GEODESIC GROUPS OF MINIMAL SURFACES

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1. Introduction. In a previous paper **(6)** we have studied those minimal surfaces which admit geodesic mappings without isometries or similarities on another, not necessarily minimal, surface. Here we determine all *pairs* of minimal surfaces which can be geodesically mapped on each other. We find that two such surfaces are either:

(i) similar Bonnet associates of each other, or

(ii) both Poisson surfaces (that is, isometric to a plane), or

(iii) both Scherk surfaces **(2)**.

In case (i) the mappings are combinations of trivial transformations with the group of self-isometries of a minimal surface **(7)**. In case (ii) the mappings are generated by the projectivities of a (complex) plane into itself, followed by isometries and similarities. Similarly in case (iii) the mappings are those of the geodesic group of a single complex Scherk surface combined with trivial transformations. The two groups in (ii) and (iii) have entirely different structures; for example, the latter is intransitive and mixed (discrete-continuous).

If one admits only non-trivial mappings transforming a real two-dimensional domain on an image of the same type, then only the real projectivities between real planes will remain. This property might be termed "geodesic rigidity of real minimal surfaces," in contrast to their "isometric flexibility."

2. Analytical formulation. First we dispose of the Poisson surfaces. Since their Gaussian curvature vanishes, any other surface onto which they can be geodesically mapped must be of constant curvature according to Beltrami's theorem. From the Weierstrass representation of a non-cylindrical minimal surface one can conclude that their curvature is never constant. Hence Poisson surfaces can be geodesically mapped only on Poisson surfaces. According to **(6**, Theorem 2**)** this also settles the case of Lie surfaces.

Dini's theorem deals with two surfaces S and S' admitting a non-trivial geodesic mapping. They must be of Liouville's type, and their respective line-elements in corresponding points can be written in suitable coordinates x, y in the forms

$$ds^2 = [A(x) + B(y)]\,(dx^2 + dy^2), \tag{1}$$

$$ds'^2 = -\left(\frac{1}{A} + \frac{1}{B}\right)\left(\frac{dx^2}{A} - \frac{dy^2}{B}\right). \tag{2}$$

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The latter can be reduced again to Liouville's form

$$ds'^2 = -\left(\frac{1}{A} + \frac{1}{B}\right)(dx'^2 + dy'^2) \tag{3}$$

by means of the transformation

$$x' = \int \frac{dx}{\sqrt{A(x)}},\ y' = \int \frac{dy}{\sqrt{-B(y)}}. \tag{4}$$

In (**6**, pp. 321–332), we showed that the most general Liouville systems on non-cylindrical minimal surfaces are of one of the following two types, (5) and (7):

$$ds^2 = G[\lambda e^{(k+2\epsilon)hx} - \mu e^{khx}]^2 e^{khx}(dx^2 + dy^2), \tag{5}$$

with the (complex) constants $G, \lambda, \mu, k, h, \epsilon$ satisfying

$$G \neq 0, h \neq 0, \lambda \neq 0, \mu \neq 0, \qquad \epsilon = \pm 1; \tag{6}$$

or

$$ds^2 = G(2\epsilon x + H)^2 e^{2kx}(dx^2 + dy^2), \tag{7}$$

with G $\neq 0$, $\epsilon = \pm 1$, H and k arbitrary constants.

We have therefore three cases to consider:

(a) both surfaces are of type (5);

(b) one is of type (5) and the other of type (7);

(c) both belong to type (7).

We shall consider in detail only (a) which is the only case that can actually arise; in a similar way one shows that cases (b) and (c) cannot occur.

