EXTENDED QUANTUM ENVELOPING ALGEBRAS OF $\mathfrak{sl}(2)$

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Abstract. In present paper we define a new kind of quantized enveloping algebra of $\mathfrak{sl}(2)$. We denote this algebra by $U_{r,t}$, where $r, t$ are two non-negative integers. It is a non-commutative and non-cocommutative Hopf algebra. If $r = 0$, then the algebra $U_{r,t}$ is isomorphic to a tensor product of the algebra of infinite cyclic group and the usual quantum enveloping algebra of $\mathfrak{sl}(2)$ as Hopf algebras. The representation of this algebra is studied.

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1. Introduction. Quantized enveloping algebras for Kac–Moody algebras were introduced independently by Drinfel’d and Jimbo [1, 3] in studying the quantum Yang–Baxter equation and two-dimensional solvable lattice models. There is a rich mathematical theory developed for these objects and their representations with connections to many areas of both mathematics and physics.

Suppose the Kac–Moody algebra is $\mathfrak{sl}(2)$. Then the usual quantum enveloping algebra is generated by $E, F, K, K^{-1}$. The four generators satisfy some relations. We obtain the extended quantum enveloping algebra $U_{r,t}$ of $\mathfrak{sl}(2)$ by adding new generators $J, J^{-1}$. $U_{r,t}$ is an algebra generated as an algebra over a field by six generators $E, F, K, K^{-1}, J, J^{-1}$. They satisfy the following relations:

\begin{align}
K^{-1}K &= KK^{-1} = JJ^{-1} = J^{-1}J = 1, \tag{1.1} \\
KEK^{-1} &= q^2E, \tag{1.2} \\
KFK^{-1} &= q^{-2}F, \tag{1.3} \\
EF - FE &= \frac{K - K^{-1}Jr}{q - q^{-1}}. \tag{1.4}
\end{align}

This algebra can be obtained from the weak quantum enveloping algebra of $\mathfrak{sl}(2)$ defined in [11]. We can introduce co-multiplication and counit on the $U_{r,t}$ to make it into a Hopf algebra. It is a non-commutative and non-cocommutative Hopf algebra. If $r = 0$, then the algebra $U_{r,t}$ is isomorphic to a tensor product of the algebra of an infinite cyclic group and the usual quantum enveloping algebra of $\mathfrak{sl}(2)$ as Hopf algebras. We will study the representation of this algebra in this paper.

Let us outline the structure of this paper. In Section 2, we give the definition of $U_{r,t}$ and obtain some properties of $U_{r,t}$. For example, we prove that $U_{r,t}$ is a Noetherian domain, a Hopf algebra. In Section 3, we study the representation of $U_{r,t}$. Using the theory developed in Section 3, we character the centre of $U_{r,t}$ in Section 4. Unlike the representation theory of usual quantum enveloping $U_q(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$, there exist...
finite-dimensional non-semisimple $U_{r,t}$-modules. But we can prove that the tensor product of two simple $U_{r,t}$-modules is semisimple, in Section 5. We also obtain a decomposition theory about the tensor product of two simple $U_{r,t}$-modules. In Section 6, we briefly discuss the representation of $U_{r,t}$ in the case where $q$ is a root of unity. In Section 7, we use the $U_{r,t}$ to construct a Hopf algebra with dimension $le^3$ for any positive integers $l, e$, where $e \geq 2$.

Throughout this paper $k$ is a fixed algebraically closed field with characteristic zero; $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ is the set of all integers. For the other undefined terms we refer to [5–7, 9].

2. The definition of $U_{r,t}$ and its basic properties. In this section, we will define the extended quantum enveloping algebra $U_{r,t}$ of the Lie algebra $\mathfrak{sl}(2)$ and study its basic properties. Recall that the three matrices $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ consist of a basis of $\mathfrak{sl}(2)$. Before giving the definition of extended quantum enveloping algebra of $\mathfrak{sl}(2)$, we introduce some notations first. Let us fix two indeterminates $q, J$.

For any integer $n$, set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{n+3} + q^{n+1}.$$

We have the following version of factorials and binomial coefficients. For integers $0 \leq k \leq n$, set $[0]! = 1$, $[k]! = [1][2] \cdots [k]$, if $k > 0$, and

$$\binom{n}{k} = \frac{[n]!}{[k]![n-k]!}.$$

With this new notation we can prove the following proposition by induction:

**Lemma 2.1.** If $x$ and $y$ are variables subject to the relation $yx = q^2 xy$, then

$$(x + y)^n = \sum_{k=0}^{n} q^{(n-k)k} \binom{n}{k} x^k y^{n-k}$$

for any positive integer $n$.

Let $k$ be an algebraically closed field with characteristic zero. We use $k_q$ to denote the fraction field of the domain $k[q, q^{-1}]$.

**Definition 2.2.** Let $r, t$ be two fixed non-negative integers. We define $U_{r,t} = U_{r,t}(\mathfrak{sl}(2))$ as the $k_q$-algebra generated by six variables $E, F, K, K^{-1}, J, J^{-1}$, where $J$ and $J^{-1}$ are in the centre of $U_{r,t}$, with the relations

$$K^{-1} K = KK^{-1} = JJ^{-1} = J^{-1}J = 1,$$

$$KEK^{-1} = q^2 E,$$

$$KFK^{-1} = q^{-2} F,$$

$$EF - FE = \frac{K - K^{-1}J^r}{q - q^{-1}}.$$
From the definition, we can prove that there is an algebra automorphism \( \omega_s \) of \( U_{r,t} \) such that\( \omega_s(E) = FJ^s, \omega_s(F) = EJ^{-s}, \omega_s(K) = K^{-1}J^r, \omega_s(K^{-1}) = KJ^{-r}, \omega_s(J) = J, \omega_s(J^{-1}) = J^{-1} \) for any integer \( s \). Moreover, we have the following proposition:

**Proposition 2.1.** There exists a unique algebra anti-automorphism \( \omega \) of \( U_{r,t} \) such that \( \omega(E) = KF, \omega(F) = EK^{-1}, \omega(K) = K, \omega(K^{-1}) = K^{-1}, \omega(J) = J, \omega(J^{-1}) = J^{-1} \).

**Proof.** To show this proposition, we only need to check the following relations:

\[
\omega(K)\omega(E) = q^{-2}\omega(E)\omega(K), \quad \omega(K)\omega(F) = q^2\omega(F)\omega(K),
\]

\[
[\omega(F), \omega(E)] = \frac{\omega(K) - \omega(K^{-1})\omega(J^r)}{q - q^{-1}} = \frac{K - K^{-1}J^r}{q - q^{-1}}.
\]

The first two relations result directly from definition. We compute the third one as

\[
[\omega(F), \omega(E)] = EK^{-1}KF - KFEK^{-1} = EF - FE = \frac{K - K^{-1}J^r}{q - q^{-1}},
\]

by relations (2.2) and (2.3). □

**Lemma 2.3.** Let \( m \geq 0 \), and \( n \in \mathbb{Z} \). The following relations hold in \( U_{r,t} \):

\[
E^mK^n = q^{-2mn}K^nE^m, \quad F^mK^n = q^{2mn}K^nF^m, \tag{2.5}
\]

\[
EF^m - F^mE = [m]F^{m-1}q^{(m-1)}K - q^{m-1}K^{-1}J^r
\]

\[
= [m]q^{m-1}K - q^{-(m-1)}K^{-1}J^r F^{m-1}, \tag{2.6}
\]

\[
E^mF - FE^m = [m]q^{-(m-1)}K - q^{-m}K^{-1}J^r E^{m-1}
\]

\[
= [m]q^{-m}K - q^{-(m+1)}K^{-1}J^r E^{m-1}. \tag{2.7}
\]

**Proof.** The first two relations result trivially from relations (2.2) and (2.3). The third one is proved by induction on \( m \) using

\[
[E, F^m] = [E, F^{m-1}]F + F^{m-1}[E, F].
\]

Similarly, we can prove (2.7). □

**Theorem 2.4.** The algebra \( U_{r,t} \) is Noetherian and has no zero divisor. The set \( \{E^I F^J K^l J^s \}_{I,J \in \mathbb{N}, l,s \in \mathbb{Z}} \) is a basis of \( U_{r,t} \).

**Proof.** Let \( A_0 = k_q[K, K^{-1}, J, J^{-1}] \). Since \( A_0 \) is a homomorphic image of a Noetherian algebra, it is a Noetherian algebra. Moreover, the family \( \{K^l J^s \}_{l,s \in \mathbb{Z}} \) is a basis of \( A_0 \).
Consider the automorphism $\alpha_1$ of $A_0$ determined by $\alpha_1(\mathcal{K}) = q^2\mathcal{K}$, $\alpha_1(\mathcal{J}) = \mathcal{J}$ and the corresponding Ore extension $A_1 = A_0[\mathcal{F}, \alpha_1, 0]$; the latter has a basis consisting of the monomials $\{\mathcal{F}^i \mathcal{K}^j \mathcal{J}^s | i \in \mathbb{N}, l, s \in \mathbb{Z}\}$.

It is easy to prove that $A_1$ is the algebra generated by $\mathcal{F}$, $\mathcal{F}^{-1}$, $\mathcal{K}$, $\mathcal{K}^{-1}$, $\mathcal{J}$, $\mathcal{J}^{-1}$ and the relations

\[ \mathcal{F} \mathcal{K} = q^2 \mathcal{F} \mathcal{K}, \quad \mathcal{F} \mathcal{J} = \mathcal{J} \mathcal{F}. \]

Define

\[ \alpha(\mathcal{F}^i \mathcal{K}^j \mathcal{J}^s) = q^{-2i} \mathcal{F}^i \mathcal{K}^j \mathcal{J}^s, \quad \delta(\mathcal{K}^l) = \delta(\mathcal{J}^l) = 0, \]

\[ \delta(\mathcal{F}^i \mathcal{K}^j \mathcal{J}^s) = \sum_{i=0}^{j-1} \mathcal{F}^{j-1} \delta(\mathcal{F})(q^{-2i} \mathcal{K}) \mathcal{K}^j \mathcal{J}^s, \]

where $\delta(\mathcal{F})(q^{-2i} \mathcal{K}) = q^{2i} (q^{-1} - q^{-2i} \mathcal{K})^{-1}$, and $j \geq 1$. We claim that $\delta$ extends to an $\alpha$-derivation of $A_1$. We must check that for all $j, m \in \mathbb{N}$, and $l_1, l_2, s_1, s_2 \in \mathbb{Z}$, we have

\[ \delta(\mathcal{F}^i \mathcal{K}^{l_1} \mathcal{J}^{s_1} \cdot \mathcal{F}^m \mathcal{K}^{l_2} \mathcal{J}^{s_2}) = \alpha(\mathcal{F}^i \mathcal{K}^{l_1} \mathcal{J}^{s_1}) \delta(\mathcal{F}^m \mathcal{K}^{l_2} \mathcal{J}^{s_2}) + \delta(\mathcal{F}^i \mathcal{K}^{l_1} \mathcal{J}^{s_1}) \mathcal{F}^m \mathcal{K}^{l_2} \mathcal{J}^{s_2}. \]

