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Rigidity in topological dynamics

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Abstract. By analogy with the ergodic theoretical notion, we introduce notions of rigidity for a minimal flow (X, T) according to the various ways a sequence T^{n_i} can tend to the identity transformation. The main results obtained are:

(i) On a rigid flow there exists a T-invariant, symmetric, closed relation $\tilde{N}$ such that (X, T) is uniformly rigid iff $\tilde{N} = \Delta$, the diagonal relation.
(ii) For syndetically distal (hence distal) flows rigidity is equivalent to uniform rigidity.
(iii) We construct a family of rigid flows which includes Körner's example, in which $\tilde{N}$ exhibits various kinds of behaviour, e.g. $\tilde{N}$ need not be an equivalence relation.
(iv) The structure of flows in the above mentioned family is investigated. It is shown that these flows are almost automorphic.

1. *Introduction*

The notion of (measure theoretical) rigidity was introduced by H. Furstenberg and B. Weiss in [**FW**]. They call the finite measure preserving system (X, B, μ, T), *rigid* if for some sequence $n_i \to \infty$ and every $f \in L_2(\mu)$, $T^{n_i} f \to f$ in $L_2(\mu)$. B. Weiss has shown that if T is rigid then there exists a subsequence n_i' of n_i and a subalgebra $A \subset L_\infty(\mu)$ which is dense in $L_2(\mu)$ and such that $\|T^{n_i'} f - f\|_\infty \to 0$ for every $f \in A$, (private communication).

In this paper we are concerned with analogous notions of rigidity in the setting of topological dynamics and their interrelation. To be specific let X be a metric compact space and $T: X \to X$ a self homeomorphism. We call the pair (X, T) a *flow*, and say that

(i) (X, T) is *weakly rigid* if for every $\varepsilon > 0$ and points $x_1 \cdots x_n \in X$, there exists $k \in \mathbb{Z} \backslash \{0\}$ such that $d(T^k x_i, x_i) < \varepsilon$, $(i = 1, 2, \ldots, n)$.
(ii) (X, T) is *rigid* (with respect to a sequence $n_k \nearrow \infty$) if $T^{n_k} x \to x \; \forall x \in X$.
(iii) (X, T) is *uniformly rigid* (w.r.t. n_k) if $\lim T^{n_k} = \text{Identity}$ uniformly on X.

Clearly (iii)$\Rightarrow$(ii)$\Rightarrow$(i). It is easy to construct an example of a rigid flow which is not uniformly rigid. (Take $X = \{r e^{i\theta}: 0 \le \theta \le 2\pi,\ r = 1 - 2^{-n},\ n = 1, 2, 3, \ldots$ or $r = 1\}$ and $Tz = z \exp(2\pi i \cdot 2^{-n})$ when $|z| = 1 - 2^{-n}$ and $Tz = z$ if $|z| = 1$; $n_k = 2^k$). The question whether (ii)$\Rightarrow$(iii) becomes more difficult if we require (X, T) to be minimal. The negative answer to this question was given by T. W. Körner who produced an example of a minimal flow, rigid with respect to some sequence $\{n_k\}$ and not uniformly rigid with respect to any sequence including $\{n_k\}$, [K].

Let (X, T) be a minimal flow, rigid w.r.t. a sequence $\{n_k\}$. Define

$$\tilde{N}=\left\{(x, x'):\ \begin{array}{c}\text{There exists a subsequence } n'_k \text{ of } n_k\\ \text{and sequences } x_k \to x \text{ and } x'_k \to x'\\ \text{such that } d(T^{n'_k}x_k, T^{n'_k}x'_k)\to 0\end{array}\right\}.$$

$\tilde{N}$ is a closed symmetric T-invariant subset of $X\times X$ and it is easy to see that (X, T) is uniformly rigid with respect to $\{n_k\}$ iff $N=\Delta$ (where $\Delta=\{(x, x): x\in X\}$). The analogous *regionally proximal relation* Q which is defined for an arbitrary minimal flow (X, T) by

$$Q=\left\{(x, x'):\ \begin{array}{c}\text{There exists a sequence } \{m_j\} \text{ and}\\ \text{sequences } x_j \to x \text{ and } x'_j \to x' \text{ with}\\ d(T^{m_j}x_j, T^{m_j}x'_j)\to 0\end{array}\right\}$$

is of fundamental importance in the abstract theory of topological dynamics. Surprisingly Q turns out to be an equivalence relation. Thus for a minimal flow (X, T) to be *equicontinuous* it is necessary and sufficient that $Q=\Delta$ and the quotient flow $(X/Q, T)$ is the largest equicontinuous factor of (X, T). A necessary and sufficient condition for (X, T) to have only trivial equicontinuous factors can be derived, namely that it is (topologically) weakly mixing. In particular when (X, T) is distal it always admits a non-trivial equicontinuous factor. (See [**F**, **V**, **P**, **Ke–R**, **E–Ke**, **B**] for these results). As we shall see minimal distal flows are weakly rigid and there are other analogies between distal and rigid flows. Thus we are naturally led to the following two questions

(1) *Is $\tilde{N}$ always an equivalence relation?*

(2) Does a minimal rigid (weakly rigid) flow always have a non-trivial uniformly rigid factor?

In order to describe our results concerning the first question we need some more definitions. In a minimal flow (X, T) let $P=\{(x, x'): \bar{\mathcal{O}}(x, x')\supset\Delta\}$,

$$L=\{(x, x'): \bar{\mathcal{O}}(x, x') \text{ contains } \Delta \text{ as a unique minimal subset}\}$$

($\bar{\mathcal{O}}$ denotes orbit closure). It is easy to see that P and L are symmetric and T-invariant, $L\subset P$ and L is an equivalence relation on X, [**C**]. P and L are called respectively the *proximal* and *syndetically proximal* relations on (X, T). (X, T) is distal iff $P=\Delta$ and we say that (X, T) is *syndetically distal* if $L=\Delta$. Thus every distal flow is syndetically distal. In § 2 we show that $\tilde{N}\subset L$. Hence a syndetically distal (and in particular a distal) rigid flow is uniformly rigid. (We wish to thank B. Weiss who is a co-author of this result for his permission to include it in our paper.) The next two sections are devoted to a general method of construction of minimal flows with various properties. In particular we retrieve Körner's example of a minimal rigid but not uniformly rigid flow in a simpler and more transparent way and construct an example in which $\tilde{N}$ is not an equivalence relation, answering question (1) above in the negative. We also investigate the structure of these examples, showing that they are almost one to one extensions of an equicontinuous flow – i.e. almost automorphic flows. In particular they admit non-trivial uniformly rigid factors. We have no complete answer to our second question. In the last section we collect some miscellaneous results about rigidity: Distal flows are weakly rigid, there are distal flows which are not rigid; (thus in general (i)$\not\Rightarrow$(ii)). Rigid flows have zero topological entropy and there exist many uniformly rigid weakly mixing minimal flows. Finally mixing flows admit only trivial rigid factors.

