Countable vector lattices

Paul F. Conrad

In his paper "On the structure of ordered real vector spaces" (Publ. Math. Debrecen 4 (1955-56), 334-343), Erdős shows that a totally ordered real vector space of countable dimension is order isomorphic to a lexicographic direct sum of copies of the group of real numbers. Brown, in "Valued vector spaces of countable dimension" (Publ. Math. Debrecen 18 (1971), 149-151), extends the result to a valued vector space of countable dimension and greatly simplifies the proof. In this note it is shown that a finite valued vector lattice of countable dimension is order isomorphic to a direct sum of σ-simple totally ordered vector spaces. One obtains as corollaries the result of Erdős and the applications that Brown makes to totally ordered spaces.

1. Notation and the statement of the main result

Throughout let \( G \) be a vector lattice over a totally ordered division ring \( D \). Then \( G \) is an abelian lattice ordered group that is also a left vector space over \( D \), and

\[
0 < d \in D, \quad 0 < g \in G \implies 0 < dg \in G.
\]

In particular, for a fixed \( 0 < d \in D \), the mapping \( g \mapsto dg \) for all \( g \in G \) is an \( \mathbb{R} \)-automorphism of \( G \). For a proof of this and the following assertions, see [3] or [5].

An \( \mathbb{R} \)-ideal is a convex \( \mathbb{R} \)-subspace of \( G \). If the order on \( G \) is archimedean, then each convex \( \mathbb{R} \)-subgroup of \( G \) is a subspace and hence an \( \mathbb{R} \)-ideal. A value of an element \( g \in G \) is an \( \mathbb{R} \)-ideal \( M \) that is maximal without \( g \). Let \( M \) be a value of \( g \) and \( M^e \) the intersection of all

Received 30 January 1974.

371
$\mathcal{L}$-ideals of $G$ that properly contain $M$. Then $g \in M^* \setminus M$ and so $M^*$ covers $M$. Moreover, $G/M$ is totally ordered with minimal convex subspace $M^*/M$. In particular, $M^*/M$ is $\omega$-simple (that is, contains no proper $\mathcal{L}$-ideals). An element of $G$ with only one value is called special. An element with a finite number of values has a unique representation as the sum of a finite number of disjoint special elements. $G$ is said to be finite-valued if each element has only finitely many values.

Let $\{G_{\gamma} \mid \gamma \in \Gamma\}$ be the set of all values of elements in $G$ and for each $\gamma \in \Gamma$, let $G_{\gamma}$ be the $\mathcal{L}$-ideal of $G$ which covers $G_{\gamma}$. Then $\Gamma$ has a natural partial order: $\alpha < \beta$ if $G_{\alpha} \subset G_{\beta}$. Let $\Sigma = \left\{\Gamma, G_{\gamma}/G_{\gamma}\right\}$ be the direct sum of the $\omega$-simple ordered vector spaces $G_{\gamma}/G_{\gamma}$. A component $v_{\gamma}$ of $v \in \Sigma$ is maximal if $v_{\gamma} \neq 0$ and $v_{\alpha} = 0$ for all $\alpha > \gamma$, $\alpha \in \Gamma$. If $0 \neq v \in \Sigma$, then $v > 0$ if all the maximal components of $v$ are positive in $G_{\gamma}/G_{\gamma}$. Then $\Sigma$ is a vector lattice over $D$. The following is the main result of this paper.

**Theorem.** If $G$ is a finite valued vector lattice of countable dimension over an ordered division ring $D$, then $G$ and $\Sigma(\Gamma, G_{\gamma}/G_{\gamma})$ are isomorphic as vector lattices.

For a non-archimedean ordered division ring $D$ the spaces $G_{\gamma}/G_{\gamma}$ can be arbitrary: for each vector space over $D$ can be totally ordered so that it is $\omega$-simple (see [2]).

**Corollary 1.** If, in addition to the above hypotheses, $D$ is archimedean, then $G \cong \Sigma(\Gamma, R_{\gamma})$, where each $R_{\gamma}$ is a subgroup of the totally ordered group $R$ of reals.

**Proof.** Since $D$ is archimedean, we may assume that $D$ is a subfield of the real field. Then each $G_{\gamma}/G_{\gamma}$ has no proper convex subgroups and so is $\omega$-isomorphic to a subgroup $R_{\gamma}$ of $R$. Clearly $\Sigma(\Gamma, G_{\gamma}/G_{\gamma}) \cong \Sigma(\Gamma, R_{\gamma})$, and hence $G \cong \Sigma(\Gamma, R_{\gamma})$.  

https://doi.org/10.1017/S000497270004106X Published online by Cambridge University Press
COROLLARY 2. If \( D = R \), then \( G \cong \Sigma(\Gamma, R) \).

