Canad. Math. Bull. Vol. 62 (1), 2019 pp. 71–74
http://dx.doi.org/10.4153/CMB-2017-038-x

Quotients of Essentially Euclidean Spaces

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Abstract. A precise quantitative version of the following qualitative statement is proved: If a finite-dimensional normed space contains approximately Euclidean subspaces of all proportional dimensions, then every proportional dimensional quotient space has the same property.

1 Introduction

Given a function λ from $(0,1)$ into the positive reals, a finite-dimensional normed space E is called λ *essentially Euclidean* provided that for every $\epsilon > 0$ there is a subspace E_ϵ of E that has dimension at least $(1-\epsilon)\dim E$ and the Euclidean distortion $c_2(E_\epsilon)$ of E_ϵ is $\le \lambda(\epsilon)$; that is, E_ϵ is $\lambda(\epsilon)$-isomorphic to the Euclidean space of its dimension. A family $\mathcal{F}$ of finite-dimensional spaces is λ essentially Euclidean provided that each space in $\mathcal{F}$ is λ essentially Euclidean, and $\mathcal{F}$ is called essentially Euclidean if it is λ essentially Euclidean for some function λ, as above. Litwak, Milman, and Tomczak-Jaegermann [LMT-J] considered the concept of essentially Euclidean, but what we are calling an essentially Euclidean family they would term a 1-*ess-Euclidean* family. The most studied essentially Euclidean families are the class of all finite-dimensional spaces that have cotype two constant less than some numerical constant, and the set of all finite-dimensional subspaces of a Banach space that has weak cotype two [Pis, Chapter 10]. However, if one is interested in the proportional subspace theory of finite-dimensional spaces, cotype two and weak cotype two are unnecessarily strong conditions, because they are conditions on all subspaces rather than on just subspaces of proportional dimension. For example, let $0 < \alpha < 1$ and let $\mathcal{F}_\alpha$ be the family $\{\ell_2^{n-n^\alpha} \oplus_2 \ell_\infty^{n^\alpha} : n = 1,2,3,\dots\}$ (throughout, we use the convention, standard in the local theory of Banach spaces, that when a specified dimension is not a positive integer, it should be adjusted to the next larger or smaller positive integer, depending on context). The cotype two constants of the spaces in $\mathcal{F}_\alpha$ are obviously unbounded and it is also well known that the family does not live in any weak cotype two space. Computing that $\mathcal{F}_\alpha$ is $\lambda_\alpha(\epsilon)$ essentially Euclidean with $\lambda_\alpha(\epsilon) \le (1/\epsilon)^{\alpha/2(1-\alpha)}$ is straightforward: First, when $n^\alpha \le \epsilon n$, the space $\ell_2^{n-n^\alpha} \oplus_2 \ell_\infty^{n^\alpha}$ has a subspace of dimension at least $(1-\epsilon)n$ that is isometrically Euclidean. On the other hand, if $\epsilon n < n^\alpha$, then the entire space $\ell_2^{n-n^\alpha} \oplus_2 \ell_\infty^{n^\alpha}$ is $n^{\alpha/2}$-Euclidean, since that is the isomorphism constant between $\ell_\infty^{n^\alpha}$ and $\ell_2^{n^\alpha}$, and $n^{\alpha/2} \le (1/\epsilon)^{\alpha/2(1-\alpha)}$.

Received by the editors October 6, 2016.
Published electronically December 5, 2018.
Author W. J. was supported in part by NSF DMS-1565826.
AMS subject classification: **46B20**, 46B07, 46B99.
Keywords: essentially euclidean.