Let (x, y) be a system of co-ordinates establishing the mapping between S and S' by association of points with the same set of co-ordinates. According to (5) it is a Liouville system on S of the form (1) with

$$A(x) = \alpha(z) = G e^{\beta z}(\lambda e^z - \mu)^2 - D, \tag{8}$$

$$B(y) = D, \qquad D \text{ a non-vanishing constant,} \tag{9}$$

where

$$z = 2\epsilon h x, \tag{10}$$

and

$$\beta = \epsilon k = \kappa - 1, \qquad \text{notation of } (\mathbf{6}, \text{ formula } (73)). \tag{11}$$

Let (x', y') be the system introduced by (4) which reduces the line-element of S' to the form (3). Since S' is by assumption also of type (5) comparison of (3) and (5) shows that there must be constants $G', \lambda', \mu', k', h', \epsilon'$ satisfying analogous conditions to (6) such that

$$\alpha(z) \cdot \alpha'(z') = -1, \tag{12}$$

whenever (4) holds. Here we defined

$$z' = 2\epsilon' h' x', \quad \beta' = \epsilon' k', \tag{13}$$

$$\alpha'(z') = G' e^{\beta' z'}(\lambda' e^{z'} - \mu')^2 + 1/D. \tag{14}$$

Because of the reciprocity of the surfaces S and S' the analytical problem may also be stated as follows: for which choice of the constants in (8) and (14) are the following expressions resulting from (4) inverse functions?

$$(15)\qquad z'(z) = \frac{\epsilon' h'}{\epsilon h}\int \frac{dz}{\sqrt{(\alpha(z))}} \quad \text{and} \quad z(z') = \frac{\epsilon h}{\epsilon' h'}\int \frac{dz'}{\sqrt{(-\alpha'(z'))}}.$$

3. Determination of the constants. From (12) we find by differentiation

$$(16)\qquad \frac{d\alpha'}{d\alpha} = -\frac{\alpha'}{\alpha},$$

and from (15)

$$(17)\qquad \frac{dz'}{dz} = \frac{\epsilon' h'}{\epsilon h}\frac{1}{\sqrt{(\alpha(z))}}.$$

Differentiating (8) and (14) with respect to their arguments z and z', dividing the results, and using (16) and (17) we obtain the following new relation between z and z'.

$$(18)\qquad \frac{\beta + 2\lambda e^z/(\lambda e^z - \mu)}{\beta' + 2\lambda' e^{z'}/(\lambda' e^{z'} - \mu')} = \frac{1}{D}\frac{\epsilon' h'}{\epsilon h}\sqrt{(\alpha(z))}.$$

We can solve (18) for z' and substitute the result in (12), deriving the following identity which contains only the variable z.

$$(19)\qquad G(\lambda')^{\beta'} e^{\beta z} R(z)^{\beta'+2} + 4\, G' D (\mu')^{\beta'+2} h'^2 \alpha^2(z)\, S^{\beta'}(z) = 0.$$

Here we made use of the following abbreviations:

$$R(z) = -\epsilon h D[\lambda(\beta+2)e^z - \beta\mu] - (\beta+2)\epsilon' h' \sqrt{(\alpha(z))}\,(\lambda e^z - \mu),$$
$$S(z) = -\epsilon h D[\lambda(\beta+2)e^z - \beta\mu] - \beta'\epsilon' h' \sqrt{(\alpha(z))}\,(\lambda e^z - \mu).$$

A discussion of (19) will now lead to the conclusion $\beta = -1$. First we see that with $\beta = -2$ equation (19) would be impossible; for $\alpha(z)$ would have at least one zero which, if substituted into (19), would lead to a contradiction. Hence we can assume $\beta \neq -2$. The function $\alpha(z)$ is then, by the assumptions (6), an entire function of order 1 which is not of the form $P(z)\exp(Az)$, ($P(z)$ a polynomial, A a constant) characteristic for such functions with at most a finite number of zeros **(1)**. Consequently $\alpha(z)$ has infinitely many different zeros. Substituting two of them (z_1 and z_2) into (19) gives

$$(20)\qquad e^{z_1} = e^{z_2} = \frac{\beta\mu}{\lambda(\beta+2)},\; z_2 - z_1 = 2\pi i n.$$

Together with $\alpha(z_1) = \alpha(z_2) = 0$, equations (20) yield $e^{2\pi i n\beta} = 1$, which shows that β is a real rational quantity $\beta = m/n \neq 0$, $n > 0$.

