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On the Regularity of the Multisublinear Maximal Functions

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Abstract. This paper is concerned with the study of the regularity for the multisublinear maximal operator. It is proved that the multisublinear maximal operator is bounded on first-order Sobolev spaces. Moreover, two key point-wise inequalities for the partial derivatives of the multisublinear maximal functions are established. As an application, the quasi-continuity on the multisublinear maximal function is also obtained.

1 Introduction

Let d be a positive integer and let $\mathbb{R}^d$ be the d-dimensional Euclidean spaces. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the centered Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

for any $x \in \mathbb{R}^d$, where $B(x,r)$ is the ball in $\mathbb{R}^d$ centered at x with radius r and $|B(x,r)|$ denotes the volume of $B(x,r)$. As is well known, the operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p \le \infty$ and maps $L^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$. In 1997, Kinnunen [6] first studied the regularity of $\mathcal{M}$ and showed that $\mathcal{M}$ is bounded on the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$ for all $1 < p \le \infty$. Subsequently, Kinnunen and Lindqvist [7] gave a local version of the original boundedness on $W^{1,p}(\Omega)$, where Ω is an open set of $\mathbb{R}^d$. This paradigm that an L^p-bound implies a $W^{1,p}$-bound was later extended to a fractional version in [8] and to a bilinear version in [2]. Later on, the continuity of $\mathcal{M}$ on $W^{1,p}(\mathbb{R}^d)$ for all $1 < p \le \infty$ was studied by Luiro in [10] (continuity is not immediate from boundedness because of the lack of linearity).

Since Kinnunen's result does not hold for the case $p = 1$, understanding the regularity at the endpoint case seems to be a deeper issue. In this regard, one of the main questions was posed by Hajlasz and Onninen in [5, Question 1]. Is the operator $f \mapsto |\nabla(\mathcal{M}(f))|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$? Here, $\nabla(f)$ denotes the weak gradient of the Sobolev function f. In 2002, Tanaka [11] showed that the non-centered maximal operator $\widetilde{\mathcal{M}}$ satisfies

$$\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} \le 2\|f'\|_{L^1(\mathbb{R})}$$

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if $f \in W^{1,1}(\mathbb{R})$, where $\widetilde{\mathcal{M}}$ is given by

$$\widetilde{\mathcal{M}}(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)|dy$$

for $x \in \mathbb{R}^d$, where the supremum is taken over all ball $B \subset \mathbb{R}^d$ containing x. Subsequently, Tanaka's result was sharpened by Aldaz and Lázaro [1], who obtained the sharp constant $C = 1$. Recently, Kurka [9] extended Tanaka's result to $\mathcal{M}$.

This paper is devoted to studying the regularity properties of the multisublinear maximal operator. Precisely, let m be a positive integer and

$$\vec{f} = (f_1, \dots, f_m) \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \times \cdots \times L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

For $0 \le \alpha < md$, the m-sublinear maximal operator $\mathscr{M}_\alpha$ is defined by

$$\mathscr{M}_\alpha(\vec{f})(x) = \sup_{r>0} |B(x,r)|^{\alpha/d-m} \prod_{j=1}^m \int_{B(x,r)} |f_j(y)|dy$$

for any $x \in \mathbb{R}^d$. For $m = 1$ and $\alpha = 0$, $\mathscr{M}_\alpha$ recovers the operator $\mathcal{M}$. For $m = 1$ and $0 < \alpha < d$, the operator $\mathscr{M}_\alpha$ recovers the classical fractional maximal operator $\mathcal{M}_\alpha$ defined by

$$\mathcal{M}_\alpha(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/d}} \int_{B(x,r)} |f(y)|dy.$$

It is well known that the following inequalities are valid:

$$\mathscr{M}_\alpha(\vec{f}) \le \prod_{i=1}^m \mathcal{M}_{\alpha_i}(f_i),$$

where $\alpha = \sum_{i=1}^m \alpha_i$ with $\alpha_i \ge 0$ $(i = 1, \dots, m)$, and

$$\|\mathscr{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} \le c(d,m,\alpha) \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^d)} \tag{1.1}$$

for $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$, provided one of the following conditions holds: (i) $\alpha = 0$, $1 \le q \le \infty$ and $1 < p_1, \dots, p_m \le \infty$; (ii) $0 < \alpha < d$, $1 \le q < \infty$ and $1 < p_1, \dots, p_m \le \infty$; (iii) $d \le \alpha < md$, $1 \le q < \infty$ and $1 < p_1, \dots, p_m < \infty$;