Let us compute the right-hand side of the above equation. We have

\[ \alpha(\mathcal{F}^i \mathcal{K}^{l_1} \mathcal{J}^{s_1}) \delta(\mathcal{F}^m \mathcal{K}^{l_2} \mathcal{J}^{s_2}) + \delta(\mathcal{F}^i \mathcal{K}^{l_1} \mathcal{J}^{s_1}) \mathcal{F}^m \mathcal{K}^{l_2} \mathcal{J}^{s_2} \]

\[ = q^{-2i} \mathcal{F}^i \mathcal{K}^{l_1} \mathcal{J}^{s_1} \mathcal{F}^m \mathcal{K}^{l_2} \mathcal{J}^{s_2} \]

\[ + \sum_{i=0}^{j-1} \mathcal{F}^{j-1} \delta(\mathcal{F})(q^{-2i} \mathcal{K}) \mathcal{K}^j \mathcal{J}^s \mathcal{F}^m \mathcal{K}^j \mathcal{J}^s \]

\[ = \sum_{i=0}^{m-1} q^{-2i} \mathcal{F}^{j+m-1} \delta(\mathcal{F})(q^{-2i} \mathcal{K}) \mathcal{K}^{l_1+\mathcal{J}^{s_1+s_2}} \]

\[ + \sum_{i=m}^{m+j-1} q^{-2i} \mathcal{F}^{j+m-1} \delta(\mathcal{F})(q^{-2i} \mathcal{K}) \mathcal{K}^{l_1+\mathcal{J}^{s_1+s_2}} \]

\[ = q^{-2i} \mathcal{F}^i \mathcal{K}^{l_1} \mathcal{J}^{s_1} \mathcal{F}^m \mathcal{K}^{l_2} \mathcal{J}^{s_2} \]

\[ = \delta(\mathcal{F}^i \mathcal{K}^{l_1} \mathcal{J}^{s_1} \cdot \mathcal{F}^m \mathcal{K}^{l_2} \mathcal{J}^{s_2}). \]

We now build an Ore extension $A_2 = A_1[\mathcal{E}, \alpha, \delta]$. Then the following relations hold in $A_2$:

\[ \mathcal{E} \mathcal{K} = \alpha(\mathcal{K}) \mathcal{E} + \delta(\mathcal{K}) = q^{-2} \mathcal{K} \mathcal{E}, \]

\[ \mathcal{E} \mathcal{J} = \alpha(\mathcal{J}) \mathcal{E} + \delta(\mathcal{J}) = \mathcal{J} \mathcal{E}, \]

and

\[ \mathcal{E} \mathcal{F} = \alpha(\mathcal{F}) \mathcal{E} + \delta(\mathcal{F}) = \mathcal{F} \mathcal{E} + \frac{\mathcal{K} - \mathcal{K}^{-1} \mathcal{J}}{q - q^{-1}}. \]
From these one easily concludes that $A_2$ is isomorphic to $U_{r,t}$. Then the properties of $U_{r,t}$ are warranted by the properties of the Ore extension.

To make the algebra $U_{r,t}$ into the Hopf algebra, we define the following three maps

$$
\Delta(E) = J^{-rt} \otimes E + E \otimes KJ^{rt}, \quad (2.12)
$$

$$
\Delta(F) = K^{-1} J^{rt(+1)} \otimes F + F \otimes J^{-rt}, \quad (2.13)
$$

$$
\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \quad (2.14)
$$

$$
\Delta(J) = J \otimes J, \quad \Delta(J^{-1}) = J^{-1} \otimes J^{-1}, \quad (2.15)
$$

$$
\varepsilon(K) = \varepsilon(K^{-1}) = \varepsilon(J) = \varepsilon(J^{-1}) = 1, \quad (2.16)
$$

$$
\varepsilon(E) = \varepsilon(F) = 0, \quad (2.17)
$$

and

$$
S(E) = -EK^{-1}, \quad S(F) = -KFJ^{-r}, \quad S(J) = J^{-1}, \quad (2.18)
$$

$$
S(J^{-1}) = J, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K. \quad (2.19)
$$

**Theorem 2.5.** Relations (2.12)–(2.19) endow $U_{r,t}$ with a Hopf algebra.

**Proof.** (a) We first show that $\Delta$ defines a morphism of algebras from $U_{r,t}$ into $U_{r,t} \otimes U_{r,t}$. It is enough to check that

$$
\Delta(K)\Delta(K^{-1}) = \Delta(K^{-1})\Delta(K) = 1 \otimes 1,
$$

$$
\Delta(J)\Delta(J^{-1}) = \Delta(J^{-1})\Delta(J) = 1 \otimes 1,
$$

$$
\Delta(K)\Delta(E)\Delta(K^{-1}) = q^2 \Delta(E),
$$

$$
\Delta(K)\Delta(F)\Delta(K^{-1}) = q^{-2} \Delta(F),
$$

$$
\Delta(E)\Delta(F) - \Delta(F)\Delta(E) = \frac{\Delta(K) - \Delta(K^{-1})\Delta(J^r)}{q-q^{-1}},
$$

and

$$
\Delta(X)\Delta(J) = \Delta(J)\Delta(X),
$$

for $X = E, F, K, K^{-1}$. We give a sample calculation for $\Delta(E)\Delta(F) - \Delta(F)\Delta(E) = \frac{\Delta(K) - \Delta(K^{-1})\Delta(J^r)}{q-q^{-1}}$ as follows:

$$
[\Delta(E), \Delta(F)] = (J^{-rt} \otimes E + E \otimes KJ^{rt})(K^{-1} J^{rt(+1)} \otimes F + F \otimes J^{-rt})
$$

$$
- (K^{-1} J^{rt(+1)} \otimes F + F \otimes J^{-rt})(J^{-rt} \otimes E + E \otimes KJ^{rt})
$$

$$
= K^{-1} J^r \otimes \frac{K - K^{-1} J^r}{q-q^{-1}} + \frac{K - K^{-1} J^r}{q-q^{-1}} \otimes K
$$

$$
= \frac{\Delta(K) - \Delta(K^{-1}) J^r}{q-q^{-1}}.
$$

(b) Next, we show that $\Delta$ is coassociative. It suffices to do it on the six generators. We give a sample calculation for $E$. On the one hand, we have

$$
(\Delta \otimes id)\Delta(E) = (\Delta \otimes id)(J^{-rt} \otimes E + E \otimes KJ^{rt})
$$

$$
= J^{-rt} \otimes J^{-rt} \otimes E + J^{-rt} \otimes E \otimes KJ^{rt} + E \otimes KJ^{rt} \otimes KJ^{rt}.
$$
On the other hand, we have
\[(id \otimes \Delta)\Delta(E) = (id \otimes \Delta)(J^{-rt} \otimes E + E \otimes KJ^{rt}) = J^{-rt} \otimes J^{-rt} \otimes E + J^{-rt} \otimes E \otimes KJ^{rt} + E \otimes KJ^{rt} \otimes KJ^{rt},\]
which is the same.

(c) It is easy to prove that \(\varepsilon\) defines a morphism of algebras from \(U_{r,t}\) to \(k_q\) and satisfies the counit axiom.

(d) It remains to see that \(S\) defines an antipode of \(U_{r,t}\). We have first to check that \(S\) is a morphism of algebras from \(U_{r,t}\) into \(U_{r,t}^{\text{opp}}\), namely the following relations hold:
\[S(K)S(K^{-1}) = S(K^{-1})S(K) = 1, \quad S(J)S(J^{-1}) = S(J^{-1})S(J) = 1,\]
\[S(K^{-1})S(E)S(K) = q^2 S(E), \quad S(K^{-1})S(F)S(K) = q^{-2} S(F),\]
\[[S(F), S(E)] = \frac{S(K) - S(K^{-1})S(J^r)}{q - q^{-1}}.\]  
(2.20)

and \(S(X)S(J) = S(J)S(X)\) for \(X = E, F, K, K^{-1}, J^{-1}\).
We only give the computation for (2.20). We have
\[[S(F), S(E)] = KFJ^{-r}EK^{-1} - EFJ^{-r} = (FE - EF)J^{-r} = \frac{S(K) - S(K^{-1})S(J^r)}{q - q^{-1}}.\]

It is easy to check that
\[\sum_{(x)} x_{(1)}S(x_{(2)}) = \sum_{(x)} S(x_{(1)})x_{(2)} = \varepsilon(x)1\]
holds when \(x\) is any of the generators \(E, F, K^{-1}, K, J, J^{-1}\). Since \(S\) is an antiautomorphism of \(U_{r,t}\), \(S\) is an antipode.

PROPOSITION 2.2. (1) If \(r = 0\), then \(U_{0,t}\) is isomorphic to \(k_q[\mathbb{Z}] \otimes U_q(\mathfrak{sl}(2))\) as Hopf algebras, where \(k_q[\mathbb{Z}]\) is the group algebra of infinite cyclic group \(\mathbb{Z}\), \(U_q(\mathfrak{sl}(2))\) is the usual quantum enveloping algebra of \(\mathfrak{sl}(2)\).
(2) We have \(S^2(u) = KuK^{-1}\) for any \(u \in U_{r,t}\).

Proof. Obvious. □

PROPOSITION 2.3. For all \(i, j \in \mathbb{N}\) and all \(l, s \in \mathbb{Z}\), we have
\[\Delta(E^iF^jK^lJ^r) = \sum_{u=0}^{i-j} \sum_{v=0}^{j} q^{(i-u)(j-v) - 2(i-u)(j-v)} \begin{bmatrix} i \cr u \end{bmatrix} \begin{bmatrix} j \cr v \end{bmatrix} \times (J^{(i+1)(j-v) - ru + s} \otimes J^{(i-u-v)+s})
\times (E^{i-u}F^u K^{l-j+v} \otimes E^v F^{j-v} K^{l+i-u}).\]
Proof. First observe that
\[
\Delta(E^i F^j K^l J^s) = \Delta(E^i) \Delta(F^j) \Delta(K^l) \Delta(J^s) \\
= (J^{-rt} \otimes E + E \otimes K J^r) (K^{-1} J^{-rt+1}) \otimes F + F \otimes J^{-rt} (K^l J^s \otimes K^l J^s).
\]