2. *The relations N and $\tilde{N}$ on a rigid flow*

Let (X, T) be a rigid flow with respect to the sequence $\{n_k\}$. Put

$$N = \left\{ (x, x') : \begin{array}{c} \text{There exist a sequence } x_k \to x \\ \text{and subsequence } \{n'_k\} \text{ of } \{n_k\} \\ \text{such that } T^{n'_k} x_k \to x' \end{array} \right\}.$$

Clearly N is closed and T-invariant. The following proposition is easy to verify.

Proposition 2.1

(1) $N \subset \tilde{N} \subset N \circ N^{-1}$ (*where* $N^{-1} = \{(x, y) : (y, x) \in N\}$).

(2) (X, T) *is uniformly rigid iff* $\tilde{N} = \Delta$. (We note that it is possible to build an example where $\tilde{N} \neq N \circ N^{-1}$.)

Proposition 2.2. *Let (X, T) be a minimal rigid flow, then*

(1) $\tilde{N} \subset L$; *thus* $(x, x') \in \tilde{N}$ *implies* $\bar{\mathcal{O}}(x, x') \subset P$.

(2) *There exists a dense G_δ subset $X_0 \subset X$ with $X_0 \times X \cap N = \Delta$ hence $X_0 \times X_0 \cap \tilde{N} = \Delta$. In particular N and $\tilde{N}$ are meagre subsets of $X \times X$.*

Proof. (1) Since L is an equivalence relation it suffices to show that $N \subset L$, for then by Proposition 2.1 $\tilde{N} \subset N \circ N^{-1} \subset L \circ L^{-1} \subset L$. Moreover since N is closed and T-invariant $N \subset L$ will follow from $N \subset P$.

Suppose then that $(x, x') \in N \backslash P$ and let $\delta = \inf\{d(T^n x, T^n x') : n \in \mathbb{Z}\} > 0$. Since $(x, x') \in N$ there exist a sequence $x_k \to x$ and a subsequence $\{n'_k\}$ of $\{n_k\}$ with $T^{n'_k} x_k \to x'$. Let $U \subset X$ be a non-empty open set. By minimality there exists l with $T^l x \in U$. Since $T^{n'_k} T^l x_k \to T^l x'$ and $T^l x_k \to T^l x$ we have for k large enough $T^l x_k \in U$ and $\delta < d(T^{n'_k} T^l x_k, T^l x_k)$.

Thus the open set $V_{k_0} = \{z : \exists k > k_0 d(T^{n_k} z, z) > \delta\}$ is dense and $B = \bigcap_{k=1}^{\infty} V_k$ is a dense G_δ subset of X. However for $z \in B$, $T^{n_k} z \to z$ a contradiction. Thus $N \subset P$ and the proof of (1) is complete.

(2) Put $V_{\varepsilon,k} = \{x : d(T^{n_j} x, x) \le \varepsilon \text{ for every } j \ge k\}$. Then $\bigcup_k V_{\varepsilon,k} = X$ and by Bair's theorem the set $B_\varepsilon = \bigcup_{k=1}^{\infty} \operatorname{int}(V_{\varepsilon,k})$ is an open dense subset of X.

We let $X_0 = \bigcap_{k=1}^{\infty} B_{1/k}$. Suppose $x \in X_0$ and $(x, x') \in N$, then there exist a subsequence n'_k of n_k and a sequence $x_k \to x$ such that $T^{n'_k} x_k \to x'$. Assume $x \neq x'$; choose l with $1/l < (1/2) d(x, x')$. Since $x \in X_0$ there exists j with $x \in \operatorname{int}(V_{1/l,j})$. For big enough k $x_k \in V_{1/l,j}$ and we have

$$2/l < d(x, x') \le d(x, x_k) + d(x_k, T^{n'_k} x_k) + d(T^{n'_k} x_k, x').$$

Sending k to infinity, we have the first and last terms on the right tend to zero while the middle term is less than $1/l$. This contradiction shows that $(X_0 \times X) \cap N = \Delta$. It follows that $N^{-1} \cap (X \times X_0) = \Delta$ and hence $N \circ N^{-1} \cap (X_0 \times X_0) = \Delta$. □

Corollary 2.3. *For a minimal syndetically distal (and hence also distal) flow, rigidity is equivalent to uniform rigidity.*

Remark. Chacon's flow is one example of a minimal syndetically distal non-distal flow. However, since it is zero dimensional it cannot be rigid (see § 6.7).

3. *Concatenation flows*

Let I denote the interval $[-1, 1]$, and let $\Omega = I^{\mathbb{Z}}$ be the compact metric space of bidirectional sequences in I, with the metric

$$d(x, y) = \sup_{n\in\mathbb{Z}} 2^{-|n|}|x_n - y_n|.$$

We call elements of I^n *n-strings* and we let $\|w, w'\| = \sup_{1\le i\le n} |w_i - w_i'|$ for w, $w' \in I^n$. If $x \in \Omega$ (or $x \in I^n$) and i, k are integers, $i \le k$, we let $x[i, k]$ be the string $w = x_i x_{i+1} \cdots x_k$. When $w \in I^n$ and $w' \in I^l$ then ww' is the $(n+l)$-string $w_1 w_2 \cdots w_n w_1' \cdots w_l'$. We say that an n-string w *appears* in $x \in \Omega$ (or $x \in I^l$) at the j-the coordinate if $x[j, j+n-1] = w$. If $W \subset I^n$, $V \subset W^2$ are closed subsets and i, $l > 2$ are integers, we define

$$C_i(W, V) = \{x \in \Omega: \forall_j \in \mathbb{Z}, x[i+jn, i+jn+2n-1] \in V\},$$

$$C(W, V) = \bigcup_{i=0}^{n-1} C_i(W, V),$$

$$C(W, V, l) = \{x \in W^l: \forall 0 \le j < l-2, x[jn+1, jn+2] \in V\}.$$

We will call W the set of n-blocks and V the set of legitimate pairs of n-blocks.