Based on the facts concerning the regularity of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, it is interesting and natural to ask whether the multisublinear maximal operator has some sort of regularity properties. The purpose of this paper is to address this problem. Our main results will be formulated in Section 2, which is organized as follows. After giving some definitions, we will show the boundedness of the multisublinear maximal operator on the first-order Sobolev spaces. Subsequently, motivated by Kinnunen and Saksman's work [8] on the one-sublinear fractional maximal operator, a key pointwise inequality for the partial derivatives of $\mathscr{M}_\alpha(\vec{f})$ will be obtained; furthermore, employing the idea in [6], we will establish another pointwise inequality for the partial derivatives. Finally, we will end the paper by proving the quasicontinuity of the multisublinear maximal operator in Section 3, which is closely related to the regularity problems studied in Section 2.

Throughout this paper, the letter c, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence, but

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independent of the essential variables. In what follows, we use the conventions

$$\prod_{i\in\varnothing} a_i = 1 \quad \text{and} \quad \sum_{i\in\varnothing} a_i = 0.$$

2 The Regularity for the Multisublinear Maximal Functions

Recall that the Sobolev space $W^{1,p}(\mathbb{R}^d)$, $1 \le p \le \infty$, consists of functions $f \in L^p(\mathbb{R}^d)$, whose first distributional partial derivatives $D_i(f)$, $i = 1, \dots, m$, belong to $L^p(\mathbb{R}^d)$. We endow $W^{1,p}(\mathbb{R}^d)$ with the norm

$$\|f\|_{1,p} = \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla(f)\|_{L^p(\mathbb{R}^d)},$$

where $\nabla(f) = (D_1(f), \dots, D_d(f))$ is the weak gradient of f. See [4] for the basic properties of Sobolev functions. The following result shows that the multisublinear maximal operator preserves first-order Sobolev spaces.

Theorem 2.1 *Let $0 \le \alpha < md$ and $\vec{f} = (f_1, \dots, f_m)$ with $f_i \in W^{1,p_i}(\mathbb{R}^d)$ for $i = 1, \dots, m$. Suppose that $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$. Then $\mathcal{M}_\alpha(\vec{f}) \in W^{1,q}(\mathbb{R}^d)$, provided one of the following conditions holds:*

(i) *$\alpha = 0$, $1 \le q \le \infty$ and $1 < p_1, \dots, p_m \le \infty$;*
(ii) *$0 < \alpha < d$, $1 \le q < \infty$ and $1 < p_1, \dots, p_m \le \infty$;*
(iii) *$d \le \alpha < md$, $1 \le q < \infty$ and $1 < p_1, \dots, p_m < \infty$.*

More precisely, there exists a constant $c = c(d, \alpha, m, p_1, \dots, p_m) > 0$ such that

$$\|\mathcal{M}_\alpha(\vec{f})\|_{1,q} \le cm \prod_{i=1}^m \|f_i\|_{1,p_i}.$$

Proof By the definition of $\mathcal{M}_\alpha$, we have

$$\begin{aligned}
(2.1)\qquad & |\mathcal{M}_\alpha(\vec{f})(x+h) - \mathcal{M}_\alpha(\vec{f})(x)| \\
&\le \sup_{r>0} |B(x,r)|^{\frac{\alpha}{d}-m} \Big| \prod_{i=1}^m \int_{B(x+h,r)} |f_i(y)|dy - \prod_{i=1}^m \int_{B(x,r)} |f_i(y)|dy \Big| \\
&= \sup_{r>0} |B(x,r)|^{\frac{\alpha}{d}-m} \Big| \prod_{i=1}^m \int_{B(x,r)} |f_i(y+h)|dy - \prod_{i=1}^m \int_{B(x,r)} |f_i(y)|dy \Big| \\
&\le \sum_{l=0}^{m-1} \sup_{r>0} |B(x,r)|^{\frac{\alpha}{d}-m} \prod_{i=1}^{l} \int_{B(x,r)} |f_i(y)|dy \prod_{i=l+2}^{m} \int_{B(x,r)} |f_i(y+h)|dy \\
&\quad \times \int_{B(x,r)} |f_{l+1}(y+h) - f_{l+1}(y)|dy
\end{aligned}$$

for any $x, h \in \mathbb{R}^d$. Thus, we get from (2.1) that

$$(2.2)\qquad |\nabla(\mathcal{M}_\alpha(\vec{f}))(x)| \le \sum_{l=1}^m \mathcal{M}_\alpha(\vec{f}^{\,l})(x)$$

for almost everywhere $x \in \mathbb{R}^d$, where $\vec{f}^{\,l} = (f_1, \dots, f_{l-1}, |\nabla(f_l)|, f_{l+1}, \dots, f_m)$. Theorem 2.1 follows from (1.1) and (2.2). ∎

We shall give two key inequalities for partial derivatives in the following theorem.