Thus if \( G \) is a real vector lattice with countable dimension, then \( G \cong \Sigma(\Gamma, R) \) if and only if \( G \) is finite valued, and if \( G \) is a finite valued real vector lattice with countable \( \Gamma \), then \( G \cong \Sigma(\Gamma, R) \) if and only if \( G \) has countable dimension.

COROLLARY 3. If \( H \) is a countable finite valued abelian \( \ell \)-group then its divisible hull \( G \) is \( \ell \)-isomorphic to \( \Sigma(\Gamma, R_\gamma) \), where each \( R_\gamma \) is a countable divisible subgroup of \( R \).

Proof. \( G \) is a countable rational vector lattice.

Note that if \( \{H_\gamma \mid \gamma \in \Gamma\} \) is the set of convex \( \ell \)-subgroups of \( H \) that are maximal without elements of \( H \), then \( G^\gamma/G_\gamma \) is the divisible hull of \( H^\gamma/H_\gamma \) for each \( \gamma \in \Gamma \). Thus if each \( H^\gamma/H_\gamma \) is a rank one group, then \( G \cong \Sigma(\Gamma, Q) \), where \( Q \) is the group of rationals.

2. The concept of an \( I \)-set

A subset \( S \) of \( G \) consisting of special elements is called an \( I \)-set if \( s_1, s_2, \ldots, s_n \in S \) have the same value \( \gamma \) and \( \sum_{i=1}^{n} d_i s_i \in G_\gamma \) for \( d_1, \ldots, d_n \in D \) implies \( d_1 = d_2 = \ldots = d_n = 0 \). Thus \( \{G_\gamma s_i \mid 1 \leq i \leq n\} \) is an independent subset of \( G^\gamma/G_\gamma \).

LEMMA 1. An \( I \)-set \( S \) is an independent subset of \( G \).

Proof. Suppose that \( s_1, \ldots, s_n \in S \) and \( d_1, \ldots, d_n \in D \) with \( \sum_{i=1}^{n} d_i s_i = 0 \). We may assume that \( d_i \neq 0 \), \( 1 \leq i \leq n \) and that the value \( \gamma \) of \( s_1 \) is maximal among the values of \( s_1, \ldots, s_n \). Then if \( \alpha \) is a value of \( s_i \), \( \alpha \neq \gamma \). Thus if \( s_1, \ldots, s_k \) are the elements with value \( \gamma \), \( \sum_{i=1}^{k} d_i s_i \in G_\gamma \) and hence \( d_1 = \ldots = d_k = 0 \).
LEMMA 2. For a vector lattice $G$ the following are equivalent:

(a) $G$ has an I-set for a basis;

(b) $G \cong \Sigma\left(\Gamma, G^\gamma/G^\gamma\right)$.

Proof. (b) $\Rightarrow$ (a) Clearly $\Sigma$ has an I-set for a basis and hence so does $G$.

(a) $\Rightarrow$ (b) Let $S$ be an I-set and a basis of $G$. If $s \in S$ with value $\gamma$, then define $\pi : S \to \Sigma\left(\Gamma, G^\alpha/G^\alpha\right)$ by $(s\pi)_\alpha = 0$ if $\alpha \neq \gamma$; $(s\pi)_\gamma = G_\gamma + s$. By linearity $\pi$ can be extended to a $D$-homomorphism $\sigma$ of $G$ into $\Sigma = \Sigma\left(\Gamma, G^\gamma/G^\gamma\right)$. We shall show that $\sigma$ is an $\ell$-isomorphism of $G$ onto $\Sigma$.

If $g \in G$ is a special element with value $\gamma$, then $g$ has a unique representation, $g = \sum_{i=1}^{n} d_i s_i$, with $0 \neq d_i \in D$, $s_i \in S$. Since $S$ is an I-set, it follows that each $s_i \in G^\gamma$ and hence $g\sigma \in \Sigma^\gamma = \{v \in \Sigma \mid v_\alpha = 0 \text{ if } \alpha > \gamma\}$. Without loss of generality, $s_1, \ldots, s_t \in G^\gamma \setminus G^\gamma$, $s_{t+1}, \ldots, s_n \in G^\gamma$. Thus

$$(g\sigma)_\gamma = \sum_{i=1}^{t} G_\gamma + d_is_i = G_\gamma + g.$$ 

Therefore $g\sigma$ is special with maximal component $(g\sigma)_\gamma = G_\gamma + g$, and if $g$ is positive, so is $g\sigma$. Also note that

$$\left(\sum_{i=1}^{t} d_i s_i\right)_{\alpha} = \begin{cases} G_{\gamma} \cdot g & \text{if } \alpha = \gamma, \\ G_{\alpha} & \text{if } \alpha \neq \gamma. \end{cases}$$

Thus it follows that $\sigma$ is a homomorphism of $G$ onto $\Sigma$.