It is easily seen that $m > 0$ must be excluded, for in that case the equation $\alpha(z) = 0$ is equivalent to

$$P(\zeta) \equiv \zeta^m(\lambda\zeta - \mu)^{2n} - \left(\frac{D}{G}\right)^n = 0,$$

where $P(\zeta)$ is a polynomial of degree $m + 2n$ in $\zeta = e^z$. According to (20)

$$\zeta_0 = \frac{\beta\mu}{\lambda(\beta + 2)}$$

is its only root. This is a double root, for one verifies that

$$P(\zeta_0) = P'(\zeta_0) = 0, \qquad P''(\zeta_0) \neq 0.$$

Hence $P(\zeta)$ must be of degree 2. But $m + 2n = 2$ contradicts $\beta \neq -2$.

Let therefore $m = -M$, $M > 0$. Now the equation $\alpha(z) = 0$ becomes the polynomial equation

$$Q(\zeta) \equiv (\lambda\zeta - \mu)^{2n} - \left(\frac{D}{G}\right)^n \zeta^M = 0.$$

Here $Q(\zeta)$ must be of degree two by a similar analysis. The case $2n = M$ is incompatible with $\beta \neq -2$, while $2n < M$ would entail $M = 2$ which is impossible because of $n > 0$. Hence we conclude $2n > M$. Under these circumstances the degree of $Q(\zeta)$ is $2n$, or $2n = 2$, $M = 1$, $\beta = -1$.

By virtue of (12) we may replace $\alpha(z)$ on the right of (18) by $-1/\alpha'(z')$, solve for z', and substitute into (12). We obtain an identity analogous to (19) in z'. A similar analysis of the roots of $\alpha'(z')$ establishes also $\beta' = -1$.

Assuming

$$\beta = \beta' = -1, \tag{21}$$

we can reduce equation (19) to a polynomial identity in e^z of the fourth degree. By comparison of the coefficients we obtain the following further conditions:

$$D = -4\lambda\mu G = \frac{1}{4\lambda'\mu' G'}, \tag{22}$$

$$\left(\frac{h'}{h}\right)^2 = -4\lambda\mu G. \tag{23}$$

Equations (21), (22), and (23) are the necessary and sufficient conditions that non-trivial geodesic mappings are possible between two surfaces with line-elements of type (5).

Since we used only the derivative of (15) there is still a constant of integration to be determined. Under the conditions (21–23) we find from (15) that

$$e^{z'} = e^C \left(\frac{1 + \sqrt{(-\lambda/\mu)}\, e^{\frac{1}{2}z}}{1 - \sqrt{(-\lambda/\mu)}\, e^{\frac{1}{2}z}}\right)^{2\epsilon\epsilon'}. \tag{24}$$

Substitution in (12) determines C, viz.

$$e^C = -\mu'/\lambda'.$$

Distinguishing the possibilities for the signs of the square roots and of ϵ and ϵ' one finds from (24) four different relations between z and z' which may be summarized as follows. Let γ be a fixed value of $\sqrt{(-\lambda/\mu)}$, δ a value of $\sqrt{(-\lambda'/\mu')}$ and define

$$Z = \gamma e^{\frac{1}{2}z}, \qquad Z' = \delta e^{\frac{1}{2}z'}. \tag{25}$$

Then our maps are described by the following bilinear transformations (denoted by e, f, g, h for reasons explained later):

$$\text{(e)}\quad Z' = \frac{1+Z}{1-Z}, \qquad \text{(f)}\quad Z' = \frac{1-Z}{1+Z},$$

$$\text{(g)}\quad Z' = -\frac{1+Z}{1-Z}, \qquad \text{(h)}\quad Z' = -\frac{1-Z}{1+Z}.$$

In every case the corresponding connection between y and y' is found from (4) and (9) to be:

$$y' = (-D)^{-\frac{1}{2}} y + y_0. \tag{26}$$

4. Geometrical interpretation. According to **(6**, p. 323**)**, the equality of β and β' indicates that two non-cylindrical minimal surfaces admitting non-trivial geodesic mappings on each other are similar-isometric, that is, they can also be mapped in a trivial way. The value $\beta = -1$ shows that both of the surfaces are Scherk surfaces, obtained from the catenoid or the right helicoid by bending and similarities. They may be represented as in **(2)**:

$$\begin{cases} x_1 = A(\sin\phi \cosh X \cos Y + \cos\phi \sinh X \sin Y), \\ x_2 = A(\sin\phi \cosh X \sin Y - \cos\phi \sinh X \cos Y), \\ x_3 = A(\sin\phi X \qquad\qquad\quad + \cos\phi\, Y). \end{cases} \tag{27}$$