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Theorem 2.2 *Let $\ell = 1, \dots, m$ and $\vec{f} = (f_1, \dots, f_m) \in L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d)$ with $p_i > 1$ for $i = 1, \dots, m$. Suppose that $0 \le \alpha < md$. Then the weak partial derivatives $D_\ell(\mathscr{M}_\alpha(\vec{f}))$ exist almost everywhere. Precisely, there exists a constant $c = c(d, \alpha)$ such that*

$$\left| D_\ell(\mathscr{M}_\alpha(\vec{f}))(x) \right| \le c \sum_{j=1}^{m} \mathscr{M}_\alpha\big(D_\ell(\vec{f})_j \big)(x) \quad a.e.\ x \in \mathbb{R}^d,$$

where $D_\ell(\vec{f})_j := (f_1, \dots, f_{j-1}, D_\ell(f_j), f_{j+1}, \dots, f_m)$.

Proof Let s_k $(k = 1, 2, \dots)$ be an enumeration of positive rational numbers. We can write

$$\mathscr{M}_\alpha(\vec{f})(x) = \sup_k |B(x, s_k)|^{\frac{\alpha}{d} - m} \prod_{j=1}^{m} \int_{B(x, s_k)} |f_j(y)| dy.$$

For $k \in \{1, 2, \dots\}$, we define the operator T_k by

$$T_k(\vec{f}) = \max_{1 \le i \le k} |B(x, s_i)|^{\frac{\alpha}{d} - m} \prod_{j=1}^{m} \int_{B(x, s_i)} |f_j(y)| dy.$$

Obviously, $\{T_k(\vec{f})\}_k$ is an increasing sequence of functions and converges to $\mathscr{M}_\alpha(\vec{f})$ pointwise. On the other hand, by the same arguments used in obtaining (2.1), we have

$$|D_\ell(T_k(\vec{f}))| \le \sum_{j=1}^{m} \mathscr{M}_\alpha(D_\ell(\vec{f})_j). \tag{2.3}$$

Since $\{T_k(\vec{f})\}_k$ converges to $\mathscr{M}_\alpha(\vec{f})$ pointwise, from this and (1.1) we know that $\{D_\ell(T_k(\vec{f}))\}_k$ converges to $D_\ell(\mathscr{M}_\alpha(\vec{f}))$ weakly in $L^q(\mathbb{R}^d)$, where

$$1/q = \sum_{i=1}^{m} 1/p_i - \alpha/d.$$

This together with (2.3) implies

$$\left| D_\ell(\mathscr{M}_\alpha(\vec{f}))(x) \right| \le c \sum_{j=1}^{m} \mathscr{M}_\alpha\big(D_\ell(\vec{f})_j \big)(x) \quad a.e.\ x \in \mathbb{R}^d.$$

This proves Theorem 2.2. ∎

Theorem 2.3 *Let $\ell = 1, \dots, m$ and $\vec{f} = (f_1, \dots, f_m) \in L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d)$ with $p_i > 1$ for $i = 1, \dots, m$. Suppose that $1 \le \alpha < \sum_{i=1}^{m} d/p_i$.*

(i) *Then the weak partial derivatives $D_\ell(\mathscr{M}_\alpha(\vec{f}))$ exist almost everywhere. Precisely, there exists a constant $c = c(d, \alpha)$ such that*

$$|D_\ell(\mathscr{M}_\alpha(\vec{f}))(x)| \le c \mathscr{M}_{\alpha - 1}(\vec{f})(x) \quad a.e.\ x \in \mathbb{R}^d.$$

(ii) *Suppose that $1/q_1 = \sum_{i=1}^{m} 1/p_i - \alpha/d$ and $1/q_2 = \sum_{i=1}^{m} 1/p_i - (\alpha - 1)/d$. Then there exists a constant $c = c(d, \alpha, m, p_1, \dots, p_m)$ such that*

$$\|\mathscr{M}_\alpha(\vec{f})\|_{L^{q_1}(\mathbb{R}^d)} \le c \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)},$$

$$\|D_\ell(\mathscr{M}_\alpha(\vec{f}))\|_{L^{q_2}(\mathbb{R}^d)} \le c \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)}.$$