Definition 3.1 Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of natural numbers, $W_0 = I$ and for every k let $W_k \subset C(W_{k-1}, V_{k-1}, n_k)$ be compact sets. Let $V_k \subset C(W_{k-1}, V_{k-1}, 2n_k)$. The subshift (X, T) where $X = \bigcap_{k=1}^{\infty} C(W_k, V_k)$ will be called the *concatenation flow of* $\{W_k\}$, $\{V_k\}$. In other words, $x \in X$ iff for every natural k there are $\{w_i\}_{i=-\infty}^{\infty} \in W_k$ s.t. $x = \cdots w_{-1} w_0 w_1 w_2 \cdots$ and s.t. for every integer i, $w_i w_{i+1} \in V_k$.

Definition 3.2. Let (X, T) be the concatenation flow of $\{W_k\}$, $\{V_k\}$ and let m_k denote the length of the strings in W_k. Given $x \in X$, by Definition 3.1 we can find integers t_k, $k = 1, 2, \ldots,$ such that $x \in C_{t_k}(W_k, V_k)$. We will call t_k a W_k *partition of* x. One can easily see that we can choose $\{t_k\}_{k=1}^{\infty}$ such that $t_{k-1} \equiv t_k (\text{mod } m_{k-1})$ for every k. Such a sequence $\{t_k\}$ will be called a *block partition of* X. If we choose $\{t_k\}$ s.t. $0 \le t_k \le m_k$ we will say that $\{t_k\}$ is a *normalized block partition of* x.

Definition 3.3. Let t be a W_k partition of $x \in X$ such that $-m_k/2 < t \le m_k/2$. $v = x[t - m_k, t + m_k - 1] \in V_k$ will be called a *central* V_k *block of* X.

The proof of the next lemma is left to the reader.

LEMMA 3.4. *Let* W_k, V_k, n_k, $k = 1, 2, \ldots,$ *and* (X, T) *be as in Definition* 3.1. *Suppose that the following condition holds: for every natural* k *there exists a* 2^{-k} *net* $\{u_1^{(k)}, u_2^{(k)}, \ldots, u_{l_k}^{(k)}\}$ *of* V_{k-1} *such that, for every* $w \in W_k$, $u_i^{(k)}$ *appears in* w *of every* $1 \le i \le l_k$. *Then* (X, T) *is a minimal flow.*

Definition 3.5. Let (X, T) be the concatenation flow of $\{W_k\}$, $\{V_k\}$, $W_k \subset I^{m_k}$, satisfying the following four conditions:

(1) For each natural k there exists $1 < r_k < m_k/2$ and $\eta_k \in I^{r_k}$ such that for every $w \in W_k$, $w[1, r_k] = \eta_k$.

(2) For every $w \in W_k$, where $w = w_1 w_2 \cdots w_{n_k}$, $w_i \in W_{k-1}$, and for every i s.t. $(i-1)m_{k-1} > r_k$, $\|w_{i-1}, w_i\| < 2^{-k}$.

(3) Let w be as in (2), then $w_1 = w_2 = w_{n_{k-1}} = w_{n_k}$.

(4) For every natural k, a 2^{-k} net of V_{k-1} appears in η_k.

We call such a flow a *concatenation flow with fixed part* $\{\eta_k\}$.

PROPOSITION 3.6. *Let (X, T) be a concatenation flow with a fixed part $\{\eta_k\}$ then (X, T) is a minimal rigid flow (with respect to the sequence m_k).*

Proof. From Lemma 3.4, it is clear that (X, T) is minimal. It is sufficient to show that for every $\varepsilon > 0$, every integer i and every $x \in X$ there exists k_0 such that for every $k > k_0$, $|x[i] - T^{m_k}x[i]| < \varepsilon$. Let $\{t_k\}$ be a block partition of x. Define $w_k^{(j)} = x[t_k + jm_k, t_k + jm_k + m_k - 1] \in W_k$ and suppose that $t_k + j_k m_k \le i < t_k + j_k m_k + m_k$, or, in other words $x[i]$ is in $w_k^{(j_k)}$.

We consider two cases.

(1) There exists no k such that $x[i]$ is contained in the fixed part η_k of $w_k^{(j_k)}$. In this case, choose k_0 such that $2^{-k_0} < \varepsilon$. For every $k > k_0$ we have by condition (2) of Definition 3.5 (or by condition (3), if $w_k^{(j_k)}$ is the last m_k string of $w_{k+1}^{(j_{k+1})}$) that $\|w_k^{(j_k)}, w_k^{(j_k+1)}\| < 2^{-k} < \varepsilon$ and thus, $|x[i] - x[i + m_k]| < \varepsilon$.

(2) There exists k_0 such that $x[i]$ is contained in the fixed part η_{k_0} of $w_{k_0}^{(j_{k_0})}$. In this case, the fact that m_{k_0} divides m_k for every $k > k_0$, implies that shifting $x[i]$ by m_k indices brings us to the same place in η_{k_0} (in another m_{k_0} block). Thus, $x[i] = x[i + m_k]$.

So in each case, we have found the required k_0. □

LEMMA 3.7. *Let (X, T) be a concatenation flow with a fixed part η_k and let $x, x' \in X$ have the same normalized block partition $\{t_k\}$, where $t_k \to \infty$ and $t_k - m_k \to -\infty$. Let $w_k = x[t_k - m_k, t_k - 1]$ and $w'_k = x'[t_k - m_k, t_k - 1]$. If for every natural k, $w_k w'_k$ appears in η_{k+1} then $(x, x') \in N$.*

Proof. Choose $y \in X$. For every natural k, η_k appears in y. Thus, we can find for each k an integer s_k such that for $x_k = T^{s_k} y$ we have

$$x_k[t_k - m_k, t_k + m_k - 1] = w_k w'_k.$$

Clearly, $x_k \to x$ and $T^{m_k} x_k \to x'$ and thus $(x, x) \in N$. □

Note that we have almost no restriction when choosing the fixed part η_k and thus we can arrange that any given $w_k w'_k$ will appear in η_k.