Since
\[
(J^{-rt} \otimes E) (E \otimes K J^r) = q^{-2} (E \otimes K J^r) (J^{-rt} \otimes E),
\]

\[
\Delta(E^i) = (J^{-rt} \otimes E + E \otimes K J^r)^i_i \\
= \sum_{u=0}^{i} q^{(i-u)} \left[ \begin{array}{c} i \\ u \end{array} \right] (J^{-rt} \otimes E)^i \otimes (E \otimes K J^r)^{i-u}
\]

\[
= \sum_{u=0}^{i} q^{(i-u)} \left[ \begin{array}{c} i \\ u \end{array} \right] (J^{-rt} \otimes E)^{i-u} \otimes (E \otimes K J^r)^{1 \otimes J^{i-u}},
\]

by Lemma 2.1. Similarly, we have
\[
\Delta(F^j) = (K^{-1} J^{-rt+1}) \otimes F + F \otimes J^{-rt}^j \\
= \sum_{v=0}^{j} q^{(j-v)} \left[ \begin{array}{c} j \\ v \end{array} \right] (F \otimes J^{-rt})^j \otimes (K^{-1} J^{-rt+1}) \otimes F
\]

\[
= \sum_{v=0}^{j} q^{(j-v)} \left[ \begin{array}{c} j \\ v \end{array} \right] (J^{r+1} \otimes J^{-rt})^j \otimes (F \otimes K^{-1} J^{-t} \otimes F
\]

\[
= \sum_{v=0}^{j} q^{(j-v)} \left[ \begin{array}{c} j \\ v \end{array} \right] (J^{r+1} \otimes J^{-rt})^j \otimes (F \otimes K^{-1} J^{-t} \otimes F
\]

Hence
\[
\Delta(E^i F^j K^l J^s) = \sum_{u=0}^{i} \sum_{v=0}^{j} q^{(i-u)+v(j-v)} \left[ \begin{array}{c} i \\ u \end{array} \right] \left[ \begin{array}{c} j \\ v \end{array} \right] \times (J^{-rt+1} \otimes J^{-rt+1}) \\
\times (E^{i-u} \otimes E^u K^{-1} \otimes K^{-1} J^{-t}) (F^v K^{-1} \otimes F^v) (K^l J^s \otimes K^l J^s)
\]

\[
= \sum_{u=0}^{i} \sum_{v=0}^{j} q^{(i-u)+v(j-v)} \left[ \begin{array}{c} i \\ u \end{array} \right] \left[ \begin{array}{c} j \\ v \end{array} \right] \times (J^{-rt+1} \otimes J^{-rt+1}) \\
\times (E^{i-u} \otimes E^u K^{-1} \otimes K^{-1} J^{-t}) (F^v K^{-1} \otimes F^v) (K^l J^s \otimes K^l J^s)
\]

By now the proof is completed. \qed

Finally in this section, we give some remarks.

Remark 2.6. Suppose \( G \) is an abelian group, and \( g, h \in G \) are two fixed elements. Then we can define a Hopf algebra \( U_{g,h} \) as follows:

1. As vector spaces \( U_{g,h} \) is isomorphic to the tensor product of \( k[G] \), the group algebra of \( G \) over the field \( k \), and \( U_q(sl(2)) \), the usual quantum enveloping algebra of
\( s(2) \), which is generated by four variables \( E, F, K, K^{-1} \). Any element of \( k[G] \) is in the centre of \( U_{g,h} \). The other generators satisfy the following relations:

\[
\begin{align*}
K^{-1}K &= KK^{-1} = 1, \quad (2.21) \\
KEK^{-1} &= q^2 E, \quad (2.22) \\
KFK^{-1} &= q^{-2} F, \quad (2.23) \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \quad (2.24)
\end{align*}
\]

(2) The other operations of Hopf algebra \( U_{g,h} \) are defined as follows:

\[
\Delta(E) = h^{-1} \otimes E + E \otimes hK \quad (2.25)
\]

\[
\Delta(F) = K^{-1}hg \otimes F + F \otimes h^{-1} \quad (2.26)
\]

\[
\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \quad (2.27)
\]

\[
\Delta(a) = a \otimes a, \quad a \in G, \quad (2.28)
\]

\[
\varepsilon(K) = \varepsilon(K^{-1}) = \varepsilon(a) = 1, \quad a \in G, \quad (2.29)
\]

\[
\varepsilon(E) = \varepsilon(F) = 0, \quad (2.30)
\]

and

\[
\begin{align*}
S(E) &= -EK^{-1}, \quad S(F) = -KFg^{-1}, \quad (2.31) \\
S(a) &= a^{-1}, \quad a \in G, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K. \quad (2.32)
\end{align*}
\]

**Remark 2.7.** By using the above method, we can construct extensions of quantum enveloping algebras of others Lie algebras (or Kac–Moody algebras [4]) by group algebras.

**Remark 2.8.** We can assume that \( q \) is an element of \( k \). If \( q^2 \neq 1 \), then \( U_{r,t} \) is a Hopf algebra over \( k \). In the remainder of this paper we always assume that \( q \) is an element in \( k \) and \( q^2 \neq 1 \).

**Remark 2.9.** One can study the dual algebra \( U^e_{r,t} \) of \( U_{r,t} \). In the case \( r = 0 \),

\[
U^e_{0,t} = \text{Hom}_k(U_{0,t}, k) \cong \text{Hom}_k(k[Z], U_q(sl(2))^e),
\]

by Proposition 2.2. Moreover, one can determine whether \( U_{r,t} \) is quasi-triangular or not.

### 3. The representation of \( U_{r,t} \)

In this section, let \( q \) be an element in the algebraically closed field \( k \) with characteristic zero. Moreover, we assume that \( q \) is not a root of unity. We shall determine all finite-dimensional simple \( U_{r,t} \)-modules in this section.

For any two elements \( \lambda, \alpha \in k \) and any \( U_{r,t} \)-module \( V \), we denote by

\[
V^{\lambda, \alpha} = \{ v \in V | Kv = \lambda v, Jv = \alpha^2 v \}.
\]

The pair \((\lambda, \alpha)\) is called a weight of \( V \) if \( V^{\lambda, \alpha} \neq 0 \).
**LEMMA 3.1.** We have $E V_{\lambda, \alpha}^\lambda \subseteq V_{q^2 \lambda, \alpha}^\lambda$ and $F V_{\lambda, \alpha}^\lambda \subseteq V_{q^{-2} \lambda, \alpha}^\lambda$.

**Proof.** For any $v \in V_{\lambda, \alpha}^\lambda$, we have

\begin{align*}
KEv &= q^2 EKv = q^2 \lambda Ev \\
JEv &= JEv = \alpha^2 Ev 
\end{align*}

and

\begin{align*}
KFv &= q^{-2} FKv = q^{-2} \lambda Fv \\
JFv &= FJv = \alpha^2 Fv
\end{align*}

So this lemma holds. \hfill \Box

**DEFINITION 3.2.** Let $V$ be a $\mathcal{U}_r$-module and $(\lambda, \alpha)$ is a pair of scalars. An element $v \neq 0$ of $V$ is the highest weight vector of weight $(\lambda, \alpha)$ if $E v = 0$, $K v = \lambda v$, and $J v = \alpha^2 v$. A $\mathcal{U}_r$-module is the highest weight module of highest weight $(\lambda, \alpha)$ if it is generated by the highest vector $v$ of weight $(\lambda, \alpha)$.

**PROPOSITION 3.1.** Any non-zero finite-dimensional $\mathcal{U}_r$-module contains a highest weight vector. Moreover the endomorphisms induced by $E$ and $F$ are nilpotent.

**Proof.** Since $k$ is algebraically closed, $V$ is finite-dimensional and $JK = KJ$, there exists a non-zero vector $w$ and $(\mu, \alpha)$ such that $K w = \mu w$, $J w = \alpha^2 w$.

If $E w = 0$, then the vector $w$ is the highest weight vector and we are done. If not, let us consider the sequence of vectors $E^n w$, where $n$ runs over the non-negative integers. According to Lemma 3.1, it is a sequence of eigenvectors with distinct eigenvalues. Consequently, there exists an integer $n$ such that $E^n w \neq 0$ and $E^{n+1} w = 0$. The vector $E^n w$ is the highest weight vector.

In order to prove that the action of $E$ on $V$ is nilpotent, it suffices to check that 0 is the only eigenvalue of $E$. Now, if $v$ is a non-zero eigenvector for $E$ with eigenvalue $\lambda \neq 0$, then so is $K^n v$ with eigenvalue $q^{-2n} \lambda$. The endomorphism $E$ would then have infinitely many distinct eigenvalues which is impossible. The same argument works for $F$. \hfill \Box

**LEMMA 3.3.** Let $v$ be a highest weight vector of weight $(\lambda, \alpha)$. Set $v_0 = v$ and $v_p = \frac{1}{[p]!} F_p v$ for $p > 0$. Then

\begin{align*}
K v_p &= q^{-2p} \lambda v_p, \\
J v_p &= \alpha^2 v_p, \\
F v_{p-1} &= [p] v_p, \\
E v_p &= \frac{q^{-(p-1)} \lambda - q^{p-1} \lambda^{-1} \alpha^{2p}}{q - q^{-1}} v_{p-1}.
\end{align*}

(3.1)
Proof. We only check equation (3.1). By Lemma 2.3, we have
\[
Ev_p = \frac{1}{[p]!} \left( F^p E + \frac{[p] F^{p-1} q^{-(p-1)} K - q^{p-1} K^{-1} J}{q - q^{-1}} \right) v_0 \\
= \frac{q^{-(p-1)} \lambda - q^{p-1} \lambda^{-1} \alpha^2}{q - q^{-1}} v_{p-1}.
\]

**Theorem 3.4.** (a) Let \( V \) be a finite-dimensional \( U_{r,t} \)-module generated by the highest weight vector \( v \) of weight \((\lambda, \alpha)\). Then

(i) \( \lambda = \epsilon q^n \alpha^n \), where \( \epsilon = \pm 1 \) and \( n \) is the integer defined by \( \dim V = n + 1 \).

(ii) Setting \( v_p = \frac{1}{[p]!} F^p v \), we have \( v_p = 0 \) for \( p > n \) and in addition the set \( \{ v = v_0, v_1, \ldots, v_n \} \) is a basis of \( V \).

(iii) The operator \( K \) acting on \( V \) is diagonalizable with \( (n + 1) \) distinct eigenvalues
\[
\{ \epsilon q^n \alpha^\prime, \epsilon q^{n+2} \alpha^\prime, \ldots, \epsilon q^n \alpha^\prime \},
\]
and the operator \( J \) acts on \( V \) by a scalar \( \alpha^2 \).

(iv) Any other highest weight vector in \( V \) is a scalar multiple of \( v \) and is of weight \((\lambda, \alpha)\).

(v) The module is simple.

(b) Any simple finite-dimensional \( U_{r,t} \)-module is generated by the highest weight vector. Two finite-dimensional \( U_{r,t} \)-modules generated by highest vectors of the same weight are isomorphic.

Proof. According to Lemma 3.3, the sequence \( \{ v_p | p \geq 0 \} \) is a sequence of eigenvectors for \( K \) with distinct eigenvalues. Since \( V \) is finite-dimensional, there is an integer \( n \) such that \( v_n \neq 0 \) and \( v_{n+1} = 0 \). Then from the formulas
\[
Ev_p = \frac{q^{-(p-1)} \lambda - q^{p-1} \lambda^{-1} \alpha^2}{q - q^{-1}} v_{p-1},
\]
we obtain \( v_m = 0 \) for all \( n > m \) and \( v_m \neq 0 \) for all \( m \leq n \). Moreover,
\[
0 = Ev_{n+1} = \frac{q^{-n} \lambda - q^n \lambda^{-1} \alpha^2}{q - q^{-1}} v_n.
\]
Hence \( \lambda^2 = q^{2n} \alpha^2 \), which is equivalent to \( \lambda = \epsilon q^n \alpha^\prime \). The rest of the proof of (i)–(iii) is easy. So we omit it.