It is also simple to check that the essentially Euclidean property passes to proportional dimensional subspaces. Suppose that E is an n-dimensional space that is λ essentially Euclidean, F is a subspace of E that has dimension αn, and $\epsilon > 0$. Take a subspace E_1 of E of dimension $(1-\epsilon\alpha)n$ with $c_2(E_1) \le \lambda(\epsilon\alpha)$. Then $\dim E_1 \cap F \ge \alpha n - \epsilon\alpha n = (1-\epsilon)\alpha n$, which implies that F is λ_F essentially Euclidean with $\lambda_F(\epsilon) \le \lambda(\epsilon\alpha)$. It is, however, not obvious that the essentially Euclidean property passes to proportional dimensional quotients; the main result of this note is that it does.

We use standard notation. We just mention that if A is a set of vectors in a normed space, $[A]$ denotes the closed linear space of A, and e_i denotes the i-th unit basis vector in a sequence space.

2 Main Result

The main tool we use is Milman's subspace of quotient theorem [Mil], [Pis, Chapters 7 & 8]. In [LMT-J] this theorem is not used directly, but the ingredients of its proof are. The theorem says that there is a function $M\colon (0,1) \to \mathbb{R}^+$ such that for every n and every $0 < \delta < 1$, if $\dim E = n$ then there is a subspace F of some quotient of E so that $\dim F = (1-\delta)n$ and $c_2(F) \le M(\delta)$. It is known that $M(\delta) \le (C/\delta)(1 + |\log C\delta|)$, as $\delta \to 0$ [Pis, Theorem 8.4].

Theorem 2.1 *Suppose that E is λ essentially Euclidean, $0 < \alpha < 1$, and Q is a quotient mapping from E onto a space F. Let $n = \dim E$ and assume that $\dim F = \alpha n$. Then F is γ essentially Euclidean, where $\gamma(\epsilon) \le \lambda(\epsilon\alpha/4)M(\epsilon/4)$; in fact, for each $\epsilon > 0$ there is a subspace E_2 of E and operators $A\colon \ell_2^{(1-\epsilon)\alpha n} \to E_2$ and $B\colon QE_2 \to \ell_2^{(1-\epsilon)\alpha n}$ such that BQA is the identity on $\ell_2^{(1-\epsilon)\alpha n}$ and $\|A\| \cdot \|B\| \le \lambda(\epsilon\alpha/4)M(\epsilon/4)$.*

Proof Set $n := \dim E$ and fix $0 < \epsilon < 1$. Let R be a quotient mapping from F onto a space G that has a subspace G_2 of dimension $(1-\epsilon/4)\alpha n$ such that $c_2(G_2) \le M(\epsilon/4)$. We want to find a subspace E_2 of E with $\dim E_2 \ge (1-\epsilon)\alpha n$ so that $RQE_2 \subset G_2$ and RQ is a "good" isomorphism on E_2, which implies that Q is also a "good" isomorphism on E_2. Since $\|R\| = \|Q\| = 1$, "good" means that $\|RQx\|$ is bounded away from zero for x in the unit sphere of E_2. Since E is λ essentially Euclidean, there is a subspace E_0 of E with $\dim E_0 = (1-\alpha\epsilon/4)n$ such that $c_2(E_0) \le \lambda(\alpha\epsilon/4)$. Put Euclidean norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E_0 and G_2, respectively, to satisfy for all $x \in E_0$ and all $y \in G_2$ the inequalities

$$\text{(2.1)} \qquad \|x\| \le \|x\|_1 \le \lambda(\alpha\epsilon/4)\|x\| \quad \text{and} \quad M(\epsilon/4)^{-1}\|y\| \le \|y\|_2 \le \|y\|.$$

Define $E_1 := E_0 \cap (RQ)^{-1}G_2$ so that $\dim E_1 := m \ge (1-\epsilon\alpha/2)n$.