Now consider an arbitrary element $0 \neq g \in G$. Then $g = g_1 + \ldots + g_n$ where the $g_i$'s are disjoint and special and hence $g\sigma = g_1\sigma + \ldots + g_n\sigma$ where the $g_i\sigma$'s are disjoint and special. In
particular, \( \sigma \) is an isomorphism. But \( g \vee 0 \) is the sum of the positive \( g_i \)'s and \( g \sigma \vee 0 \) is the sum of the positive \( g_i \sigma \)'s. Therefore 
\[(g \vee 0) \sigma = g \sigma \vee 0 \] and so \( \sigma \) is an \( L \)-isomorphism of \( G \) onto \( \Sigma \).

3. Proof of the theorem

Let \( G \) be a finite valued vector lattice with countable dimension. For a subset \( T \) of \( G \), \((T)\) will denote the subspace of \( G \) generated by \( T \). Let \( \{b_1, b_2, \ldots\} \) be a basis of \( G \) consisting of special elements.

Suppose that \( A = \{a_1, \ldots, a_m\} \) is an \( I \)-set such that 
\[\langle a_1, \ldots, a_m \rangle \supseteq \langle b_1, \ldots, b_n \rangle.\] We show that \( A \) can be extended to an \( I \)-set \( A' = \{a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+t}\} \) so that 
\[\langle a_1, \ldots, a_{m+t} \rangle \supseteq \langle b_1, \ldots, b_n, b_{n+1} \rangle.\] Thus it will follow that \( G \) has an \( I \)-set for a basis and so by Lemma 2, \( G \cong \Sigma \langle \Gamma, G/\mathcal{G} \rangle \).

If \( b_{n+1} \in \langle a_1, \ldots, a_m \rangle \), or if \( \{a_1, \ldots, a_m, b_{n+1}\} \) is an \( I \)-set then the result follows. Suppose that this is not the case and let 
\[Y = \langle a_1, \ldots, a_m, b_{n+1} \rangle.\] Since \( \{a_1, \ldots, a_m\} \) is an \( I \)-set and 
\(\{a_1, \ldots, a_m, b_{n+1}\} \) is not, we may assume that \( a_1, \ldots, a_s, b_{n+1} \) have the same value, \( G_y \). And there exist \( d_1, \ldots, d_s, d \in D \) with \( d \neq 0 \) such that 
\[\sum_{i=1}^s d_i a_i + d b_{n+1} \in G_y.\] Now \( \sum_{i=1}^s \left(\frac{d_i}{d}\right)a_i + b_{n+1} = x \) can be written as 
\[x = x_1 + \ldots + x_k, \] where \( x_i \)'s are disjoint special elements. Then the values of \( x \) are the values of the \( x_i \)'s and so the value of \( x_i \) is contained in \( G_y \), for each \( i \), \( 1 \leq i \leq k \). If 
\(\{a_1, \ldots, a_m, x_1, \ldots, x_k\} \) is an \( I \)-set, then 
\[\langle a_1, \ldots, a_m, x_1, \ldots, x_k \rangle \supseteq Y \supseteq \langle b_1, \ldots, b_n, b_{n+1} \rangle \] and so we are done. If not, then again without loss of generality, 
\(a_{s+1}, \ldots, a_{s+r}, x_1\) have the same value \( \alpha \), and there exist 
\(d_1', \ldots, d_r' \), \(d' \in D \), \(d' \neq 0 \) such that 
\[\sum_{i=1}^r d_i' a_{s+i} + d' x_1 \in G_\alpha.\] Let
\[ x' = \Sigma (d'_i/d')a_{e+i} + x_1. \]

Repeating the above argument, several times as necessary, there exists \( y \in Y \) such that \( y = y_1 + \ldots + y_t \), where \( y_1, \ldots, y_t \) are disjoint special, \( \{a_1, \ldots, a_m, y_1, \ldots, y_t\} \) is an \( I \)-set, and

\( (a_1, \ldots, a_m, y_1, \ldots, y_t) \supseteq (a_1, \ldots, a_m, b_{n+1}) \supseteq (b_1, \ldots, b_n, b_{n+1}). \)

References


Department of Mathematics,
University of Kansas,
Lawrence, Kansas,
USA.