4. *The structure of concatenation flows with fixed part*

In this section we show that certain concatenation flows with fixed part are almost automorphic.

LEMMA 4.1. *Let (X, T) be a concatenation flow with a fixed part $\{\eta_k\}$. Suppose that, for a natural number k, a string $w_k w'_k$ appears in η_k such that $w_k, w'_k \in W_{k-1}$ and $\|w_k, w'_k\| > 2^{-k}$. Under these conditions every $x \in X$ has only one normalized W_k partition.*

Proof. Let $x \in X$ and k such that in η_k appears the string $w_k w'_k$, where $\|w_k, w'_k\| > 2^{-k}$

and w_k, $w'_k \in W_{k-1}$. Let r_k be the length of the string η_k and let $1 \le i_k < r_k$ be the minimal index such that $|\eta_k[i_k] - \eta_k[i_k + m_k - 1]| > 2^{-k}$.

Suppose that x has two normalized W_k partitions t and t', and $t < t'$. We consider the two following cases:

(1) $t' - t < r_k$. We have $x[t, t + r_k - 1] = \eta_k = x[t', t' + r_k - 1]$ and so $|x[t + i_k] - x[t + i_k + m_{k-1}]| > 2^{-k}$. Since outside the fixed part η_k, $|x[i] - x[i + m_{k-1}]| < 2^{-k}$, t' must be such that $x[t + i_k]$ lies in the fixed part η_k of a W_k block in the t' partition. This implies that $t' \le t + i_k < t' + r_k$. (As $t' - t < r_k$ we cannot have $t' - m_k \leqq t + i_k < t' - m_k + r_k$.) Suppose $t + i_k = t' + j_k$. Then $j_k < i_k$ because $t < t'$. But

$$|\eta_k[j_k] - \eta_k[j_k + m_{k-1}]| = |x[t + i_k] - x[t + i_k + m_{k-1}]| > 2^{-k}$$

and this contradicts the minimality of i_k.

(2) $t' - t \ge r_k$. We have $|x[t' + i_k] - x[t' + i_k + m_{k-1}]| > 2^{-k}$. From this it follows that also in the t partition $x[t' + i_k]$ must be in η_k and thus $t + m_k \le t' + i_k < t + m_k + r_k$. (As $t' - t \ge r_k$, we cannot have $t \le t' + i_k < t + r_k$.) Suppose $t'_k + i_k = t_k + m_k + j_k$ then $j_k < i_k$ and again we have a contradiction. □

Definition 4.2. Let (X, T) be a minimal concatenation flow of $\{W_k\}$, $\{V_k\}$ where $W_k \in I^{m_k}$ and where each $x \in X$ has only one normalized block partition. $\hat{X}$ will be the set of all sequences $\{t_k\}$ where $\{t_k\}$ is a normalized block partition of some $x \in X$. On $\hat{X}$ we will define the following metric: for $\{t_k\}$, $\{t'_k\} \in \hat{X}$, $d(t_k, t'_k) = 2^{-n}$ where $n = \min_{j=1,2\cdots} \{j \mid t_j \ne t'_j\}$. We will define $T : \hat{X} \to \hat{X}$ as follows: for $\{t_k\} \in \hat{X}$, $T\{t_k\} = \{t'_k\}$ where $t'_k = t_k + 1 (\text{mod } m_k)$. Clearly, $(\hat{X}, T)$ is an equicontinuous flow since $d(T\{t_k\}, T\{t'_k\}) = d(\{t_k\}, \{t'_k\})$.

PROPOSITION 4.3. *Let (X, T) be a concatenation flow with a fixed part η_k. Suppose that there exists a natural number k_0 such that for every $k \ge k_0$ there exists w_k, $w'_k \in W_{k-1}$ such that $\|w_k, w'_k\| > 2^{-k}$ and such that the string $w_k w'_k$ appears in η_k. Then*
(1) *$(\hat{X}, T)$ is the maximal equicontinuous factor of (X, T).*
(2) *$P = Q = \{(x, x') \in X \times X \mid x$ and x' have the same block partition$\}$.*
(3) $(\hat{X}, T)$ is an almost 1-1 factor of (X, T).

Proof. By Lemma 4.1 for every $k > k_0$ and every $x \in X$ there is a unique normalized W_k partition t_k of x. Let $\{t'_k\}_{k=1}^{\infty}$ be a normalized block partition of x. For $k \ge k_0$, $t'_k = t_k$ and for $k < k_0$, $t'_k \equiv t'_{k_0} (\text{mod } m_k)$. So $\{t'_k\}$ is unique and $(\hat{X}, T)$ is well defined.

Define $p : X \to \hat{X}$, $p(x) = \{t_k\}$, where $\{t_k\}$ is the normalized block partition of x. Clearly $p(Tx) = T(px)$. We will now prove that p is continuous. Let $x \in X$, $\{x_n\} \in X$, $x_n \to x$. From Lemma 4.1, we have that $\{C_i(W_k, V_k)\}_{i=0}^{m_k - 1}$ are pairwise disjoint sets for $k \ge k_0$.

Let $\varepsilon = \min_{0 \le i_1 < i_2 < m_k} d(C_{i_1}(W_k, V_k), C_{i_2}(W_k, V_k))$. Choose n_0 such that for every $n > n_0$, $d(x_n, x) < \varepsilon$. Suppose that $x \in C_{i_0}(W_k, V_k)$, then, for $n > n_0$, $x_n \in C_{i_0}(W_k, V_k)$. But this implies that, for $n > n_0$, $d(p(x_n), p(x)) < 2^{-k}$ and so p is continuous.

We will now prove that $(\hat{X}, T)$ is an almost one to one factor of (X, T).