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Proof Suppose that $f_j \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ for $j = 1, \dots, m$. First we shall claim that $\mathscr{M}_\alpha(\vec{f})$ is Lipschitz continuous on $\mathbb{R}^d$. Thus it suffices to prove that there exists a constant c that depends only on f_j $(j = 1, \dots, m)$ and d, m, α such that

$$\left|\mathscr{M}_\alpha(\vec{f})(x+h) - \mathscr{M}_\alpha(\vec{f})(x)\right| \le c|h| \tag{2.4}$$

for any $x, h \in \mathbb{R}^d$. To this end, by (2.1) we have

$$\begin{aligned}\left|\mathscr{M}_\alpha(\vec{f})(x+h) - \mathscr{M}_\alpha(\vec{f})(x)\right| &\le \sum_{l=1}^m \sup_{r>0} |B(x,r)|^{\frac{\alpha}{d}-m} \int_{B(x,r)} |f_l(y+h) - f_l(y)|dy \\ &\quad \times \prod_{i=1}^{l-1} \int_{B(x,r)} |f_i(y+h)|dy \prod_{j=l+1}^m \int_{B(x,r)} |f_j(y)|dy \\ &:= \sum_{l=1}^m \mathscr{M}_\alpha^l(\vec{f})(x,h).\end{aligned}$$

For fixed $l \in \{1, \dots, m\}$ and any $x, h \in \mathbb{R}^d$, to prove (2.4), it suffices to prove that there exists a constant c that depends only on f_j $(j = 1, \dots, m)$ and d, α such that

$$|\mathscr{M}_\alpha^l(\vec{f})(x,h)| \le c|h|. \tag{2.5}$$

Observe that

$$\left|\mathscr{M}_\alpha^l(\vec{f})(x,h)\right| \le \left(\mathscr{M}_\alpha^{l,1}(\vec{f})(x,h) + \mathscr{M}_\alpha^{l,2}(\vec{f})(x,h)\right), \tag{2.6}$$

where

$$\begin{aligned}\mathscr{M}_\alpha^{l,1}(\vec{f})(x,h) &\le \sup_{0<r\le 2} |B(x,r)|^{\frac{\alpha}{d}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_l(y+h) - f_l(y)|dy \\ &\quad \times \prod_{i=1}^{l-1} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_i(y+h)|dy \prod_{j=l+1}^m \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_j(y)|dy\end{aligned}$$

and

$$\begin{aligned}\mathscr{M}_\alpha^{l,2}(\vec{f})(x,h) &\le \sup_{r>2} |B(x,r)|^{\frac{\alpha}{d}-m} \int_{B(x,r)} |f_l(y+h) - f_l(y)|dy \\ &\quad \times \prod_{i=1}^{l-1} \int_{B(x,r)} |f_i(y+h)|dy \prod_{j=l+1}^m \int_{B(x,r)} |f_j(y)|dy.\end{aligned}$$

By our assumption and the mean value theorem for differentials, it is easy to see that

$$|\mathscr{M}_\alpha^{l,1}(\vec{f})(x,h)| \le 2^\alpha c(\vec{f})|h|. \tag{2.7}$$

It remains to estimate $\mathscr{M}_\alpha^{l,2}(\vec{f})(x,h)$. We consider the following two cases. If $|h| > 1$, since $\alpha < md$, by our assumption we have

$$|\mathscr{M}_\alpha^{l,2}(\vec{f})(x,h)| \le c(\vec{f}) \le c(\vec{f})|h|. \tag{2.8}$$

If $|h| \le 1$, we also assume that the function f_l is supported in $B(0, r_l)$. Thus, by the mean value theorem for differentials, we obtain

$$\int_{B(x,r)} |f_l(y+h) - f_l(y)|dy \le \int_{B(0,r_l+1)} |f_l(y+h) - f_l(y)|dy \le c(f_l)|h|,$$

which implies

$$|\mathscr{M}_\alpha^{l,2}(\vec{f})(x,h)| \le c(\vec{f})|h|. \tag{2.9}$$

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Thus (2.5) follows from (2.6)–(2.9). This proves our claim. We denote

$$D^+(\mathscr{M}_\alpha(\vec{f}))(x) = \limsup_{|h|\to 0} \frac{\mathscr{M}_\alpha(\vec{f})(x+h) - \mathscr{M}_\alpha(\vec{f})(x)}{|h|}$$