(iv) Let \( v^\prime \) be another highest weight vector. It is an eigenvector for the action of \( K \) and \( J \); hence it is a scalar multiple of some vector \( v_i \). But the vector \( v_i \) is killed by \( E \) if and only if \( i = 0 \).

(v) Let \( V' \) be a non-zero \( U_{r,t} \)-submodule of \( V \) and let \( v^\prime \) be the highest weight vector of \( V' \). Then \( v^\prime \) also is the highest weight vector for \( V \). By (iv), \( v^\prime \) has to be a non-zero scalar multiple of \( v \). Therefore \( v \) is in \( V' \). Since \( v \) generates \( V \), we must have \( V = V' \), which proves that \( V \) is simple.

(b) By Proposition 3.1, any simple finite-dimensional \( U_{r,t} \)-module \( V \) contains a highest weight vector \( v \). Let \( V' \) be the submodule of \( V \) generated by \( v \). Since \( V \) is simple, \( V = V' \) and hence \( V \) is generated by the highest weight vector \( v \). The rest results of (b) follow from (a).

\( \square \)
We denote the \((n+1)\)-dimensional simple \(U_{r,t}\)-module-generated highest weight vector \(v\) by \(V_{\epsilon,n,\alpha}\), where \(v\) satisfies
\[
Ev = 0, \quad Jv = \alpha^2 v, \quad Kv = \epsilon q^n \alpha^r v.
\]

Let \(\rho_{\epsilon,n,\alpha}\) be the corresponding morphism of algebras from \(U_{r,t}\) to \(\text{End}(V_{\epsilon,n,\alpha})\).

Observe that the formulas of Lemma 3.3 may be rewritten as follows for \(V_{\epsilon,n,\alpha}\):
\[
Kv_p = \epsilon q^{n-2p} \alpha^r v_p, \quad Jv_p = \alpha^2 v_p, \quad Fv_{p-1} = [p]v_p,
\]
and
\[
Ev_p = \epsilon \frac{q^{n-(p-1)} \alpha^r - q^{p-1-n} \alpha^r}{q - q^{-1}} v_{p-1} = \epsilon \alpha' [n - p + 1] v_{p-1}.
\] (3.2)

As a special case, we have \(V_{\epsilon,0,\alpha} = k\). The morphism \(\rho_{\epsilon,0,\alpha}\) is given by
\[
\rho_{\epsilon,0,\alpha}(K) = \epsilon \alpha^r, \quad \rho_{\epsilon,0,\alpha}(E) = \rho_{\epsilon,0,\alpha}(F) = 0, \quad \rho_{\epsilon,0,\alpha}(J) = \alpha^2.
\]

**Lemma 3.5.** There exists an element \(C\) of the centre of \(U_{r,t}\) acting by 0 on \(V_{\epsilon,0,\alpha}\) and by a non-zero scalar on \(V_{\epsilon,n,\alpha}\) when \(n\) is an integer greater than zero, and \(\epsilon, \epsilon' = \pm 1\).

**Proof.** Define \(C = C_p - \epsilon \frac{\alpha'(q+q^{-1})}{(q-q^{-1})^2}\), where \(C_p = EF + \frac{q^{-1}K + qK^{-1}J}{(q-q^{-1})^2}\). First we show that \(C_p\) is in the centre of \(U_{r,t}\). Let us calculate \(KC_pK^{-1}\) and \(EC_p\).
\[
KC_pK^{-1} = KEFK^{-1} + \frac{q^{-1}K + qK^{-1}J}{(q-q^{-1})^2} = EF + \frac{q^{-1}K + qK^{-1}J}{(q-q^{-1})^2} = C_p.
\]
Since
\[
[E,F] = \frac{K - K^{-1}J}{q - q^{-1}}, \quad C_p = FE + \frac{qK + q^{-1}K^{-1}J}{(q-q^{-1})^2}.
\]
Hence
\[
EC_p = EFE + E\frac{qK + q^{-1}K^{-1}J}{(q-q^{-1})^2} = C_p F.
\]
Similarly we can prove \(FC_p = C_p F\). So \(C_p\) is in the centre of \(U_{r,t}\). Consequently \(C\) is in the centre of \(U_{r,t}\).

\(C\) acts on \(V_{\epsilon,0,\alpha}\) by
\[
\frac{q \epsilon \alpha^r + q^{-1} \epsilon \alpha^r}{(q-q^{-1})^2} - \epsilon \frac{q \alpha^r + q^{-1} \alpha^r}{(q-q^{-1})^2} = 0.
\]
Since \( C \) acts on \( V_{e',n,\alpha} \) by

\[
\beta = \frac{q^{n+1}e'\alpha' + q^{-n}e'\alpha'}{(q - q^{-1})^2} - \epsilon q\alpha' + q^{-1}\alpha' = 0,
\]

we have to show that \( \beta \neq 0 \) when \( n > 0 \). If \( \beta = 0 \), we would have \( (q^{n+2} - \epsilon e')(q^n - \epsilon e') = 0 \), which would be contrary to the assumption, that \( q \) is not a root of unity.

**Theorem 3.6.** When \( q \) is not a root of unity, any two-dimensional \( U_{r,t} \)-module \( V \) is isomorphic to either \( V_{e,0,\alpha} \oplus V_{e,0,\beta} \) or \( V_{e,1,\alpha} \) or a module \( V(\alpha, \epsilon, y) \) with basis \( \{v_1, v_2\} \) such that \( \rho(E) = \rho(F) = 0 \), and \( \rho(J) = (\alpha^2 \ y \ \epsilon) \), \( \rho(K) = (\epsilon \alpha' \ y) \), where \( \rho \) is the algebra homomorphism determined by \( V(\alpha, \epsilon, y) \).

**Proof.** Suppose \( V \) is simple. Then \( V \) is isomorphic to \( V_{e,1,\alpha} \) by Theorem 3.4. Otherwise there exists a proper submodule \( V' \) of \( V \). Since the dimension of \( V' \) is equal to one, we can assume that \( \{v_1, v_2\} \) is a basis of \( V \) satisfying

\[
Kv_1 = \epsilon\alpha'v_1, \quad Kv_2 = \epsilon\beta'v_2 + xv_1, \\
Jv_1 = \alpha^2v_1, \quad Jv_2 = \beta^2v_2 + yv_1.
\]

Since \( \epsilon\beta'(\beta^2v_2 + yv_1) + x\alpha^2v_1 = JKv_2 = KJv_2 = \beta^2(\epsilon\beta'v_2 + xv_1) + y\epsilon\alpha'v_1, \ x(\alpha^2 - \beta^2) = y(\epsilon\beta' - \epsilon\alpha') \).

If \( \epsilon\alpha' \neq \epsilon\beta' \) and \( \alpha^2 \neq \beta^2 \), then \( v_1, v_2' = v_2 + \frac{x}{\beta^2 - \epsilon\alpha'}v_1 = v_2 + \frac{y}{\beta^2 - \epsilon\alpha'}v_1 \) is another basis of \( V \). Since \( K v_2' = \epsilon\beta'v_2' \) and \( J v_2' = \beta^2v_2' \), \( V = kv_1 \oplus kv_2' \) is a direct sum of \( U_{r,t} \)-modules.

If \( \alpha^2 \neq \beta^2 \) and \( \epsilon\beta' = \epsilon\alpha' \), then \( x = 0 \). Let \( v_2' = v_2 + \frac{y}{\beta^2 - \epsilon\alpha'}v_1 \). Then \( J v_2' = \beta^2v_2' \) and \( K v_2' = \epsilon\beta'v_2' \). Consequently \( V = kv_1 \oplus kv_2' \) is a direct sum of \( U_{r,t} \)-modules.

Next we assume that \( \epsilon\alpha' = \epsilon\beta' \), and \( \alpha^2 = \beta^2 \). Since \( Ev_1 \) is an eigenvector for \( K \) with eigenvalue \( \epsilon q^2\alpha' \neq \epsilon\alpha' \), it is zero. Let us prove that \( Ev_2 \) is zero too. Indeed, writing \( Ev_2 = \lambda v_1 + \mu v_2 \), we have

\[
epsilon\alpha'\lambda v_1 + \mu(\epsilon\alpha'v_2 + xv_1) = KEv_2 = q^2EKv_2 = q^2(\epsilon\alpha'v_2 + xv_1) = q^2\epsilon\alpha'(\lambda v_1 + \mu v_2).
\]

Hence

\[
\begin{align*}
\epsilon\alpha'\lambda + \mu x &= q^2\epsilon\alpha'\lambda, \\
\mu \epsilon\alpha' &= q^2\mu \epsilon\alpha'.
\end{align*}
\]

Since \( q^2 \neq 1 \), we obtain \( \lambda = \mu = 0 \) from (3.3). One can show in a similar way that \( F \) acts as zero on \( V \). Since \( [E, F] \) acts as zero, we have \( K = K^{-1}J^r \) on \( V \). In particular, since \( K^{-1}J^r v_2 = \epsilon\alpha^{-r}v_2 - x\alpha^{-2}v_1 \),

\[
J^rK^{-1}v_2 = \epsilon\alpha^{-r}J^r v_2 - x\alpha^{-2}v_1 = \epsilon\alpha^{-r}v_2 + (\epsilon r \alpha^{-2} - x)v_1.
\]

Hence \( \epsilon r y \alpha^{-2} - x = x \) and \( x = \frac{y^2}{2} \epsilon\alpha^{-2} \). So \( \rho(E) = \rho(F) = 0 \), and \( \rho(J) = (\alpha^2 y), \rho(K) = (\epsilon \alpha' \ 2) \epsilon\alpha' = 2) \), where \( \rho \) is the algebra homomorphism determined by \( V(\alpha, \epsilon, y) \).

**Remark 3.7.** If \( y \neq 0 \), then \( V(\alpha, \epsilon, y) \) is not a semisimple \( U_{r,t} \)-module.
Remark 3.8. Suppose that the submodule $V'$ of a module $V$ is simple of dimension greater than 1 and the dimension of $V/V_1$ is 1. Then there exists a one-dimensional module $V_2$ such that $V = V_1 \oplus V_2$. In fact, let the one-dimensional quotient module $V/V'$ has weight $(\epsilon \alpha', \alpha)$. Let us consider the operator

$$C = C_p - \epsilon \frac{q \alpha' + q^{-1} \alpha'}{(q - q^{-1})^2},$$

it acts by zero on $V/V'$. Consequently, we have $CV \subseteq V'$. On the other hand, $C$ acts on $V'$ as multiplication by a scalar $y \neq 0$. It follows that $\frac{1}{y} C$ is the identity on $V'$. Therefore the map $\frac{1}{y} C$ is a projector of $V$ onto $V'$. This projector is a $U_{t,t}$-linear since $C$ is central. Let $V_2 = \text{ker}(\frac{1}{y} C)$. Then $V = V' \oplus V_2$.