Let $\{y_k\} = y \in \hat{X}$ be such that $y_k = 0$ for each natural k, and let $x \in X$ be such that $x \in p^{-1}(y)$. For every k, $x[0, m_k - 1] \in W_k$ and thus $x[0, r_k - 1] = \eta_k$. By (3) in Definition 3.5

$$x[-m_{k-1}, -1] = x[0, m_{k-1} - 1] = \eta_k[1, m_{k-1}].$$

But this determines x. Thus $p^{-1}(y)$ contains only one point and $(\hat{X}, T)$ is an almost one to one factor of (X, T).

It also follows that $(\hat{X}, T)$ is the maximal equicontinuous factor of (X, T), because an equicontinuous flow has no proper almost one to one factors. Clearly now $P \subset Q \subset R$ where $R = \{(x, x') \mid p(x) = p(x')\}$. But if $(x, x') \in R$, they are proximal because, for each natural k, η_k appears at the same places in x and x', so $P = Q = R$. □

COROLLARY 4.4. *If $N \neq \Delta$ in a concatenation flow X with fixed part, then every $x \in X$ has a unique normalized block partition. Moreover, $(x, x') \in N$ implies that x and x' have the same block partition.*

Proof. Suppose $x_k \to x$ and $T^{m_k} x_k \to x'$. Choose k big enough such that $\|x[i] - x'[i]\| > 2^{-k+2}$ for some i. Let n_0, $n_0 > k$, be such that for $n > n_0$ $\|x_n[i] - x[i]\| < 2^{-k}$ and $\|x_n[i + m_n] - x'[i]\| < 2^{-k}$. Then $\|x_n[i] - x_n[i + m_n]\| > 2^{-k} > 2^{-n}$ and $x_n[i]$ and $x_n[i + m_n]$ must lie in η_{n+1}. Hence the conditions of Proposition 4.3 are fulfilled (where w_k is the W_n block in η_{n+1} that contains $x[i]$, and w'_k is the W_n block that contains $x[i + m_n]$). □

5. *Examples*

We will now build concrete examples of minimal rigid but not uniformly rigid concatenation flows with fixed part. The first example is an example where N is not an equivalence relation and where for every $x \in X$ there exists at most one $x' \neq x \in X$ such that $(x, x') \in N$.

Example 5.1. Let $f: I \to I$ be a monotonic ascending and continuous function. Choose $q, q' \in I$, $q > q'$. We will define by induction numbers m_k and continuous monotonic functions $w_k: I \to I^{m_k}$ as follows: $w_0(s) = f(s)\ \forall s \in I$. Assume that we defined $w_{k-1}(s): I \to I^{m_{k-1}}$. Choose $\varepsilon > 0$ s.t. $|s - s'| < \varepsilon$ implies $\|w_{k-1}(s), w_{k-1}(s')\| < 2^{-k}$ and choose $0 = s_1 < s_2 < \cdots < s_{l_k} = 1$ s.t. $|s_{i+1} - s_i| < \varepsilon$, $1 \leq i \leq l_k$ and s.t. there exist $l_k \geq r > r' \geq 1$ with $s_r = q$, $s_{r'} = q'$. Define

$$\eta_k = w_{k-1}(0) w_{k-1}(s_1) w_{k-1}(s_2) \cdots w_{k-1}(s_r) w_{k-1}(s_{r'}) w_{k-1}(s_{r'+1}) \cdots$$
$$w_{k-1}(s_{l_k}) w_{k-1}(s_{l_k - 1}) \cdots w_{k-1}(0) w_{k-1}(0).$$

Define $w_k(s) = \eta_k w_{k-1}(ss_1) w_{k-1}(ss_2) \cdots w_{k-1}(ss_{l_k}) w_{k-1}(ss_{l_k-1}) \cdots w_{k-1}(0) w_{k-1}(0)$ and let m_k be the length of $w_k(s)$.

Define $W_k = \{w_k(s) \mid s \in I\}$ and

$$V_k = \{w_k(ss_i) w_k(ss_j) \mid s \in I, 1 \leq i < l_k, j = i+1, i-1\} \cup \{w_k(q) w_k(q')\}.$$

Define (X, T) as the concatenation flow defined by $\{W_k\}$, $\{V_k\}$.

PROPOSITION 5.2. *(X, T) in Example 5.1 is a non-empty concatenation flow with a fixed part and thus is minimal and rigid.*

Proof. Clearly for each k, $C(W_k, V_k)$ is not empty and thus $X \neq \varnothing$. η_k contains $w_{k-1}(s_i) w_{k-1}(s_{i+1})$ for every $1 \leq i < l_k$ and contains $w_{k-1}(q) w_{k-1}(q')$. By the definition of $\{s_i\}$, η_k contains a 2^{-k} net of V_{k-1}. η_k is a fixed part of W_k. By Proposition 3.6 (X, T) is minimal and rigid. □

LEMMA 5.3. *(X, T) is not uniformly rigid with respect to m_k.*

Proof. Let x, $x' \in X$ be as follows.

We define by induction on k $x[i_k, j_k]$, $x'[i_k, j_k]$, where $i_k < i_{k-1}$ and $j_k > j_{k-1}$. For $k = 0$, let $i_0 = j_0 = 0$, and let $x[i_0, j_0] = w_0(q)$, $x'[i_0, j_0] = w_0(q')$. Assume that $x[i_k, j_k](x'[i_k, j_k])$ is defined and equals $w_k(q)(w_k(q'))$. Consider $w_{k+1}(q) = \eta_{k+1} w_k(s_1 q) \cdots w_k(s_{i_k} q) \cdots w_k(0)$. As $s_{i_k} = 1$, $w_k(q)$ appears in $w_{k+1}(q)$ (outside of η_{k+1}).

Define i_{k+1}, j_{k+1} and $x[i_{k+1}, j_{k+1}]$ such that $x[i_{k+1}, j_{k+1}] = w_{k+1}(q)$ and such that $x[i_k, j_k]$ is the above mentioned appearance of $w_k(q)$ in $w_{k+1}(q)$. Define $x'[i_{k+1}, j_{k+1}]$ to be $w_{k+1}(q')$. Notice that $x'[i_{k+1}, j_{k+1}]$ is well defined, since $w_k(q')$ $(=x'[i_k, j_k])$ appears in $w_{k+1}(q')$ exactly in the same place as $w_k(q)$ appears in $w_{k+1}(q)$.