for every $x \in \mathbb{R}^d$. Below we shall prove the following claim

$$D^+(\mathscr{M}_\alpha(\vec{f}))(x) \le c\mathscr{M}_{\alpha-1}(\vec{f})(x) \tag{2.10}$$

for every $x \in \mathbb{R}^d$, where $c = c(d, m, \alpha)$. Indeed, by the definition of D^+, we can choose a sequence $\{h_k\}_k \subset \mathbb{R}^d$ such that

$$D^+(\mathscr{M}_\alpha(\vec{f}))(x) - \frac{1}{k} \le \frac{\mathscr{M}_\alpha(\vec{f})(x+h_k) - \mathscr{M}_\alpha(\vec{f})(x)}{|h_k|}$$

and $\lim_{k\to\infty} |h_k| = 0$. Then we have

$$\Big(D^+(\mathscr{M}_\alpha(\vec{f}))(x) - \frac{1}{k}\Big)|h_k| + \mathscr{M}_\alpha(\vec{f})(x) \le \mathscr{M}_\alpha(\vec{f})(x+h_k). \tag{2.11}$$

By the definition of $\mathscr{M}_\alpha(\vec{f})$, we can choose a sequence of positive numbers $\{r_j\}_{j\ge 1}$ such that

$$\mathscr{M}_\alpha(\vec{f})(x+h_k) < |B(x+h_k, r_j)|^{\frac{\alpha}{d}-m} \prod_{i=1}^m \int_{B(x+h_k,r_j)} |f_i(y)|dy + \frac{1}{j}. \tag{2.12}$$

It is obvious that

$$\mathscr{M}_\alpha(\vec{f})(x) > |B(x, |h_k| + r_j)|^{\frac{\alpha}{d}-m} \prod_{i=1}^m \int_{B(x,|h_k|+r_j)} |f_i(y)|dy. \tag{2.13}$$

Note that $B(x+h_k, r_j) \subset B(x, |h_k| + r_j)$. Thus we get from (2.11)–(2.13) that

$$\Big(D^+(\mathscr{M}_\alpha(\vec{f}))(x) - \frac{1}{k}\Big)|h_k| \le c(d, m, \alpha)(r_j^{\alpha-md} - (r_j + |h_k|)^{\alpha-md}) \prod_{i=1}^m \int_{B(x,|h_k|+r_j)} |f_i(y)|dy + \frac{1}{j}. \tag{2.14}$$

By the mean-value theorem of differentials, we get

$$r_j^{\alpha-md} - (r_j + |h_k|)^{\alpha-md} \le (md - \alpha) r_j^{\alpha-1-md}|h_k|. \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$D^+(\mathscr{M}_\alpha(\vec{f}))(x) \le \frac{1}{k} + c(d, m, \alpha)\Big(\frac{r_j + |h_k|}{r_j}\Big)^{md+1-\alpha} \mathscr{M}_{\alpha-1}(\vec{f})(x) + \frac{1}{j|h_k|}. \tag{2.16}$$

Fixing $k = 1, 2, \ldots$, we can choose a large j such that $j|h_k| > k$. Thus (2.16) leads to

$$D^+(\mathscr{M}_\alpha(\vec{f}))(x) \le \frac{2}{k} + c(d, m, \alpha)\Big(\frac{r_j + |h_k|}{r_j}\Big)^{md+1-\alpha} \mathscr{M}_{\alpha-1}(\vec{f})(x). \tag{2.17}$$

Letting $k \to \infty$, (2.17) implies (2.10). Now we suppose that $f_i \in L^{p_i}(\mathbb{R}^d)$. Fixing $i \in \{1, \ldots, m\}$, we choose a sequence $f_i^l \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $f_i^l \to f_i$ in $L^{p_i}(\mathbb{R}^d)$ as $l \to \infty$. Let $\vec{g^k} = (f_1^k, \ldots, f_m^k)$. Obviously $\{\vec{g^k}\}_k$ converges to $\vec{f}$ pointwise. Since $\mathscr{M}_\alpha(\vec{f})$ is Lipschitz, it is differentiable at almost every $x \in \mathbb{R}^d$. Equation (2.10) yields

$$|D_\ell(\mathscr{M}_\alpha(\vec{g^k}))(x)| \le D^+(\mathscr{M}_\alpha(\vec{g^k}))(x) \le c\mathscr{M}_{\alpha-1}(\vec{g^k})(x), \ a.e.\ x \in \mathbb{R}^d,$$