Now take an orthonormal basis $e_1, e_2, \ldots, e_m$ for the Euclidean space $(E_1, \|\cdot\|_1)$ so that $RQe_1, RQe_2, \ldots, RQe_m$ is orthogonal in the Euclidean space $(G_2, \|\cdot\|_2)$ and ordered so that $\|RQe_1\|_2, \|RQe_2\|_2, \ldots, \|RQe_m\|_2$ is decreasing. We next compute that $\|RQe_j\|_2$ is large for $j := (1-\epsilon)\alpha n$. Now $\|RQe_j\|_2$ is the norm of the restriction to $E_3 := [e_j, e_{j+1}, \ldots, e_{(1-\alpha\epsilon/2)n}]$ of the operator RQ when it is considered as an operator from the Euclidean space $(E_1, \|\cdot\|_1)$ to the Euclidean space $(G_2, \|\cdot\|_2)$, and $\dim E_3 = m - j + 1 \ge n\alpha\epsilon/2 + 1$, which is strictly larger than the dimension of the kernel of RQ, because it has dimension at most $(1-\alpha)n + \epsilon\alpha n/4$. By the definition of quotient norms, the norm of $RQ_{|E_3}$ when RQ is considered as an operator from E_1 to G_2 under their original norms is the maximum over points x in the unit sphere of E_3 of the distance from x to the kernel of RQ. By a well-known consequence of the Borsuk–Ulam antipodal mapping theorem (first observed in [KKM]; see also [Day]), this distance is one. In view of the relationship (2.1), we deduce that $\|RQe_j\|_2 \geq \lambda(\alpha\epsilon/4)^{-1}M(\epsilon/4)^{-1}$. Also by (2.1), the norm of RQ is at most one when considered as an operator from $(E_1, \|\cdot\|_1)$ to $(G_2, \|\cdot\|_2)$. Finally, set $E_2 := [e_1, e_2, \dots, e_j]$ and let U_1 be the restriction to E_2 of RQ, considered as a mapping onto RQE_2. We have just shown that the identity on ℓ_2^j factors through U_1 with factorization constant at most $\lambda(\alpha\epsilon/4)^{-1}M(\epsilon/4)^{-1}$, hence it factors with the same constant through the restriction of Q to E_2, considered as an operator from E_2 to QE_2. ∎

Theorem 2.2 gives an improvement of the qualitative version of Theorem 2.1 when $E = \ell_p^n$, $1 \leq p < 2$. For $S \subset \{1, \dots, n\}$, let ℓ_p^S be the span in ℓ_p^n of the unit vector basis elements $\{e_i : i \in S\}$.

Theorem 2.2 *There is a function $g\colon (0,1)^2 \to (1,\infty)$ so that for all $1 \leq p < 2$, all natural numbers n, and all $\epsilon \in (0,1)$, the following is true. If Q is a quotient mapping from ℓ_p^n onto a normed space F and $\dim F = \alpha n$, then there is a subset S of $1, 2, \dots, n$ of cardinality $(1-\epsilon)\alpha n$ such that $\|(Q_{|\ell_p^S})^{-1}\| \leq g(\alpha, \epsilon)$.*