Clearly, $i_{k+1} < i_k$ and $j_{k+1} > j_k$, which implies that all coordinates of x, x' are defined. Also x, x' have the same block partition. The construction implies that $x \neq x'$ since $x[0] = w_0(q) = f(q) \neq f(q') = w_0(q') = x'[0]$. For each k, $w_k(q)$ is a central W_k block of x, and $w_k(q')$ is a central W_k block of x'. $w_k(q) w_k(q')$ appears in η_k and thus by Lemma 3.7. $(x, x') \in N$ and by Proposition 2.1 (X, T) is not uniformly rigid. □

PROPOSITION 5.4. *Let X be as in Example* 5.1 *then*

(1) *For each $x \in X$ there is at most one $x' \neq x$ such that $(x, x') \in N$.*

(2) *If $x \neq x'$ and $(x, x') \in N$ then for every integer i $x[i] \geq x'[i]$. Thus $(x', x) \notin N$ and N is not an equivalence relation.*

Proof. Let $(x, x') \in N$ and $x \neq x'$. Let m_{j_k} be a subsequence of m_k and $\{x_k\} \in X$, $x_k \to x$ such that $T^{m_{j_k}} x_k \to x'$. Choose i such that $x[i] \neq x'[i]$. For each natural number l choose i_l such that $x \in C_{i_l}(W_l, V_l)$ and such that $i_l \leq i \leq i_l + m_l$ i.e., $x[i_l, i_l + m_l - 1]$ is the W_l block x in which $x[i]$ lies.

By 4.4. also $x' \in C_{i_l}[W_l, V_l]$ and for n big enough $x_n \in C_{i_l}(W_l, V_l)$. Denote $v_l = x[i_l, i_l + m_l - 1]$, $v'_l = x'[i_l, i_l + m_l - 1]$, and denote $v_l^n = x_n[i_l, i_l + m_l - 1] \in W_l$, $v_l'^n = (T^{m_{j_n}} x_n)[i_l, i_l + m_l - 1] \in W_l$. Choose k s.t. $2^{-k} < |x[i] - x'[i]|$. $x[i]$ is contained in v_k and $v_k^n \to v_k$. $x'[i]$ is contained in v'_k and $v_k'^n \to v'_k$. As $|x[i] - x'[i]| > 2^{-k}$; we have, for large enough n, $\|v_k^n, v_k'^n\| > 2^{-k} > 2^{-j_n}$. For $j_n > k$, v_k^n appears in $v_{j_n}^n$ and $v_k'^n$ appears in $v_{j_n}'^n$. Thus for n big enough we have $\|v_{j_n}^n, v_{j_n}'^n\| > 2^{-j_n}$. $v_{j_n}^n$ and $v_{j_n}'^n$ are two adjacent W_{j_n} blocks in x_n. Thus the only possibility of their distance to be greater than 2^{-j_n} is that $v_{j_n}^n$ lies in η_{j_n+1} and that $v_{j_n}^n = w_{j_n}(q)$, $v_{j_n}'^n = w_{j_n}(q')$. We have v_k^n appearing in v_{k+1}^n appearing in $\cdots$ appearing in $v_{j_n-1}^n$ appearing in $v_{j_n}^n$. Thus for $k \leq l < j_n$, $v_l^n \neq v_l'^n$ which implies that v_l^n does not lie in η_{l+1}. (Otherwise $v_l'^n$ would have also been in η_{l+1} in the same place which implies $v_l^n = v_l'^n$). If we look at the definition of $w_l(q)$, this implies that there exists some number s_n such that $v_k^n = w_k(s_n q)$. Since $v_k'^n$ lies in $v_{j_n}'^n$ in the same place as v_k^n lies in $v_{j_n}^k$, we have $v_k'^n = w_k(s_n q')$. Let $s = \overline{\lim}\, s_n$ then $w_k(sq) = \lim_{n\to\infty} v_k^n = v_k$ and $v'_k = \lim_{n\to\infty} v_k'^n = w_k(sq')$. Thus, we have

(*) $x[i_k, i_k + m_k - 1] = w_k(sq)$

(**) $x'[i_k, i_k + m_k - 1] = w_k(sq')$.

As each coordinate in $w_k(sq)$ is greater than the corresponding one in $w_k(sq')$ we have $x[i] > x'[i]$ which proves (2).

Let r_k denote the length of η_k then $i > i_k + r_k$ (because $x[i_k, i_k + r_k - 1] = x'[i_k, i_k + r_k - 1] = \eta_k$ and $x[i] \neq x'[i]$) and also $i_k + m_k - m_{k-1} > i$ (because

$$x[i_k + m_k - m_{k-1}, i_k + m_k - 1] = x'[i_k + m_k - m_{k-1}, i_k + m_k - 1] = w_{k-1}(0)).$$

Thus $i_k \to -\infty$ and $i_k + m_k \to \infty$ and (**) defines x' uniquely. □

In the first example, we had N small with respect to Q. Now, we will show an example where $N = Q$.

Example 5.5. Define (X, T) as in Example 5.1 changing η_k to be as follows

$$\eta_k = w_{k-1}(0) w_{k-1}(0) \eta'_k w_{k-1}(0) w_{k-1}(0),$$

where η'_k is the concatenation of $\{w_{k-1}(s_j) w_{k-1}(s_i)\}_{1 \le i,j \le l_k}$, and adding to V_k the set $\{w_k(s_i) w_k(s_j)\}_{1 \le i,j \le l_k}$.

PROPOSITION 5.6. *Let (X, T) be as in Example 5.5 and let x and x' have the same block partition then $(x, x') \in N$.*

Proof. Let $x, x' \in X$, $x \neq x'$ have the same normalized block partition $\{t_k\}$. The sequence $\{t_k\}$ is not bounded, because otherwise, as $x[t_k, t_k + r_k - 1] = \eta_k = x'[t_k, t_k + r_k - 1]$ and $x[t_k - m_{k-1}, t_k - 1] = w_{k-1}(0) = x'[t_k - m_{k-1}, t_k - 1]$, we would have $x = x'$ - a contradiction. Thus we may assume that $\{t_k\}$ is not bounded from below. (Otherwise, as $t_k < m_k/2$ we have $t_k - m_k$ is not bounded from below. Replacing t_k by $t_k - m_k$ we have t_k bounded). As $t_k < m_k/2$, $t_k + m_k$ is not bounded from above. Fix k. There exists r such that for s_r, $\|x[t_k, t_k + m_k - 1], w_k(s_r)\| < 2^{-k}$ and r' such that for $s_{r'}$, $\|x'[t_k, t_k + m_k - 1], w_k(s_{r'})\| < 2^{-k}$.