Here $A \neq 0$ and ϕ are arbitrary constants, and X, Y is a special system of Liouville coordinates. For the image surface we have a similar representation, and $X = X'$, $Y = Y'$ establishes the trivial mapping mentioned above. Besides X, Y there are other Liouville systems on S which are all given by transformation of the type

$$X = cx + x_0, \qquad Y = \pm cy + y_0$$

or

$$X = cy + y_0, \qquad Y = \pm cx + x_0.$$

Identifying such a general x, y system with the system introduced in (5) and comparing the line-element obtained from (27) with (5) we see that equations (25) go over into

$$Z = e^X, \qquad Z' = e^{X'}, \tag{28}$$

while (26), if (22) and (23) are taken into account, transforms into

$$Y' = \pm iY + Y_0, \qquad Y_0 = \text{const.} \tag{29}$$

Here (X, Y) and $(X'\ Y')$ are two corresponding points in the representations (27), and the non-trivial mappings are given by (e, f, g, h) and (29).

Inspection of the line-element of (27) shows that, besides the trivial map described by $X = X'$, $Y = Y'$, there are three more classes of such isometric-similar maps. With the notation (28) they are all listed as follows:

$$(a)\quad Z' = Z, \qquad (b)\quad Z' = -Z,$$
$$(c)\quad Z' = \frac{1}{Z}, \qquad (d)\quad Z' = \frac{-1}{Z},$$

and in every case

(30) $$Y' = \pm Y + Y_0.$$

It is easy to see that the non-trivial transformations carry certain real lines into real curves, but no real two-dimensional domain into a similar domain, and hence follows the geodesic rigidity mentioned in the introduction.

Since the geodesics of (27) can be expressed by

$$\tanh X = \text{cn}\,(\pm k^{-1} Y + C; k) \qquad k,\ C, \text{ constants of integration},$$

our mappings illustrate certain transformations of the Jacobian elliptic functions.

5. Structure of the geodesic group of a Scherk surface. All geodesic mappings of a Scherk surface onto itself form a non-abelian, intransitive, mixed discrete-continuous group $\mathfrak{G}$. Writing the transformations (29) and (30) in the form

(31) $$Y' = i^r\, Y + \eta,$$

we may describe every element of $\mathfrak{G}$ by a symbol $\phi = (\varphi, r, \eta)$, where φ denotes one of the bilinear transformations $(a, \ldots, h)$, r is a residue class (mod 4), and η a complex number; if φ is an element from the set (a, b, c, d) r can take only the values 0 or 2, and for φ belonging to (e, f, g, h), r must be either 1 or 3. The product of ϕ by $\phi' = (\varphi', r', \eta')$ is given by

$$\phi\phi' = (\varphi\varphi', r + r', i^r\eta' + \eta),$$

where $\Theta\varphi'$ is the composite of the corresponding bilinear functions.

The transformations φ form a group which is isomorphic to the dihedral group $\mathfrak{D}_4$ of order 8. Denoting by $\mathfrak{L}$ the group of the linear substitutions of a complex variable of the form (31) we find that $\mathfrak{G}$ is an invariant subgroup of the direct product $\mathfrak{D}_4 \times \mathfrak{L}$.

The proper combinations (φ, r) of the first two symbols form a non-abelian group $\mathfrak{F}$ of order 16, the "finite part of $\mathfrak{G}$." It is a normal subgroup of $\mathfrak{D}_4 \times \mathfrak{C}_4$. Its identity element is $E = (a, 0)$, and it may be generated by the two elements

$$P = (e, 1), \qquad R = (b, 0)$$

with the defining relations

$$P^4 = R^2 = (PR)^4 = (P^2R)^2 = E.$$

In fact, $\mathfrak{F}$ is $(4, 4|2, 2)$ in the notation of Coxeter **(4**, p. 81; **5**, p. 421**)**, that is it is the "rotation" group of the regular map $\{4, 4\}_{2,0}$ of four squares covering a torus. By naming the 8 edges as in Fig. 1, we can express the generators as permutations:

$$P = (1\ 2\ 3\ 4)\ (5\ 6\ 7\ 8), \qquad R = (1\ 5)\ (3\ 7).$$

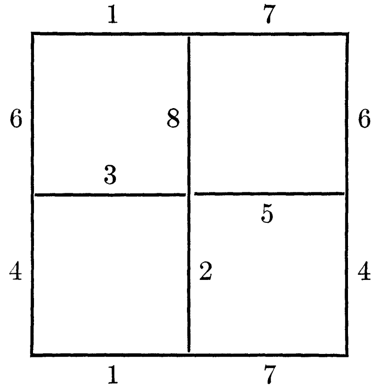

FIGURE 1

According to **(3)** the group $\mathfrak{F}$ can also be generated by the three elements P, R, and $Q = (d, 0)$ with the relations

$$P^4 = Q^2 = R^2 = E, \qquad PQ = QP, \qquad QR = RQ, \qquad PR = RPQ.$$

Every subgroup of $\mathfrak{G}$ induces in $\mathfrak{F}$ a subgroup, and conversely every subgroup of $\mathfrak{F}$ gives rise to subgroups of $\mathfrak{G}$; in the same way invariant subgroups correspond to each other. $\mathfrak{F}$ contains the following 21 proper subgroups which are all abelian (cyclic groups are listed by their generators):

Order 2:

$$\mathfrak{C}_2^1 = (a, 2),\ \mathfrak{C}_2^2 = (b, 0),\ \mathfrak{C}_2^3 = (b, 2),$$
$$\mathfrak{C}_2^4 = (c, 0),\ \mathfrak{C}_2^5 = (c, 2),\ \mathfrak{C}_2^6 = (d, 0),\ \mathfrak{C}_2^7 = (d, 2).$$

Order 4: There are 7 Kleinian groups D_2 and 4 cyclic groups C_4, viz.

$$\mathfrak{D}_2^1 = \mathfrak{C}_2^1 \times \mathfrak{C}_2^2, \qquad \mathfrak{D}_2^2 = \mathfrak{C}_2^1 \times \mathfrak{C}_2^4, \qquad \mathfrak{D}_2^3 = \mathfrak{C}_2^1 \times \mathfrak{C}_2^6,$$
$$\mathfrak{D}_2^4 = \mathfrak{C}_2^2 \times \mathfrak{C}_2^4, \qquad \mathfrak{D}_2^5 = \mathfrak{C}_2^2 \times \mathfrak{C}_2^5, \qquad \mathfrak{D}_2^6 = \mathfrak{C}_2^3 \times \mathfrak{C}_2^4, \qquad \mathfrak{D}_2^7 = \mathfrak{C}_2^3 \times \mathfrak{C}_2^5;$$
$$\mathfrak{C}_4^1 = (e, 1), \qquad \mathfrak{C}_4^2 = (e, 3), \qquad \mathfrak{C}_4^3 = (f, 1), \qquad \mathfrak{C}_4^4 = (g, 1).$$

Order 8:

$$\mathfrak{U}^1 = \mathfrak{C}_4^1 \times \mathfrak{C}_2^1, \qquad \mathfrak{U}^2 = \mathfrak{C}_4^3 \times \mathfrak{C}_2^6, \qquad \mathfrak{U}^3 = \mathfrak{C}_2^1 \times \mathfrak{C}_2^2 \times \mathfrak{C}_2^6.$$

The invariant proper subgroups of $\mathfrak{F}$ are:

$$\mathfrak{C}_2^1, \qquad \mathfrak{C}_2^6 \quad (= \text{commutator group}), \qquad \mathfrak{C}_2^7,$$
$$\mathfrak{D}_2^3\ (= \text{centre}), \qquad \mathfrak{D}_2^4, \qquad \mathfrak{D}_2^7, \qquad \mathfrak{U}^1, \qquad \mathfrak{U}^2, \qquad \mathfrak{U}^3.$$

The trivial geodesic mappings form a non-abelian invariant subgroup of $\mathfrak{G}$ the finite part of which is $\mathfrak{U}^3$. The product of any two non-trivial mappings is trivial; in particular the non-trivial mappings may be considered as "square roots" of isometries.

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