Sketch of proof The main point is the observation made in [JS, Theorem 2.1] that the proof of [BKT, Theorem 2.1] by Bourgain, Kalton, and Tzafriri shows that there is a constant $c > 0$ so that if Q is a quotient mapping from ℓ_p^n, $1 \leq p < 2$, onto a space of dimension at least βn, then there is a subset S of $1, 2, \dots, n$ of cardinality at least $c^{1/\beta} n$ so that $\|(Q_{|\ell_p^S})^{-1}\| \leq c^{-1/\beta}$. Given a quotient mapping Q on ℓ_p^n whose range has dimension αn and given $0 < \epsilon < 1$, apply the observation iteratively with $\beta := (1-\epsilon)\alpha$. At step one set $Q_1 := Q$ and get $S_1 \subset \{1, 2, \dots, n\}$ of cardinality at least $c^{1/\beta} n$ so that $\|((Q_1)_{|\ell_p^{S_1}})^{-1}\| \leq c^{-1/\beta}$. At step two take the quotient mapping Q_2 on ℓ_p^n whose kernel is the span of the kernel of Q_1 and $\{e_i\}_{i\in S_1}$ and get $S_2 \subset \{1, 2, \dots, n\}$ of cardinality at least $c^{1/\beta} n$ so that $\|((Q_2)_{|\ell_p^{S_2}})^{-1}\| \leq c^{-1/\beta}$. Necessarily, S_1 and S_2 are disjoint. More importantly, from the definition of the norm in a quotient space we see that in $Q\ell_p^n$, the norm of the projection from $Q[e_i]_{i\in S_1\cup S_2}$ onto $Q[e_i]_{i\in S_1}$ that annihilates $Q[e_i]_{i\in S_1}$ is controlled by $c^{-1/\beta}$, which implies that the norm of $(Q_{|\ell_p{}^{S_1\cup S_2}})^{-1}$ is also controlled. Then let Q_3 be the quotient mapping on ℓ_p^n whose kernel is the span of the kernel of Q and $\{e_i\}_{i\in S_1\cup S_2}$ and use the observation to get S_3. The iteration stops once the dimension of the kernel of Q_k is larger than $(1-\beta n)$, which happens after fewer than $c^{-1/\beta}$ steps; say, after k steps. By construction you can estimate the basis constant of $(Q[e_i]_{i\in S_m})_{m=i}^{k-1}$, so that Q will be a good isomorphism on $[e_i : i \in \cup_{m=1}^{k-1} S_m]$, because it is a good isomorphism on each $[e_i : i \in S_m]$ for $1 \leq m < k$. ∎

Remark 2.3. It is interesting to have the best estimates for γ in Theorem 2.1 and for g in Theorem 2.2. In Theorem 2.1 we gave the estimate for $\gamma(\epsilon)$ that the method gives and we think that this might be the order of the best estimate. We did not do the same in Theorem 2.2, because we think that a different argument is probably needed to obtain the best estimate for $g(\alpha, \epsilon)$.

https://doi.org/10.4153/CMB-2017-038-x Published online by Cambridge University Press

References

[BKT] J. Bourgain, N. J. Kalton, and L. Tzafriri, *Geometry of finite-dimensional subspaces and quotients of L_p.* In: Geometric aspects of functional analysis (1987–88), Lecture Notes in Math., 1376, Springer, Berlin, 1989, pp. 138–175. http://dx.doi.org/10.1007/BFb0090053

[Day] M. M. Day, *On the basis problem in normed spaces.* Proc. Amer. Math. Soc. **13**(1962), 655–658. http://dx.doi.org/10.1090/S0002-9939-1962-0137987-7

[JS] W. B. Johnson and G. Schechtman, *Very tight embeddings of subspaces of L_p, $1 \leq p < 2$, into ℓ_p^n.* Geom. Funct. Anal. **13**(2003), no. 4, 845–851. http://dx.doi.org/10.1007/s00039-003-0432-9

[KKM] M. G. Krein, D. P. Milman, and M. A. Krasnosel'ski, *On the defect numbers of linear operators in Banach space and some geometric questions.* (Russian) Sbornik Trudov Inst. Acad. NAUK Uk. SSR **11**(1948), 97–112.

[LMT-J] A. E. Litvak, V. D. Milman, and N. Tomczak-Jaegermann, *Essentially-Euclidean convex bodies.* Studia Math. **196**(2010), no. 3, 207–221. http://dx.doi.org/10.4064/sm196-3-1

[Mil] V. D. Milman, *Almost Euclidean quotient spaces of subspaces of a finite-dimensional normed space.* Proc. Amer. Math. Soc. **94**(1985), no. 3, 445–449. http://dx.doi.org/10.1090/S0002-9939-1985-0787891-1

[Pis] G. Pisier, *The volume of convex bodies and Banach space geometry.* Cambridge Tracts in Mathematics, 94, Cambridge University Press, Cambridge, 1989. http://dx.doi.org/10.1017/CBO9780511662454

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