Let $y \in X$. $w_k(s_r) w_k(s_{r'})$ appears in η_{k+1} thus there exists n such that $T^n(y)[t_k, t_k + m_k - 1] = w_k(s_r)$ and $T^{n+m_k} y[t_k, t_k + m_k - 1] = w_k(s_{r'})$. Define $x_k = T^n y$. As t_k is not bounded from below and $t_k + m_k$ is not bounded from above, we have $x_k \to x$ and $T^{m_k} x_k \to x'$. □

Example 5.7. We will use the notations of Example 5.1. Let $q'', q, q' \in I$ $q'' > q > q'$. Let $s_1, s_2 \cdots s_{l_k}$ be as in Example 5.1 and such that there exist $r'' > r > r'$ with $s_{r''} = q''$, $s_r = q$, $s_{r'} = q'$. Define

$$\eta_{k+1} = w_k(0) w_k(s_1) w_k(s_2) \cdots w_k(s_r) w_k(s_{r'}) w_k(s_{r'+1})$$
$$\cdots w_k(s_{r''-1}) w_k(s_{r''}) w_k(s_{r''}) w_k(s_{r''-1}) \cdots$$
$$w_k(s_{r'}) w_k(s_{r''}) w_k(s_{r''+1}) \cdots w_k(s_{l_k}) w_k(s_{l_{k-1}}) \cdots w_k(s_1)$$

and define

$$w_{k+1}(t) = \eta_{k+1} w_k(s_1 t) \cdots w_k(0),$$
$$W_{k+1} = \{w_{k+1}(t) \mid t \in [0, 1]\},$$
$$V_{k+1} = \{w_k(tt_i) w_k(tt_{i+1}),\ t \in [0, 1]\}_{i=1,l_k} \cup \{w_k(t_{i+1}t) w_k(t_i t),\ t \in [0, 1]\}_{i=1,l_k}$$
$$\cup \{w_k(q) w_k(q')\} \cup \{w_k(q') w_k(q'')\}.$$

PROPOSITION 5.8. *Let (X, T) be as in Example 5.7 then the relation $\tilde{N}$ (with respect to m_k) is not an equivalence relation.*

Proof. We will only sketch the proof. We will find x, x', x'' such that $(x, x') \in \tilde{N}$,

$(x', x'') \in \tilde{N}$. But $(x, x'') \notin \tilde{N}$. Let x, x', x'' be such that for every k their central W_k block are $w_k(q)$, $w_k(q')$, $w_k(q'')$ respectively.

In a similar way to the proof of Proposition 5.4, one can show the following Lemmas:

(1) Let x_k'' be a sequence such that $x_k \to x''$ then $T^{m_k} x_k'' \to x''$.

(2) Let x_k be a sequence such that $x_k \to z$ and such that for a subsequence m_k' of m_k $T^{m_k'} x_k \to x''$ then $z = x'$ or $z = x''$.

From (1) and (2) it follows that $(x, x'') \notin \tilde{N}$. But clearly from Lemma 3.7, we have $(x, x') \in N \subset \tilde{N}$ and $(x', x'') \in N \subset \tilde{N}$. □

6. *Miscellaneous results*

LEMMA 6.1. *(X, T) is weakly rigid iff the identity map e is a limit point of the enveloping semigroup $E(X)$ of (X, T).*

Proof. Follows directly from the definition of weak rigidity. □

COROLLARY 6.2. *Every minimal distal flow is weakly rigid.*

Proof. By [E] (X, T) is distal iff $E(X)$ is a group. If e is isolated in $E(X)$ then X is finite, hence rigid. □

Consider the distal minimal flow on $\mathbf{T}^2$ defined by $T(x, y) = (x + \alpha, y + x)$, where $\mathbf{T} = [0, 1]$, α an irrational number in $\mathbf{T}$ and addition is mod 1. We claim that this flow is not rigid. In fact if T^{n_k} tend pointwise to the identity for some *sequence* $\{n_k\}$, then since $T^n(x, y) = (x + n\alpha, y + nx + [n(n-1)/2]\alpha)$ we have $n_k x \to 0$ for every $x \in \mathbf{T}$, an obvious absurdity. Thus, in general, weak rigidity does not imply rigidity.

PROPOSITION 6.3. *The topological entropy of a rigid flow is zero.*

Proof. Let (X, T) be rigid with respect to the sequence $\{n_k\}$. Then $T^{n_k} x \to x$ for every $x \in X$ and by the dominated convergence theorem we have for every invariant probability measure η on X and every $f \in L_2(\eta)$. $\lim T^{n_k} f = f$ in $L_2(\eta)$. Thus the measure preserving system (X, η, T) is rigid in the measure theoretical sense. It follows from [**FW**] that it has zero entropy. Thus the variational principle implies that the topological entropy of (X, T) is zero as well. □

One consequence of Furstenberg's structure theorem for distal flows is that the topological entropy of such flow vanishes [**Ke**]. Since every minimal distal flow is weakly rigid, this suggests the possibility that weak rigidity is a sufficient condition for zero entropy. We do not know whether this is true.

Considering the second question posed in the introduction, we observe that since an equicontinuous flow is uniformly rigid, a counter example to this question should be looked for among the weakly mixing flows. Recall that (X, T) is (strongly) mixing if for every two non-empty open subsets U, V of X the set $N(U, V) = \{n \in \mathbb{Z}: T^n U \cap V \neq \emptyset\}$ has a finite complement.