Theorem 3.9. The dual module $V^*_{e,n,\alpha}$ of the simple $U_{t,t}$-module $V_{e,n,\alpha}$ is a simple module, and $V^*_{e,n,\alpha} \cong V_{e,n,\alpha^{-1}}$.

Proof. Since $U_{t,t}$ is a Hopf algebra, the dual of any $U_{t,t}$-module is still an $U_{t,t}$-module. First we prove that $V$ is a simple module if and only if $V^* := \text{Hom}_k(V,k)$ is a simple module. Since $V$ is finite dimensional, $V \cong V^{**}$. We only need to verify the implication that $V^*$ is simple if $V$ is simple. Let $L$ be a non-zero submodule of $V^*$. If $L \neq V^*$, then $W = \{ x \in V | f(x) = 0 \text{ for all } x \in L \} \neq 0$. For any $x \in W$ and any $f \in L$, we have $f(Kx) = (K^{-1}f)(x) = 0$. Hence $W$ is a submodule of $V$. Consequently, $W = V$. So $L = 0$. This is contrary to our original assumption. Hence $V^*$ is simple. Now suppose $V_{e,n,\alpha}$ is spanned by $\{v_0, \ldots, v_n\}$ with relations

$$K v_p = \epsilon q^{n-2p} \alpha' v_p, \quad J v_p = \alpha^2 v_p, \quad F v_{p-1} = [p] v_p,$$

and

$$E v_p = \epsilon \frac{q^{n-(p-1)} \alpha' - q^{n-1-p} \alpha'}{q - q^{-1}} v_{p-1} = \epsilon \alpha' [n - p + 1] v_{p-1}.$$

Let $\{v_0^*, \ldots, v_n^*\}$ be the dual basis of $\{v_0, \ldots, v_n\}$. Then

$$(E v_n^*)(v_i) = -v^*_n(EK^{-1}v_i) = \epsilon \alpha' \frac{q^{2i-n}q^{n-1}[n-1]}{q - q^{-1}} v_n^*(v_{i-1}) = 0,$$

$$(K v_n^*)(v_i) = v_n^*(K^{-1}v_i) = q^{2i-n} \epsilon \alpha' \alpha^{-r} v_n^*(v_i) = q^n \epsilon \alpha' \alpha^{-r} v_n^*(v_i)$$

and

$$(J v_n^*)(v_i) = v_n^*(J^{-1}v_i) = \alpha^{-2} v_n^*(v_i).$$

Thus, $v_n^*$ is the highest weight vector with weight $(\epsilon \alpha' \alpha^{-r}, \alpha^{-1})$ of $V_{e,n,\alpha}$ and hence $V^*_{e,n,\alpha} \cong V_{e,n,\alpha^{-1}}$. \qed

Finally in this section, for any given finite-dimensional semisimple $U_{t,t}$-module $V$, we construct a scalar product, i.e. a non-degenerated symmetric bilinear form $(,)$ on $V$ such that

$$(xv, v') = (v, \omega(x)v')$$

(3.4)
for all $x \in U_{r,t}$ and $v, v' \in V$. The linear map $\omega$ has been defined in Proposition 2.1. This is done in the following theorem:

**Theorem 3.10.** On the simple $U_{r,t}$-module $V_{\epsilon,n,\alpha}$ generated by the highest weight vector $v$, there exists a unique scalar product such that $(v, v) = 1$. If we define the vectors $v_i := \frac{1}{[i]} F^i v$ for all $i \geq 0$, then they are pairwise orthogonal and we have

$$(v_i, v_j) = q^{(i+1-n)} \binom{n}{i} \delta_{ij}.$$

**Proof.** Let us first assume that there exists a scalar product on $V_{\epsilon,n,\alpha}$ such that $(v, v) = 1$. Next we will show that $(v_i, v_j) = q^{(i+1-n)[i]} \delta_{ij}$. By definition and (3.4) we have

$$(v_i, v_j) = 1 \frac{1}{[i]} (F^i v, v_j) = 1 \frac{1}{[i]} (v, \omega(F^i) v_j) = 1 \frac{1}{[i]} (v, (E K^{-1})^i v_j).$$

By (2.5) we can prove that $(E K^{-1})^i = q^{(i+1)} K^{-i} E^i$ for any $i > 0$. Consequently, the vector $\omega(F^i) v_j$ is a scalar multiple of $E^i v_j$, which is equal to zero as soon as $i > j$. Therefore $(v_i, v_j) = 0$ if $i > j$. By symmetry, we also have $(v_i, v_j) = 0$ if $i < j$.

We need the formula

$$(E^i v_j) = (\epsilon \alpha') \frac{[n-j+i]}{[n-j]} v_{j-i}$$

to compute $(v_i, v_i)$. We have

$$(v_i, v_i) = \frac{1}{[i]} q^{(i+1)} (v, K^{-i} E^i v_i)$$

$$= (\epsilon \alpha') q^{(i+1)} \frac{[n]}{[i]! [n-i]!} (v, K^{-i} v)$$

$$= q^{(i+1-n)} \frac{[n]}{[i]! [n-i]!}.$$ 

This proves the uniqueness of the scalar product. Let us now prove its existence.

Clearly, there exists a non-degenerate symmetric bilinear form such that

$$(v_i, v_j) = q^{(i+1-n)} \binom{n}{i} \delta_{ij}.$$ (3.5)

We have to check that it satisfies relation (3.4). It is enough to check this for $x = E, F, K, K^{-1}, J$ and $J^{-1}$. We shall do this for $x = E$ and $x = F$, since the other computations are easy.

For the case $x = E$. On the one hand, we have

$$(E v_i, v_j) = \epsilon \alpha'[n-i+1](v_{i-1}, v_j) = \epsilon \alpha' q^{(i-1)(i-n)} \frac{[n]}{[i-1]! [n-i]!} \delta_{i-1, j}.$$
One the other hand, by Proposition 2.1 and by (3.4), we have
\[
(v_i, \omega(E)v_j) = (v_i, KFv_j) = [j + 1](v_i, K_{v_{j+1}})
= \epsilon \alpha' q^{i(j+1)-n-2(j+1)[j + 1]} \frac{[n]!}{[i]! [n - j]!} \delta_{y_{j+1}}
= \epsilon \alpha' q^{i(1)(i-n)} \frac{[n]!}{[i]! [n - i]!} \delta_{y_{j+1}}
= (Ev_i, v_j).
\]

For the case \( x = F \). On the one hand, we have
\[
(Fv_i, v_j) = [i + 1](v_{i+1}, v_j) = q^{i(i+1)(i+2-n)} \frac{[n]!}{[i]! [n - i - 1]!} \delta_{y_{i+1}}.
\]

One the other hand, by Proposition 2.1 and by (3.4), we have
\[
(v_i, \omega(F)v_j) = (v_i, EK^{-1}v_j) = \epsilon \alpha' q^{2j-n}(v_i, Ev_j)
= q^{2j-n}[n - j + 1](v_i, v_{j-1})
= q^{i(i+1-n)+2(i+1-n)[n - j + 1]} \frac{[n]!}{[i]! [n - j - 1]!} \delta_{y_{j-1}}
= q^{i(i+1)(i+2-n)} \frac{[n]!}{[i]! [n - j - 1]!} \delta_{y_{j-1}}
= (Fv_i, v_j).
\]

This completes the proof of this theorem. \( \square \)

4. The Harish-Chandra homomorphism and the centre of \( U_{r,t} \). Our objective in this section is to describe the centre \( Z \) of \( U_{r,t} \) in case \( q \) is not a root of unity. We assume this throughout this section.

Let us fix \((\lambda, \alpha)\), where \(\alpha \lambda \neq 0\). Consider an infinite-dimensional vector space \( V(\lambda, \alpha) \) with denumerable basis \( \{v_i | i \in \mathbb{N} \} \). For \( p \geq 0 \), set
\[
\begin{cases}
Kv_p = q^{-2p} \lambda v_p, & Jv_p = \alpha^2 v_p, \\
Ev_{p+1} = \frac{q^{p\lambda} - q^{-\lambda} \alpha^2}{q^{-p} - q^{-\lambda} \alpha^2} v_p, & Fv_p = [p + 1]v_{p+1}.
\end{cases}
\]  

(4.1)

\[
K^{-1}v_p = q^2 v_p, \quad J^{-1}v_p = \alpha^{-2} v_p.
\]  

(4.2)

**Lemma 4.1.** Relations in (4.1) and (4.2) define a \( U_{r,t} \)-module structure on \( V(\lambda, \alpha) \). The element \( v_0 \) generates \( V(\lambda, \alpha) \) as a \( U_{r,t} \)-module and is the highest weight vector of weight \((\lambda, \alpha)\).

**Proof.** Immediate computation yield
\[
K^{-1}Kv_p = KK^{-1}v_p = v_p, \quad J^{-1}Jv_p = JJ^{-1}v_p = v_p,
\]
\[
KEK^{-1}v_p = q^2 Ev_p, \quad KFK^{-1}v_p = q^2 Fv_p.
\]
\[
\begin{align*}
[E, F]_{v_p} &= (p + 1) \frac{q^{-p} \lambda - q^p \lambda^{-1} \alpha^{2r}}{q - q^{-1}} - [p] \frac{q^{-p+1} \lambda - q^{p-1} \lambda^{-1} \alpha^{2r}}{q - q^{-1}} v_p \\
&= \frac{q^{-2p} \lambda - q^{2p} \lambda^{-1} \alpha^{2r}}{q - q^{-1}} v_p \\
&= \frac{K - K^{-1} J^r}{q - q^{-1}} v_p.
\end{align*}
\]

(4.3)

This shows that the relations in (4.1) and (4.2) define a \( U_{r,t} \)-module structure on \( V(\lambda, \alpha) \). The proof is complete.

Let \( U^K \) be the subalgebra of \( U_{r,t} \) of all elements commuting with \( K \).

**Lemma 4.2.** An element of \( U_{r,t} \) belongs to \( U^K \) if and only if it is of the form

\[
\sum_{i \geq 0} F^i P_i E^i,
\]

where \( P_0, P_1, \ldots \) are elements of \( k[K, K^{-1}; J, J^{-1}] \).