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where $c = c(d, \alpha, m)$. We have from (1.1) that

$$\|D_\ell(\mathscr{M}_\alpha(\vec{g^k}))\|_{L^{q_2}(\mathbb{R}^d)} \le c\|\mathscr{M}_{\alpha-1}(\vec{g^k})\|_{L^{q_2}(\mathbb{R}^d)} \le c\prod_{i=1}^m \|f_i^k\|_{L^{p_i}(\mathbb{R}^d)},$$

which tells us that $\{D_\ell(\mathscr{M}_\alpha(\vec{g^k}))\}_k$ is a bounded sequence in $L^{q_2}(\mathbb{R}^d)$.

On the other hand, we have

$$\begin{aligned}(2.18)\qquad &\mathscr{M}_\alpha(\vec{g^k})(x) - \mathscr{M}_\alpha(\vec{f})(x)| \\ &\le \sup_{r>0}|B(x,r)|^{\frac{\alpha}{d}-m}\Big|\prod_{i=1}^m\int_{B(x,r)}|f_i^k(y)|dy - \prod_{i=1}^m\int_{B(x,r)}|f_i(y)|dy\Big| \\ &\le \sum_{l=0}^{m-1}\sup_{r>0}|B(x,r)|^{\frac{\alpha}{d}-m}\prod_{i=1}^l\int_{B(x,r)}|f_i(y)|dy\prod_{j=l+2}^m\int_{B(x,r)}|f_j^k(y)|dy \\ &\quad\times\int_{B(x,r)}|f_{l+1}^k(y) - f_{l+1}(y)|dy \\ &\le \sum_{l=1}^m \mathscr{M}_\alpha(\vec{F_l^k})(x)\end{aligned}$$

for any $x \in \mathbb{R}^d$, where $\vec{F_l^k} = (f_1, \dots, f_{l-1}, f_l^k - f_l, f_{l+1}^k, \dots, f_m^k)$. Then, combining (1.1) with Minkowski's inequality implies that

$$\begin{aligned}&\|\mathscr{M}_\alpha(\vec{g^k}) - \mathscr{M}_\alpha(\vec{f})\|_{L^{q_1}(\mathbb{R}^d)} \le \\ &\qquad c(d,m,\alpha)\sum_{l=1}^m\prod_{i=1}^l\|f_i\|_{L^{p_i}(\mathbb{R}^d)}\|f_l^k - f_l\|_{L^{p_l}(\mathbb{R}^d)}\prod_{j=l+1}^m\|f_j^k\|_{L^{p_j}(\mathbb{R}^d)},\end{aligned}$$

which tells us that $\mathscr{M}_\alpha(\vec{g^k}) \to \mathscr{M}_\alpha(\vec{f})$ in $L^{q_1}(\mathbb{R}^d)$ as $k \to \infty$. Thus we can claim that $D_\ell(\mathscr{M}_\alpha(\vec{f}))$ belongs to $L^{q_2}(\mathbb{R}^d)$, and $D_\ell(\mathscr{M}_\alpha(\vec{g^k})) \to D_\ell(\mathscr{M}_\alpha(\vec{f}))$ weakly in $L^{q_2}(\mathbb{R}^d)$ as $k \to \infty$. Using this and (2.18), we conclude that

$$|D_\ell(\mathscr{M}_\alpha(\vec{f}))(x)| \le c\mathscr{M}_{\alpha-1}(\vec{f})(x) \quad a.e.\ x \in \mathbb{R}^d.$$

This implies (i). Part (ii) follows from (i) and (1.1). This completes the proof of Theorem 2.3. ∎

3 A Capacity Inequality and Quasicontinuity for the Multisublinear Maximal Functions

This section is devoted to studying the continuity of $\mathscr{M}_\alpha$. First, let us recall some notation.

Definition 3.1 For $1 < p < \infty$, the Sobolev p-capacity of the set $E \subset \mathbb{R}^d$ is defined by

$$C_p(E) := \inf_{f\in\mathcal{A}(E)}\int_{\mathbb{R}^d}(|f(y)|^p + |\nabla(f)(y)|^p)dy,$$

where

$$\mathcal{A}(E) = \{f \in W^{1,p}(\mathbb{R}^d) : f \ge 1 \text{ on a neighbourhood of } E\}.$$

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We set $C_p(E) = \infty$ if $\mathcal{A}(E) = \varnothing$. It was shown in [3] that the Sobolev p-capacity is a monotone and a contably subadditive set function. Also, it is outer measure over $\mathbb{R}^d$.