PROPOSITION 6.4. *A strongly mixing minimal flow admits only trivial rigid factors.*

Proof. Suppose (X, T) is minimal and strongly mixing; then every factor of (X, T) have these same properties. Thus in order to prove our proposition, it suffices to show that (X, T) is trivial if in addition to the above it is also rigid. Assume therefore that (X, T) is rigid with respect to $\{n_k\}$. Let $(x, x')\in X\times X$ and U and V neighbourhoods of x and x' respectively. By strong mixing there is k_0 such that $T^{n_k}U\cap V\neq\emptyset$ for $k\geq k_0$. Thus it is possible to find a subsequence $\{n'_k\}$ of $\{n_k\}$ and a sequence $x_k\to x$ for which $T^{n'_k}x_k\to x'$. In other words $X\times X\subset N$. But by Proposition 2.2 $N\subset L\subset P$. Since for every $x\in X$ $Tx\neq x$ implies $(x, Tx)\notin P$ we conclude that X has only one point. □

We next show the existence of plenty of minimal, uniformly rigid, weakly mixing flows. Let (Z, σ) be a minimal flow rigid with respect to a sequence $\{n_k\}$. Let Y be a compact metric space. Let $X = Z\times Y$ and let $\mathcal{H}(X)$ be the group of homeomorphisms of X. Put

$$\mathcal{O}(\sigma)=\{G^{-1}\circ\sigma\circ G\colon G\in\mathcal{H}(X)\}$$

(we identify σ with $\sigma\times\mathrm{id}$), and let

$$V_{k_0,\varepsilon}=\{T\in\bar{\mathcal{O}}(\sigma)\colon \exists k\geq k_0\quad d(T^k,\mathrm{id})<\varepsilon\}.$$

Clearly $V_{k_0,\varepsilon}$ is an open dense subset of $\bar{\mathcal{O}}(\sigma)$, and $\mathcal{R}_1=\bigcap_k V_{k,1/k}$ is a residual subset of $\bar{\mathcal{O}}(\sigma)$. For each $T\in\mathcal{R}_1$, (X, T) is uniformly rigid w.r.t. some subsequence of $\{n_k\}$. If we assume that the identity path component of $\mathcal{H}(Y)$ acts minimally on Y we get from [**GW**, Th. 1] the existence of a residual subset $\mathcal{R}_2\subset\bar{\mathcal{O}}(\sigma)$ such that each member of $\mathcal{R}_2$ acts minimally on X. Finally assuming further that $Z=\mathbf{T}$ and $\sigma=R_\alpha$, an irrational rotation, we get from [**GW**, Th. 5] a residual subset $\mathcal{R}_3$ of $\bar{\mathcal{O}}(\sigma)$ every member of which acts weak mixingly on X. Taking in the latter case $\mathcal{R}=\mathcal{R}_1\cap\mathcal{R}_2\cap\mathcal{R}_3$ we have:

PROPOSITION 6.5. *Let $Z=\mathbf{T}$, $\sigma=R_\alpha$ an irrational rotation. Let Y be a non-trivial compact metric space, let $X=Z\times Y$ and let $\mathcal{H}_0(Y)$ be the identity path component of $\mathcal{H}(Y)$. If the action of $\mathcal{H}_0(Y)$ on Y is minimal then there exists a residual subset $\mathcal{R}$ of $\bar{\mathcal{O}}(\sigma)$ such that for each $T\in\mathcal{R}$ the flow (X, T) is uniformly rigid, minimal and weakly mixing.* □

We conclude by showing that weak rigidity implies equicontinuity in zero dimensional flows.

LEMMA 6.6. *Let $Y\subset\{0,1\}^{\mathbf{Z}}=\Omega$ be a T-invariant closed subset of the two shift (Ω, T). If (Y, T) is infinite then there exist points $y_0, y_1, z_0, z_1\in Y$ such that $y_0(0)=z_0(0)=0$, $y_1(0)=z_1(0)=1$, $y_0(i)=y_1(i)$ for $i<0$ and $z_0(i)=z(i)$ for $i>0$.*

Proof. Let k be the minimal length such that for every block B of length k whenever B appears in $y\in Y$ say, $y[n, n+k-1]=B$, then always $y(n+k)=0$ (or always $y(n+k)=1$).

If $k<\infty$ then each $y\in Y$ is periodic of period $\leq L$, where L is the number of k-blocks appearing in Y. This implies that Y is finite contradicting our assumption. Thus $k=\infty$; let $|B_i|=k_i$, a sequence n_i and points $y_1^{(i)}$, $y_0^{(i)}\in Y$ be chosen with $y_\varepsilon^{(i)}[n_i, n_i+k_i-1]=B_i$, $\varepsilon=0$, 1 and $y_0^{(i)}(n_i+k_i)=0$, $y_1^{(i)}(n_i+k_i)=1$. We can assume the existence of the limits $y_0=\lim T^{-(n_i+k_i)}y_0^{(i)}$, $y_1=\lim T^{-(n_i+k_i)}y_1^{(i)}$. Then $y_0(0)=0$, $y_1(0)=1$ and $y_0(i)=y_1(i)$ for $i<0$. The existence of z_0 and z_1 follows similarly. □

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PROPOSITION 6.7. *Let (Y, T) be a minimal weakly rigid flow.*
(i) *If (Y, T) is a subshift then Y is finite.*
(ii) *If Y is zero dimensional then (Y, T) is equicontinuous.*

Proof. (i) Let $\{T^{n_\alpha}\}$ be a net converging to the identity pointwise on Y. We can assume that $n_\alpha > 0\ \forall\alpha$. If Y is infinite, there exist $y_0, y_1 \in Y$ as in Lemma 6.6. However $\lim T^{n_\alpha} y_0 = \lim T^{n_\alpha} y_1$ contradicting our assumption that $\lim T^{n_\alpha} = \mathrm{id}$.

(ii) Let $V \subset Y$ be a clopen set then the map $\pi_V : Y \to \Omega$, $\pi_V(y) = \{1_V(T^n y)\}_{n\in\mathbf{Z}}$ is a homomorphism of (Y, T) into (Ω, T). Since the image $(\pi(Y), T)$ is a weakly rigid subshift, (i) implies it is finite. Since by assumption the maps π_V, V clopen in Y, separate points in Y, we have $(Y, T) = \lim_{\leftarrow} (\pi_V(Y), T)$. This implies the equicontinuity of (Y, T). □

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