**Proof.** This is a consequence of the fact that \( \{F^i K^j J^s E^l | i, j \in \mathbb{N}, l, s \in \mathbb{Z} \} \) is a basis of \( U_{r,t} \) and that

\[
K(F^i K^j J^s E^l)K^{-1} = q^{2(i-j)} F^i K^j J^s E^l.
\]

\[\square\]

**Lemma 4.3.** We have \( I = U_{r,t}E \cap U^K = FU_{r,t} \cap U^K \) and

\[
U^K = k[K, K^{-1}; J, J^{-1}] \oplus I.
\]

**Proof.** Let \( u = \sum_{i \geq 0} F^i P_i E^i \in U_{r,t} \) be an element of \( U^K \). If \( u \) also lies in \( U_{r,t}E \), then \( P_0 = 0 \). Hence \( u \) belongs to \( FU_{r,t} \cap U^K \) and conversely. Since the form \( \sum_{i \geq 0} F^i P_i E^i \) is unique for any element of \( U^K \), we get the desired direct sum.

\[\square\]

It results from \( I = U_{r,t}E \cap U^K = FU_{r,t} \cap U^K \) that \( I \) is a two-sided ideal and the projector \( \varphi \) from \( U^K \) onto \( k[K, K^{-1}; J, J^{-1}] \) is a morphism of algebras. The map \( \varphi \) is called the Harish-Chandra homomorphism. It permits one to express the action of the centre \( Z \) on the highest weight module.

**Proposition 4.1.** Let \( V(\lambda, \alpha) \) be the highest weight module of \( U_{r,t} \) with highest weight \( (\lambda, \alpha) \). Then, for any central element \( z \in Z \) and any \( v \in V \), we have

\[
zv = \varphi(z)(\lambda, \alpha^2)v.
\]

Recall that \( \varphi(z) \) is a Laurent polynomial in \( K, J \), and \( \varphi(z)(\lambda, \alpha^2) \) is its value at \( K = \lambda \) and \( J = \alpha^2 \).

**Proof.** Let \( v_0 \) be the highest weight vector generating \( V(\lambda, \alpha) \) and \( z \) a central element of \( U_{r,t} \). The element \( z \) can be written in the form

\[
z = \varphi(z) + \sum_{i>0} F^i P_i E^i.
\]
Since
\[
\begin{cases}
Er_0 = 0, \\
Jv_0 = \alpha^2 v_0, \\
Kv_0 = \lambda v_0,
\end{cases}
\]
we get \( ev_0 = \phi(x)(\lambda, \alpha^2)v_0. \) If \( v \) is an arbitrary element of \( V(\lambda, \alpha) \), we have \( v = xv_0 \) for some \( x \in U_{r,l} \), hence \( zv = xzv_0 = \phi(z)(\lambda, \alpha^2)v. \)

**Lemma 4.4.** Let \( z \in Z \). If \( \phi(z) = 0 \), then \( z = 0. \)

**Proof.** Let \( z \) be an element in the centre such that \( \phi(z) = 0 \). Assume \( z \neq 0. \) Since \( z \in U^K \), we can assume that \( z = \sum_{i=k}^l F^iP_iE^i \in FU_{r,l} \) for some \( k \geq 1 \), where \( P_k, P_{k+1}, \ldots, P_l \) are non-zero Laurent polynomials in \( K \) and \( J \). Consider a Verma module \( V(\lambda, \alpha) \), The relations in (4.1) and (4.2) show that \( E v_0 = 0 \) if and only if \( p = 0 \). Let us apply \( z \) to the vector \( v_k \) of \( V(\lambda, \alpha) \). On the one hand
\[
zv_k = \phi(x)(\lambda, \alpha^2)v_k = 0.
\]
On the other hand, we get
\[
zv_k = F^k P_k E^k v_k = cP_k(\lambda, \alpha^2)v_k,
\]
where \( c \) is a non-zero constant. It follows that \( P(\lambda, \alpha^2) = 0 \) for any non-zero \( \lambda \) and \( \alpha \). Thus \( P_k = 0 \). This is impossible. \( \Box \)

**Theorem 4.5.** When \( q \) is not a root of unity, the centre \( Z \) of \( U_{r,l} \) is a polynomial algebra generated by the element \( C_p \) over the algebra \( k[J, J^{-1}] \). The restriction of Harish-Chandra homomorphism to \( Z \) is an isomorphism onto the subalgebra of \( k[K, K^{-1}, J^{-1}, J] \) generated by \( qK + q^{-1}K^{-1}J \).

**Proof.** For any integer \( n > 0 \), consider the Verma module \( V(q^{n-1}\alpha', \alpha) \) for any non-zero element \( \alpha \). By (4.1) we have \( Ev_n = 0 \). Thus \( v_n \) is the highest weight vector of weight \( (q^{n-1}-1)\alpha', \alpha) \). By Proposition 4.1 a central element \( z \) acts on the module generated by \( v_n \) as the multiplication by scalar \( \phi(z)(q^{n-1}-1)\alpha', \alpha^2) \); but since \( v_n \) is in \( V(q^{n-1}\alpha', \alpha) \), the element \( z \) also acts as the scalar \( \phi(q^{n-1}\alpha', \alpha^2) \). Thus
\[
\phi(z)(q^{n-1}\alpha', \alpha^2) = \phi(z)(q^{-(n+1)}\alpha', \alpha^2)
\]
for any \( \alpha \neq 0 \) and any \( n > 0 \). Suppose \( \phi(z) = P(K, K^{-1}, J, J^{-1}) \). Then (4.4) implies
\[
P(q^{n-1}\alpha', q^{-(n-1)}\alpha', \alpha^2, \alpha^{-2}) = P(q^{-(n+1)}\alpha', q^{n+1}\alpha, \alpha^2, \alpha^{-2}).
\]
Let
\[
\psi_\alpha(x) = P(q^{-1}\alpha'x, q\alpha'x^{-1}, \alpha^2, \alpha^{-2}).
\]
Then \( \psi_\alpha(q^n) = \psi_\alpha(q^{-n}) \) for any integer \( n \) by (4.5). Hence
\[
\psi_\alpha(x) = \sum_{i \geq 0} a_i(\alpha)(x + x^{-1})^i,
\]
where \( a_i(\alpha) \in k[\alpha, \alpha^{-1}] \). Therefore
\[
\psi_\alpha(qK\alpha') = \sum_{i \geq 0} a_i(\alpha)(qK\alpha')^i + q^{-1}K^{-1}\alpha')(qK\alpha') = P(K, K^{-1}, \alpha^2, \alpha^{-2}),
\]
for any non-zero $\alpha$. Since

$$P(K, K^{-1}, (-\alpha)^2, (-\alpha)^{-2}) = P(K, K^{-1}, \alpha^2, \alpha^{-2})$$

$$\sum_{i \geq 0} a_i(\alpha)\alpha^{-ri}(qK + q^{-1}K^{-1}\alpha^{2r})^i = \sum_{i \geq 0} a_i(-\alpha)\alpha^{-ri}(qK + q^{-1}K^{-1}\alpha^{2r})^i.$$ 

Hence $a_i(\alpha) = a^{\alpha^2}b_i(\alpha^2)$. So

$$\sum_{i \geq 0} b_i(\alpha^2)(qK + q^{-1}K^{-1}\alpha^{2r})^i = P(K, K^{-1}, \alpha^2, \alpha^{-2}). \quad (4.7)$$

Consequently,

$$\varphi(z) = \sum_{i \geq 0} c_i(J, J^{-1})(qK + q^{-1}K^{-1}J^r)^i.$$ 

Since $\varphi(C_p) = \frac{qK + q^{-1}K^{-1}J^r}{(q^{-1}q)^{r/2}}$, $\varphi(J) = J$ and $\varphi(J^{-1}) = J^{-1}$, $\varphi$ is a surjective map from $Z$ to the subalgebra of $k[K, K^{-1}, J, J^{-1}]$ generated by $qK + q^{-1}K^{-1}J^r$. Using Lemma 4.4, we obtain the proof of the remaining results of this theorem. \hfill \Box

5. The generalized quantum Clebsch–Gordan formula. We now prove a generalized quantum Clebsch–Gordan formula for the finite-dimensional simple $U_{\epsilon,1}$-modules. Since

$$V_{\epsilon,n,\alpha} \simeq V_{\epsilon,0,\alpha} \otimes V_{1,n,1},$$

and $V_{1,n,1}$ can view a module over $U_{\epsilon,1}/(J - 1) \simeq U_q(\mathfrak{sl}(2))$, we get the following lemma by using the quantum Clebsch–Gordan formula for the usual quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$.

**Lemma 5.1.** Let $n \geq m$ be two non-negative integers. There exists an isomorphism of $U_{\epsilon,1}$-modules

$$V_{\epsilon,n,\alpha} \otimes V_{\epsilon',n,\beta} \simeq V_{\epsilon,0,\alpha} \otimes V_{\epsilon',0,\beta} \otimes V_{\epsilon',n-m,\alpha\beta}.$$ 

**Proof.** It is obvious that $V_{\epsilon,0,\alpha} \otimes V_{\epsilon',0,\beta} \simeq V_{\epsilon,0,\alpha\beta}$. Thus this lemma follows from the above remark. \hfill \Box

In the remainder of this section, we always assume that $n \geq m$ and $\epsilon = \epsilon' = 1$. In the case $\alpha' = 1$, we can determine the all highest weight vectors of $V_{\epsilon,n,\alpha} \otimes V_{\epsilon',n,\beta}$ in the following lemma.

**Lemma 5.2.** Let $v^{(0)}$ be the highest weight vector of weight $(q^n\alpha', \alpha)$ in $V_{1,n,\alpha}$ and $v^{(m)}$ be the highest weight vector of weight $(q^m\beta', \beta)$ in $V_{1,m,\beta}$. Let us define $v_p^{(n)} = \frac{1}{[p]!}F_p v^{(n)}$, $v_p^{(m)} = \frac{1}{[p]!}F_p v^{(m)}$, for all $p \geq 0$. Suppose $\alpha'^{r} = 1$. Then

$$v^{(n+m-2p)} = \sum_{i=0}^{p} (-1)^i \frac{[m-p+i][n-i]}{[m-p][n]} q^{-i(m-2p+i+1)} \beta^{2n(m-i)} v^{(n)} \otimes v^{(m)}_{p-i}$$

is the highest weight vector of weight $(q^{n+m-2p}\beta', \alpha\beta)$. 

