}(y) = y'$ and $\eta'_{\tau, y} = \phi$. It is evident that $(\eta'_{\tau, \#}, \eta'_{\tau})$ is a link morphism. It remains to check the commutativity of the diagram (7.17.1). Let $x = (A, \iota_A, \lambda_A, \alpha_{K^p}) \in \mathbf{Sh}_{K'}(G'_{\mathfrak{S}})_{k_0, \tau}$ be a point above y and $x' = (A', \iota_{A'}, \lambda_{A'}, \alpha'_{K^p}) \in \mathbf{Sh}_{K'}(G'_{\mathfrak{S}'})_{k_0, \eta(\tau)}$ be the image of x under $\eta'_{\#}$. We need to prove that $\pi_{\eta(\tau)}(x') = y'$. Let $y'' = (B'', \iota_{B''}, \lambda_{B''}, \beta''_{K^p}) \in \mathbf{Sh}_{K'}(G'_{\mathfrak{S}'(\eta(\tau))})_{k_0, \eta(\tau)}$ denote temporarily the point $\pi_{\eta(\tau)}(x')$. Denote by $\psi: A \rightarrow B$ and $\psi': A' \rightarrow B''$ the quasi-isogenies given by Theorem 5.8. Then we have $\psi_*(\tilde{D}^\circ_{A, \tilde{\tau}_0^-}) = \tilde{D}^\circ_{B, \tilde{\tau}_0^-}$ and $\psi'_*(\tilde{D}^\circ_{A', \tilde{\tau}_0^-}) = \tilde{D}^\circ_{B'', \tilde{\tau}_0^-}$ by § 5.15, and $\eta'^{\#}_*(\tilde{D}^\circ_{A, \tilde{\tau}_0^-}) = p^{t_{\tilde{\tau}_0^-}} \tilde{D}^\circ_{A', \tilde{\tau}_0^-}$ by Definition 7.5(3) as η is a straight line at τ_0^- . Consider the quasi-isogeny $\psi' \circ \eta'^{\#} \circ \psi^{-1} \circ \phi^{-1}: B' \rightarrow B''$. It induces an isomorphism $\tilde{D}^\circ_{B', \tilde{\tau}_0^-} \xrightarrow{\sim} \tilde{D}^\circ_{B'', \tilde{\tau}_0^-}$. But the other components of the Dieudonné modules are determined by that at $\tilde{\tau}_0^-$, as $\mathfrak{S}'(\eta(\tau))_\infty = \Sigma_\infty$. It follows that $\psi' \circ \eta'^{\#} \circ \psi^{-1} \circ \phi^{-1}: B' \rightarrow B''$ is an isomorphism compatible with all structures, i.e. $y'' = y'$. The computation of the indentation degree is the same as that in the case when $\#\Sigma_\infty - \#\mathfrak{S}_\infty > 2$.

- (b) In the case $\tau = \tau_0^+ = \tau_0^-$, the construction is similar. Let $\tilde{\tau}_0$ be the lift of τ_0 in $\Sigma_{E, \infty/q}$ and $\eta(\tilde{\tau}_0) = \sigma^{m(\tau_0)}\tilde{\tau}_0$. Put $M^\circ_{\eta(\tilde{\tau}_0)} := p^{-t_{\tilde{\tau}_0}} F^{m(\tau_0)}(\tilde{D}^\circ_{B, \eta(\tilde{\tau}_0)})$ and $M_{\sigma^i \eta(\tilde{\tau}_0)} := p^{-\delta_i} F^i(M^\circ_{\eta(\tilde{\tau}_0)})$ for each integer i with $1 \leq i \leq g-1$, where δ_i is the number of integers j with $1 \leq j \leq i$ such that $\sigma^j \eta(\tilde{\tau}_0) \in \tilde{\mathfrak{S}}'(\eta(\tau))_\infty$. Let $M^\circ_{\mathfrak{q}} = \bigoplus_{0 \leq i \leq g-1} M_{\sigma^i \eta(\tilde{\tau}_0)}$ and $M_{\mathfrak{q}} \subseteq \tilde{D}^\circ_{B, \mathfrak{q}}[1/p]$ be the dual lattice. As in case (a) above, such a lattice $M^\circ := M^\circ_{\mathfrak{q}} \oplus M_{\mathfrak{q}}$ gives rise to an $\overline{\mathbb{F}}_p$ -point $y' = (B', \iota_{B'}, \lambda_{B'}, \beta'_{K^p})$ together with a p -isogeny $\phi: B \rightarrow B'$ compatible with all structures. We define the desired link morphism $(\eta'_{\tau, \#}, \eta'_{\tau})$ such that $\eta'_{\tau, \#}(y) = y'$ and $\eta'_{\tau, y} = \phi$. The commutativity of (7.17.1) is proved by the same arguments as in (a). We leave the details to the reader.

(3) We note that, if $\tilde{\tau}^-$ is the lifting of τ^- not contained in $\tilde{\mathfrak{S}}(\tau)_\infty$, then we have a canonical isomorphism $\mathcal{O}_\tau(1) \cong \text{Lie}(\mathbf{A}'_{\mathfrak{S}, k_0})^\circ_{\tilde{\tau}^-}$ by the construction of π_τ . Similarly, one has $\mathcal{O}_{\eta(\tau)}(1) \cong \text{Lie}(\mathbf{A}'_{\mathfrak{S}'(\eta(\tau)), k_0})^\circ_{\eta(\tilde{\tau}^-)}$. Now the isomorphism (7.17.2) follows immediately from Theorem 7.16(1)(b). We prove now the second part of (3). By the construction of η'_τ , it follows from Theorem 7.16(1)(b) that $\eta'^{\#}_\tau: \mathbf{A}'_{\mathfrak{S}(\tau), k_0} \rightarrow \eta'^{\#}_{\tau, \#}(\mathbf{A}'_{\mathfrak{S}'(\eta(\tau)), k_0})$ induces, for any $\tilde{\tau}' \in \Sigma_{E, \infty}$ lifting an element $\Sigma_\infty - \mathfrak{S}(\eta(\tau))$, an isomorphism

$$\eta'^{\#}_{\tau, \#}(\text{Lie}(\mathbf{A}'_{\mathfrak{S}'(\eta(\tau)), k_0})^\circ_{\tilde{\tau}'}) \cong \begin{cases} \text{Lie}(\mathbf{A}'_{\mathfrak{S}(\tau), k_0})^\circ_{\sigma^{-m(\tau_0)}\tilde{\tau}'} & \text{if } \tilde{\tau}' \text{ lifts } \eta(\tau_0), \\ \text{Lie}(\mathbf{A}'_{\mathfrak{S}(\tau), k_0})^\circ_{\tilde{\tau}'} & \text{otherwise.} \end{cases}$$

If $\tau \in \{\tau_0, \tau_0^+\}$ or equivalently $\tau_0 \in \mathfrak{S}(\tau)$, then the first case above never happens. Therefore, by Proposition 4.7, we see that $\eta'_{\tau, \#}$ induces an isomorphism of tangent spaces between $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}'})_{k_0}$ and $\mathbf{Sh}_{K'}(G'_{\mathfrak{S}'(\eta(\tau))})_{k_0}$. Since $\eta'_{\tau, \#}$ is bijective on the closed points by the definition of link morphism, $\eta'_{\tau, \#}$ is actually an isomorphism. If $\tau \notin \{\tau_0, \tau_0^+\}$ or equivalently $\tau_0 \notin \mathfrak{S}(\tau)$, we conclude by the same arguments as in the proof of Theorem 7.16(1)(d). \square

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