Definition 3.2 A function f is p-quasi-continuous in $\mathbb{R}^d$ if for every $\epsilon > 0$, there exists a set $F \subset \mathbb{R}^d$ such that $C_p(F) < \epsilon$ and the restriction of f to $\mathbb{R}^d \backslash F$ is continuous and finite. A property holds p-quasi-everywhere if it holds outside a set of the Sobolev p-capacity zero.

Remark 3.3 As is well known, each Sobolev function has a quasi-continuous representative; that is, for each $u \in W^{1,p}(\mathbb{R}^d)$ there is a p-quasi-continuous function $v \in W^{1,p}(\mathbb{R}^d)$ such that $u = v$ *a.e.* in $\mathbb{R}^d$. We remark that this representative is unique in the following sense. If v and w are p-quasi-continuous and $v = w$ *a.e.*, then $w = v$ p-quasi-everywhere in $\mathbb{R}^d$; see [3] for more details.

In 1997, J. Kinnunen proved that $\mathcal{M}(f)$ is p-quasi-continuous if $f \in W^{1,p}(\mathbb{R}^d)$ for any $1 < p < \infty$. Motivated by J. Kinnunen's work [6], we shall establish the following theorem.

Theorem 3.4 *Let $0 \le \alpha < md$ and $\vec{f} = (f_1, \dots, f_m)$. Suppose that $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$ and $p_i > 1$, $i = 1, \dots, m$. If $f_i \in W^{1,p_i}(\mathbb{R}^d)$, $i = 1, \dots, m$, then $\mathscr{M}_\alpha(\vec{f})$ is q-quasi-continuous.*

Proof Let us begin by proving a capacity inequality that can be used in studying the pointwise behaviour of Sobolev functions by the standard methods (see [3]). For $\lambda > 0$, we denote

$$O_\lambda = \{x \in \mathbb{R}^d : \mathscr{M}_\alpha(\vec{f})(x) > \lambda\}.$$

Obviously, O_λ is an open set. We get from Theorem 2.1 that

$$\begin{aligned} C_q(O_\lambda) &\le \frac{1}{\lambda^q} \int_{\mathbb{R}^d} (|\mathscr{M}_\alpha(\vec{f})(x)|^q + |\nabla(\mathscr{M}_\alpha(\vec{f}))(x)|^q)dx \\ &\le \frac{\|\mathscr{M}_\alpha(\vec{f})\|_{1,q}}{\lambda^q} \le \frac{A^q}{\lambda^q} \prod_{i=1}^m \|f_i\|_{1,p_i}, \end{aligned} \tag{3.1}$$

where $A = c(d, m, \alpha, p_1, \dots, p_m)$. Suppose that $\vec{g} = (g_1, \dots, g_m)$ with $g_j \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $j = 1, \dots, m$. From the claim in the proof of Theorem 2.3, we know that $\mathscr{M}_\alpha(\vec{g})$ is continuous. For each $f_j \in W^{1,p_j}(\mathbb{R}^d)$, $j = 1, \dots, m$, we can choose a sequence of function $\{f_j^k\}_k \subset \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $f_j^k \to f_j$ in $W^{1,p_j}(\mathbb{R}^d)$. This means that there exists a large positive integer K_0 such that

$$\|f_j^k - f_j\|_{1,p_j} \le 2^{-2k} A^{-1}, \quad j = 1, \dots, m \tag{3.2}$$

whenever $k \ge K_0$. For $k \ge 1$, we set $\vec{g^k} = (f_1^k, \dots, f_m^k)$ and

$$E_k = \{x \in \mathbb{R}^d : |\mathscr{M}_\alpha(\vec{f})(x) - \mathscr{M}_\alpha(\vec{f^k})(x)| > 2^{-k}\}.$$

Using (2.18) we have

$$|\mathscr{M}_\alpha(\vec{f})(x) - \mathscr{M}_\alpha(\vec{g^k})(x)| \le \sum_{l=1}^m \mathscr{M}_\alpha(\vec{F_l^k})(x) \tag{3.3}$$

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for every $x \in \mathbb{R}^d$, where $\vec{F}_l^k$ is as in (2.18). Thus we obtain from (3.3) that

$$E_k \subset \Big\{ x \in \mathbb{R}^d : \sum_{l=1}^{m} \mathscr{M}_\alpha(\vec{F}_l^k)(x) > 2^{-k} \Big\}. \tag{3.4}$$