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Proof. It is clear that $v_i^{(n)} \otimes v_{p-i}^{(m)}$ has weight $(q^{n+m-2p} \beta', \alpha \beta)$. Let us check that $E_{V(n+m-2p)} = 0$. Recall that

$$\Delta(E) = J^{-rt} \otimes E + E \otimes KJ^{rt}.$$ 

It follows that

$$E_{V(n+m-2p)} = \sum_{i=0}^{p} (-1)^i \frac{[m-p+i][n-i]}{[m-p][n]} q^{-i(m-2p+i+1)}[m-p+i+1]$$

$$\times \beta^{2r(n-i)+r} v_i^{(n)} \otimes v_{p-i}^{(m)}$$

$$+ \sum_{i=0}^{p} (-1)^i \frac{[m-p+i][n-i]}{[m-p][n]} q^{-i(m-2p+i+1)+(m-2p+2)}[n-i+1]$$

$$\times \beta^{2r(n-i)+r} v_{i-1}^{(n)} \otimes v_{p-i}^{(m)}$$

$$= \sum_{i=0}^{p} (-1)^i \frac{[m-p+i][n-i+1]}{[m-p][n]} q^{-(i-1)(m-2p+i)}(\beta^{2r(n-i)+1+r}$$

$$- \beta^{2r(n-i)+r})v_i^{(n)} \otimes v_{p-i}^{(m)}$$

$$= 0.$$ 

Thus this lemma is true. \qed

We wish to go one step further and address the following problem. We now have two bases of $V_{1,n,a} \otimes V_{1,m,b}$ at our disposal. They are of different natures, the first one, adapted to the tensor product, is the set

$$\{v_i^{(n)} \otimes v_j^{(m)} | 0 \leq i \leq n, 0 \leq j \leq m\};$$

the second one, formed by the vectors

$$v_k^{(n+m-2p)} = \frac{1}{[k]!} e^k v_k^{(n+m-2p)}$$

with $0 \leq p \leq m$ and $0 \leq k \leq n + m - 2p$, is better adapted to the $U_{r,t}$-module structure. Comparing both bases leads us to the so-called generalized quantum Clebsch–Gordan coefficients $v_i^{(n)} v_j^{(m)} v_k^{(n+m-2p)}$ defined for $0 \leq p \leq m$, and $0 \leq k \leq n + m - 2p$ by

$$v_k^{(n+m-2p)} = \sum_{0 \leq i \leq n, 0 \leq j \leq m} \begin{pmatrix} n & m & n+m-2p \\ i & j & k \end{pmatrix} v_i^{(n)} v_j^{(m)}.$$ 

In particular,

$$\begin{pmatrix} n & m & n+m-2p \\ i & j & 0 \end{pmatrix} = (-1)^i \frac{[m-p+i][n-i]}{[m-p][n]} q^{-i(m-2p+i+1)} \beta^{2r(n-i)}$$

$$= \begin{pmatrix} n & m & n+m-2p \\ i & j & 0 \end{pmatrix} \beta^{2r(n-i)}.$$
where \([\genfrac{}{}{0pt}{}{n}{i} \genfrac{}{}{0pt}{}{m}{j} \genfrac{}{}{0pt}{}{n+m-2p}{0}]\) is the usual quantum Clebsch–Gordan coefficients, is also called quantum 3j-symbols in the physics literature.

**Proposition 5.1.** Fix \(p\) and \(k\). The vector \(v_k^{(n+m-2p)}\) is a linear combination of vectors of the form \(v_i^{(n)} \otimes v_{p-i+k}^{(m)}\). Therefore we have \(v_i^{(n)} \otimes v_{p-i+k}^{(m)} = 0\) when \(i + j \neq p + k\). We also have the induction relation
\[
\begin{align*}
\left\{ \begin{array}{ccc}
n & m & n + m - 2p \\
j + 1 & k + 1 & \\
 \end{array} \right\} = & \frac{\lfloor j + 1 \rfloor q^{2i-n}}{k + 1} \left\{ \begin{array}{ccc}
n & m & n + m - 2p \\
j & k & \\
 \end{array} \right\} \\
+ & \frac{\lfloor i \rfloor}{k + 1} \left\{ \begin{array}{ccc}
n - 1 & m & n + m - 2p \\
j + 1 & k & \\
 \end{array} \right\} \beta^{2rt}.
\end{align*}
\]

**Proof.** This goes by induction on \(k\). The assertion holds for \(k = 0\) by Lemma 5.2. Supposing
\[
v_k^{(n+m-2p)} = \sum_i x_i v_i^{(n)} \otimes v_{p-i+k}^{(m)},
\]
we have
\[
[k + 1] v_{k+1}^{(n+m-2p)} = F v_k^{(n+m-2p)}
\]
\[
= \sum_i x_i [(J_i^{(i+1)} K^{-1}) v_i^{(n)} \otimes F v_{p-i+k}^{(m)} + F v_i^{(n)} \otimes J_i^{(i+1)} v_{p-i+k}^{(m)}]
\]
\[
= \sum_i x_i [(p - i + k + 1) q^{2i-n} v_i^{(n)} \otimes v_{p-i+k+1}^{(m)}]
\]
\[
+ \frac{\lfloor i + 1 \rfloor \beta^{2rt}}{k + 1} v_{i+1}^{(n)} \otimes v_{p-i+k}^{(m)}
\]
\[
= \sum_i (x_i [p i) \frac{\lfloor j \rfloor q^{2i-n} + x_i [i] \beta^{2rt}}{k + 1} \times v_i^{(n)} \otimes v_{p-i+k+1}^{(m)}.
\]
The rest follows easily. \(\Box\)

We now prove some orthogonality relations for the generalized quantum Clebsch–Gordan coefficients. Let us equip \(V_{1,n,\alpha}\) and \(V_{1,m,\beta}\) with the scalar product \((, , )\) defined in Section 4. Consider the symmetric bilinear form on \(V_{1,n,\alpha} \otimes V_{1,m,\beta}\) given by
\[
(v_1 \otimes v_1', v_2 \otimes v_2') = (v_1, v_2)(v_1', v_2'),
\]
where \(v_1, v_2 \in V_{1,n,\alpha}\) and \(v_1', v_2' \in V_{1,m,\beta}\).

**Proposition 5.2.** (a) We have
\[
v_k^{(n+m-2p)} = \frac{1}{[k]!} \sum_{i=0}^{p} \sum_{s=0}^{k} (-1)^i [m - p + i][n - i][s + i][p + k - i - s]!
\]
\[
\times q^{-(m-2p+i+1)+(k-s)(s+2l-n)} \beta^{2rt(n-i-s)} v_i^{(n)} \otimes v_{p-i-2l+k-s}^{(m)}.
\]
(b) \(v_k^{(n+m-2p)}, v_l^{(n+m-2q)} = 0\) whenever \(p + k = q + l\).
Proof. Since \( \Delta(F) = J^{(t+1)} K^{-1} \otimes F + F \otimes J^{-rt} \),

\[
\Delta(F^k) = \sum_{s=0}^k q^{s(k-s)} \left[ \frac{k}{s} \right] (J^{n(t+1)(k-s)} F^s K^{-(k-s)} \otimes J^{-rt s} F^{-s}).
\]

Hence

\[
v_k^{(n+m-2p)} = \frac{1}{[k]!} \sum_{i=0}^{p} \sum_{s=0}^{k} (-1)^i \left[ \frac{k}{s} \right] \frac{[m-p+i][n-i]!}{[m-p][n]!} \times
\]

\[
q^{-i(n-2p+i+1)+(k-s)} \beta^{2r(n-i)} \times F^s K^{-(k-s)} v_i^{(n)} \otimes J^{-rt s} F^{-s} v_p^{(m)}
\]

\[
= \frac{1}{[k]!} \sum_{i=0}^{p} \sum_{s=0}^{k} (-1)^i \left[ \frac{k}{s} \right] \frac{[m-p+i][n-i]!}{[m-p][n]!} \times
\]

\[
q^{-i(n-2p+i+1)+(2i-n+s)(k-s)} \beta^{2r(n-i)-2rs} F^s v_i^{(n)} \otimes F^{k-s} v_p^{(m)}
\]

By Theorem 3.10, \( (v_{i+s}^{(n)}, v_{i+s}^{(n)}) = 0 \) either \( i+s \neq j+u \) or \( p+k - i-s \neq q+l-j-u \). If \( i+s = j+u \) and \( p+k - i-s = q+l-j-u \), then \( p+k = q+l \). Hence \( v_k^{(n+m-2p)} \), \( v_i^{(n+m-2q)} \) = 0 whenever \( p+k \neq q+l \). \( \square \)

Remark 5.3. Similarly to [3], one can study the categorification of tensor products of arbitrary finite-dimensional irreducible modules over the \( U_{r,t} \).

6. In the case \( q \) is a root of unity. Our main aim is to find all finite-dimensional simple \( U_{r,t} \) in the case when the parameter \( q \) is a root of unity \( \neq \pm 1 \). Denote by \( d \) the order of \( q \), i.e. the smallest integer greater than 1 such that \( q^d = 1 \). Since we assume \( q^2 \neq 1, d > 2 \). Define

\[
e = \begin{cases} 
\frac{d}{2} & \text{if } d \text{ is odd} \\
\frac{d}{2} & \text{when } d \text{ is even}
\end{cases}
\]

It is easy to check that \( [n] = 0 \) if and only if \( n \equiv 0(\text{mod } e) \).

Lemma 6.1. The elements \( E^e, F^e \) and \( K^e \) belong to the centre of \( U_{r,t} \).

Proof. \( K^e \) commutes with \( E \) and \( F \) because \( q^{2e} = 1 \). So \( K^e \) is in the centre of \( U_{r,t} \). Since \( [e] = 0 \),

\[
[E^e, F^e] = [e] \frac{q^{(e-1)} K - q^{e-1} K^{-1} F^e}{q - q^{-1}} E^{e-1} = 0.
\]

Moreover \( KE^e K^{-1} = (KEK^{-1})^e = (q^2 E)^e = E \). So \( E^e \) belongs to the centre of \( U_{r,t} \). Similar arguments can be applied to \( F^e \). \( \square \)

Lemma 6.2. There is no simple finite-dimensional \( U_{r,t} \) module of dimension greater than \( e \).
Proof. Let us assume that there exists a simple finite-dimensional module greater than \( e \). We shall prove that \( V \) has a non-zero submodule of dimension less than or equal to \( e \). Hence, a contradiction.

(a) Suppose there exists a non-zero vector \( v \in V \) such that \( K v = \lambda v, J v = \alpha^2 v \) and \( F v = 0 \). We claim that the subspace \( V' \) spanned by \( v, E v, \ldots, E^{e-1} v \) is a submodule of dimension less than or equal to \( e \). It is enough to check that \( V' \) is stable under the action of generators \( E, F, K, J \). This is clear for \( K, J \). Let us prove that \( V' \) is stable under the action of \( E \). The vector \( E(E^p v) = E^{p-1} v \) belongs to \( V' \) if \( p < e - 1 \). If \( p = e - 1 \), then the action of \( E^e \) on the irreducible module \( V \) is given by a scalar \( c \), as \( E^e \) is in the centre of \( U_{r,t} \). So \( E(E^{e-1}) v = cv \) belongs to \( V' \). Finally, \( V' \) is stable under the \( F \) by \( F v = 0 \) and Lemma 2.3.

(b) Suppose there is no common eigenvector \( v \) of \( K \) and \( J \) satisfying \( F v = 0 \). We claim that the subspace \( V' \) spanned by \( v, F v, \ldots, F^{e-1} v \) is a submodule of \( V \), where \( v \) satisfies \( K v = \lambda v, J v = \alpha^2 v \). Since \( F^e \) is in the centre of \( U_{r,t} \), \( F^e v = cv \) for some \( c \in k \) and \( c \neq 0 \). Thus \( V' \) is stable under the action of \( F \). It is easy to prove that \( V' \) is stable under the actions of \( J, K \). Let us show that \( V' \) is stable under the action of \( E \). Recall that \( C_p = E F + \frac{q^{-1} K + \alpha^{-1} J}{q + q^{-1}} \) is in the centre of \( U_{r,t} \). Hence there exists \( a \in k \) such that \( C_p w = aw \) for any vector \( w \in V \). Hence \( E v = \frac{1}{c} E F^e v = \frac{1}{c} (C_p - \frac{q^{-1} K + \alpha^{-1} J}{q + q^{-1}}) F^{e-1} v = \frac{1}{c} (a - \frac{q^{-1} \lambda^{-1} \alpha^2}{q + q^{-1}} - \frac{q^{p+1} \lambda^{-1} \alpha^2}{q - q^{-1}}) F^{e-1} v \). For any \( p \geq 0 \), \( EF^{p+1} v = ([p + 1] \frac{q^{-1} K + \alpha^{-1} J}{q + q^{-1}} F^p + F^{p+1} E) v = (\frac{q^{p+1} \lambda^{-1} \alpha^2}{q - q^{-1}} [p + 1] + a - \frac{q^{p} \lambda^{-1} \alpha^2}{q + q^{-1}}) F^{e-1} v \).

Theorem 6.3. Any non-zero simple finite-dimensional \( U_{r,t} \) is isomorphic to a module of the form

(i) \( V_{r,n,a} \) with \( 0 \leq n < e - 1 \),

(ii) \( V_{\lambda,a,a} \), where \( V_{\lambda,a} \) has a basis \( \{v_0, v_1, \ldots, v_{e-1}\} \) such the action of the generators of \( U_{r,t} \) given by

\[
K v_p = q^{2p} v_p, \quad 0 \leq p \leq e - 1, \tag{6.1}
\]
\[
J v_p = \alpha^2 v_p, \quad 0 \leq p \leq e - 1, \tag{6.2}
\]
\[
F v_{p+1} = \frac{q^{-p} \lambda^{-1} \alpha^2}{q - q^{-1}} [p + 1] v_p, \quad 0 \leq p < e - 1, \tag{6.3}
\]
\[
E v_p = v_{p+1}, \quad 0 \leq p < e - 1, \tag{6.4}
\]
\[
F v_0 = 0, \quad E v_{e-1} = av_0. \tag{6.5}
\]

(iii) \( V_{\lambda,a,b} \), where \( b \neq 0 \) and \( V_{\lambda,a,b} \) has a basis \( \{v_0, v_1, \ldots, v_{e-1}\} \) such the action of the generators of \( U_{r,t} \) given by

\[
K v_p = q^{-2p} v_p, \quad 0 \leq p \leq e - 1, \tag{6.6}
\]
\[
J v_p = \alpha^2 v_p, \quad 0 \leq p \leq e - 1, \tag{6.7}
\]
\[
E v_{p+1} = \left( \frac{q^p \lambda^{-1} - q^{-p} \alpha^{-1} \alpha^2}{q - q^{-1}} [p + 1] + ab \right) v_p, \quad 0 \leq p < e - 1, \tag{6.8}
\]
\[
F v_p = v_{p+1}, \quad 0 \leq p < e - 1, \tag{6.9}
\]
\[
F v_{e-1} = bv_0, \quad E v_0 = av_{e-1}. \tag{6.10}
\]
Proof. Suppose the simple module \( V \) with \( \dim V < e \). Then we can prove \( V \) is isomorphic to \( V_{e,n,a} \), as we have done in the proof of Theorem 3.4.

Suppose the simple module \( V \) with \( \dim V = e \). Then we can obtain that \( V \) is isomorphic to either \( V_{\lambda,a,a} \), or \( V_{\lambda,a,a,b} \) from the proof of Lemma 6.2 \( \square \)

REMARK 6.4. In Sections 3 and 6, we describe the irreducible representations of \( U_{r,t} \). An irreducible representation of the quantum group \( U_q(\mathfrak{sl}(2)) \) can be realized in terms of the space of functions on some algebraic varieties [2]. We will study the representations of \( U_{r,t} \) on some spaces of functions, and establish the relations between the representations of \( U_{r,t} \) and hypergeometric series as in refs. [7, 10] in the future paper.

7. Finite-dimensional Hopf algebra. The basic problem in the theory of Hopf algebras is to classify finite-dimensional Hopf algebras (see [8] and references therein). So one need to construct various Hopf algebras. Our main aim in this section is to construct a kind of finite-dimensional Hopf algebras by using the algebra \( U_{r,t} \). We assume that the parameter \( q \) is a root of unity \( \neq \pm 1 \). The definitions of \( e \) and \( q \) were given in Section 6.

LEMMA 7.1. Let \( U' = U_{r,t}/(E^e, F^e) \). Then \( U' \) has a basis \( \{E^i F^j K^m J^n \mid 0 \leq i, j \leq e - 1, m, n \in \mathbb{Z} \} \).

Proof. From Theorem 2.4, we know that \( U' \) is generated by \( \{E^i F^j K^m J^n \mid 0 \leq i, j \leq e - 1, m, n \in \mathbb{Z} \} \). We only need to prove the elements in \( \{E^i F^j K^m J^n \mid 0 \leq i, j \leq e - 1, m, n \in \mathbb{Z} \} \) are linearly independent. Suppose

\[
Z = \sum_{0 \leq i,j \leq e-1} a_{ijmn} E^i F^j K^m J^n = 0.
\]

Let \( V \) be a \( U_{r,t} \)-module with basis \( \{v_0, v_1, \ldots, v_{e-1} \} \) such that \( E v_{e-1} = 0, E v_i = v_{i+1} \) for \( 0 \leq i < e - 1 \), \( F v_{p+1} = \frac{q^{-p} - q^{-p+1}}{q^{-p+1} - q^{-p}} [p+1] v_p \) for \( 0 \leq p < e - 1 \), and \( F v_0 = 0 \), \( K v_p = q^{p} v_p \), \( J v_p = \alpha^2 v_p \), where \( \lambda \) is neither zero nor a root of unity. Then

\[
Z v_{e-1} = \sum_{1 \leq i \leq e-1} a_{i-1mn} \alpha^2 n \lambda^m v_i = 0.
\]

Hence

\[
\sum_{r_1 \leq m \leq s_1} \left( \sum_{r_2 \leq n \leq s_2} a_{i-1mn} \alpha^2 \right) \lambda^m = 0, \quad (7.1)
\]

for any \( 0 \leq i \leq e - 1 \). Writing (7.1) for \( s_1 - r_1 + 1 \) distinct elements \( \lambda \in k \), we get a linear system whose determinant is not equal to zero. Consequently,

\[
\sum_{r_2 \leq n \leq s_2} a_{i-1mn} \alpha^2 = 0, \quad (7.2)
\]

for any \( m \). Similarly we can prove \( a_{i-1mn} = 0 \) for any \( n \) from (7.2).

Next, we apply \( Z \) to the vector \( v_{e-2} \). We get \( a_{i-2mn} = 0 \) for all \( i, m, n \) by the same argument as above. Applying \( Z \) successively to the vector \( v_{e-2} \) down to \( v_0 \), one shows that all coefficients \( a_{ijmn} \) vanish. \( \square \)
Lemma 7.2. Let $U'' = U_{r,t}/(E^e, F^e, K^e - 1)$. Then $U''$ has a basis \{\(E^iF^jK^mJ^n\)\(0 \leq i, j, m \leq e - 1, n \in \mathbb{Z}\)}.

Proof. We use $d(Z)$ (resp. $\delta(Z)$) to denote the degree in $K$ (resp. $K^{-1}$) of the element $Z \in U'$. It is clear that the set \{\(E^iF^jK^mJ^n\)\(0 \leq i, j, m \leq e - 1, n \in \mathbb{Z}\)} span the algebra $U''$. It remains to check that they are linearly independent. If

$$Z = \sum_{0 \leq i, j, m \leq e - 1, n \leq s_1} a_{ijmn}E^iF^jK^mJ^n = 0$$

in $U''$, then in $U'$

$$Z = (K^e - 1)Y$$

$$= \sum_{0 \leq i, j \leq e - 1, m, n \in \mathbb{Z}} b_{ijmn}E^iF^jK^{m+e}J^n,$$

where $Y = \sum_{0 \leq i, j \leq e - 1, m, n \in \mathbb{Z}} b_{ijmn}E^iF^jK^mJ^n$. Since

$$Z = \sum_{0 \leq i, j, m \leq e - 1, n \leq t_1} a_{ijmn}E^iF^jK^mJ^n,$$

\(0 \leq \delta(Z) \leq d(Z) < e\). From (7.3) we obtain $d(Z) = d(Y) + e$ and $\delta(Z) = \delta(Y)$. Thus \(d(Y) = d(Z) - e < 0 \leq \delta(Z) = \delta(Y)\). This is impossible, hence $Z = 0$ in $U'$. Therefore all coefficients $a_{ijmn}$ vanish. \(\square\)

Lemma 7.3. Let $U_{r,t,1} = U_{r,t}/(E^e, F^e, K^e - 1, J^l - 1)$. Then $U_{r,t,1}$ has a basis \{\(E^iF^jK^mJ^n\)\(0 \leq i, j, m \leq e - 1, 0 \leq n \leq l - 1\)}.

Proof. The proof is similar to that of Lemma 7.2. \(\square\)

Theorem 7.4. Let $U_{r,t,1} = U_{r,t}/(E^e, F^e, K^e - 1, J^l - 1)$. Then $U_{r,t,1}$ has a unique Hopf algebra structure such that the canonical projection from $U_{r,t}$ to $U_{r,t,1}$ is a morphism of Hopf algebras. Moreover the dimension of $U_{r,t,1}$ is equal to $le^3$.

Proof. We only need to check that

$$\Delta(E^e) = \Delta(F^e) = \Delta(K^e) - 1 = \Delta(J^l) - 1 = 0,$$

$$\varepsilon(E^e) = \varepsilon(F^e) = \varepsilon(K^e - 1) = \varepsilon(J^l - 1) = 0,$$

$$S(E^e) = S(F^e) = S(K^e) - 1 = S(J^l) - 1 = 0.$$

The only non-trivial computations concern the vanishing \(\Delta(E^e) = \Delta(F^e) = 0\). Following Proposition 2.3,

$$\Delta(E^e) = \sum_{u=0}^{e} q^{u(e-u)} \left[^e\atop u\right] (J^{-ru}E^{e-u} \otimes J^{r(e-u)}E^u).$$

Since \(\left[^e\atop u\right] = 0\) for $0 < u < e$, \(\Delta(E^e) = E^e \otimes J^{rie} + J^{-rte} \otimes E^e\). Thus \(\Delta(E^e) = 0\) as $E^e = 0$. One can prove that \(\Delta(F^e) = 0\) in a similar way.
By Lemma 7.3, we obtain a Hopf algebra $U_{r,t,l}$ with dimension $le^3$. □

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