We get from (3.1), (3.2), and (3.4) that

$$\begin{aligned} C_q(E_k)^{1/q} &\le 2^k A \sum_{l=1}^{m} \prod_{i=1}^{l-1} \|f_i\|_{1,p_i} \|f_l^k - f_l\|_{1,p_l} \prod_{j=l+1}^{m} \|f_j^k\|_{1,p_j} \\ &\le 2^{-k} \sum_{l=1}^{m} \prod_{\substack{i=1 \\ i\neq l}}^{m} \|f_i\|_{1,p_i} \end{aligned} \tag{3.5}$$

for $k \ge K_0$. Let $G_i = \bigcup_{k=i}^{\infty} E_k$ with $i \ge K_0$. Then by subadditivity and (3.5) we have

$$C_q(G_i)^{1/q} \le \sum_{k=i}^{\infty} 2^{-k} \sum_{l=1}^{m} \prod_{\substack{i=1 \\ i\neq l}}^{m} \|f_i\|_{1,p_i} < \infty$$

whenever $i \ge K_0$, which implies

$$\lim_{i\to\infty} C_q(G_i) = 0. \tag{3.6}$$

On the other hand, we have for $x \in \mathbb{R}^d \backslash G_i$,

$$|\mathscr{M}_\alpha(\vec{f})(x) - \mathscr{M}_\alpha(\vec{g^k})(x)| \le 2^{-k}$$

whenever $k \ge i$, which implies that the sequence of functions $\{\mathscr{M}_\alpha(\vec{g^k})\}_k$ converges to $\mathscr{M}_\alpha(\vec{f})$ uniformly in $\mathbb{R}^d \backslash G_i$. Theorem 3.4 follows from this and (3.6). ∎

Remark 3.5 When $m = 1$ and $\alpha = 0$, Theorem 3.4 implies [6, Theorem 4.1]. As a consequence, we obtain that $\mathcal{M}_\alpha(f)$ is q-quasi-continuous if $f \in W^{1,p}(\mathbb{R}^d)$ and $1/q = 1/p - \alpha/d$ for $1 < p < d/\alpha$.

References

[1] J. M. Aldaz and J. Pérez Lázaro, *Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities.* Trans. Amer. Math. Soc. **359**(2007), no. 5, 2443–2461. http://dx.doi.org/10.1090/S0002-9947-06-04347-9

[2] E. Carneiro and D. Moreira, *On the regularity of maximal operators.* Proc. Amer. Math. Soc. **136**(2008), no. 12, 4395–4404. http://dx.doi.org/10.1090/S0002-9939-08-09515-4

[3] H. Federer and W. Ziemer, *The Lebesgue set of a function whose distribution derivatives are p-th power summable.* Indiana Univ. Math. J. **22**(1972/73), 139–158. http://dx.doi.org/10.1512/iumj.1973.22.22013

[4] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Second ed., Grundlehren der Mathematischen Wissenschaften, 224, Springer-Verlag, Berlin, 1983.

[5] P. Hajlasz and J. Onninen, *On boundedness of maximal functions in Sobolev spaces.* Ann. Acad. Sci. Fenn. Math. **29**(2004), no. 1, 167–176.

[6] J. Kinnunen, *The Hardy-Littlewood maximal function of a Sobolev function.* Israel J. Math. **100**(1997), 117–124. http://dx.doi.org/10.1007/BF02773636

[7] J. Kinnunen and P. Lindqvist, *The derivative of the maximal function.* J. Reine. Angew. Math. **503**(1998), 161–167.

[8] J. Kinnunen and E. Saksman, *Regularity of the fractional maximal function.* Bull. London Math. Soc. **35**(2003), no. 4, 529–535. http://dx.doi.org/10.1112/S0024609303002017

[9] O. Kurka, *On the variation of the Hardy-Littlewood maximal function.* arxiv:1210.0496v1

https://doi.org/10.4153/CMB-2014-070-7 Published online by Cambridge University Press

[10] H. Luiro, *Continuity of the maximal operator in Sobolev spaces.* Proc. Amer. Math. Soc. **135**(2007), no. 1, 243–251. http://dx.doi.org/10.1090/S0002-9939-06-08455-3
[11] H. Tanaka, *A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function.* Bull. Austral. Math. Soc. **65**(2002), no. 2, 253–258. http://dx.doi.org/10.1017/S0004972700020293

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