Can. J. Math., Vol. XXII, No. 6, 1970, pp. 1217–1223

HOMOLOGY OF DELETED PRODUCTS OF CONTRACTIBLE 2-DIMENSIONAL POLYHEDRA. III

C. W. PATTY

1. Introduction.

Our aim in this paper is to continue our investigation of the homology of deleted products of finite, contractible, 2-dimensional polyhedra. In [**1**], we observed that if X is such a polyhedron, then a homeomorph of X can be constructed by starting with a 2-simplex and appending n-simplexes ($n = 1, 2$). In this paper, we are concerned with those polyhedra which have the property that if they are constructed as above, then at some stage we are forced to add to X_{i-1} a 2-simplex τ at two of its 1-faces, $\langle u_3, u_1\rangle$ and $\langle u_3, u_2\rangle$, where there is a simple closed curve S in $\partial(\mathrm{St}(u_3, X_{i-1}))$ such that u_1 and u_2 are not in S but every sequence of 1-simplexes in $\partial(\mathrm{St}(u_3, X_{i-1}))$ from u_1 to u_2 intersects S. The cone over the complete graph on five vertices and the cone over the houses-and-wells figure are examples of such spaces.

Acknowledgements. I wish to express my appreciation to A. H. Copeland, Jr. for valuable communications concerning spaces which have the above property. I am also indebted to the referee for many suggestions which resulted in a shorter and more readable paper.

2. Notation.

The notation introduced in [**1**; **2**] is used in this paper. We shall use the notation "$\simeq$" for having the same homotopy type, and $S^2 \vee S^2$ will denote the union of two 2-spheres with a single point in common. Also we shall denote $A - \bigcup_{i=1}^{n} \mathrm{St}(u_i, A)$ by $A[u_1, \ldots, u_n]$ and

$$\mathrm{Cl}(\mathrm{St}(u_m, A)) - \bigcup_{i=1}^{n} \mathrm{St}(u_i, A)$$

by $A[u_m|u_1, \ldots, u_n]$. We shall also use the map $\rho\colon X \times X \to X \times X$ defined by $\rho(x, y) = (y, x)$.

Throughout this paper, with the exception of § 4, we assume that A is a finite, contractible, 2-dimensional polyhedron such that A^* is connected, B is a 2-simplex, $A \cap B = s_1 \cup s_2$, where s_1 and s_2 are 1-simplexes of A and B, $s_1 \cap s_2 = \{u_3\}$, $X = A \cup B$, and if u_i is the vertex of s_i different from u_3, then there is a simple arc in $\partial(\mathrm{St}(u_3, A))$ joining u_1 to u_2.

3. One way of adding 3-dimensional homology.

Definition 1. If S is a simple closed curve in $\partial(\mathrm{St}(u_3, A))$ such that neither

Received March 23, 1970 and in revised form, July 8, 1970. This research was supported by NSF grants GP–7943 and GP–15357.

u_1 nor u_2 are in S but every simple arc in $\partial(\mathrm{St}(u_3, A))$ joining u_1 to u_2 meets S, then we say that A has *property* P *at* u_3 *with respect to* (S, u_1, u_2) provided:

(1) there is a simple arc R in $\partial(\mathrm{St}(u_3, A))$ joining u_1 to u_2 such that if $R_1, R_2, \ldots, R_q$ are the components of the subset of R consisting of those 1-simplexes of R which meet S, then, for each $i = 1, 2, \ldots, q$, there is an arc R_i' in the 1-skeleton of $A[u_3]$ from the first vertex of R_i to the last vertex of R_i such that $R_i \cup R_i'$ bounds a 2-chain c_i in $A[u_3]$, and

(2) there is a simple closed curve S' in the 1-skeleton of $A[u_3]$ such that S' is disjoint from R and c_i for any i and $S \cup S'$ bounds a 2-chain d in $A[u_3]$, where $d \cap R_i' = \emptyset$ for each i and d is disjoint from a 1-simplex r_i of R if $r_i \cap S = \emptyset$.

Let s denote the 1-face of B which is not in A. Then, if

$$\Gamma = (A[u_1, u_2, u_3] \times B) \cup (A[u_3|u_1, u_2] \times s),$$

we have $P(X^*) = P(A^*) \cup \Gamma \cup \rho(\Gamma)$.

THEOREM 1. *If A has property* P *at u_3 with respect to (S, u_1, u_2) and z is the 2-cycle which assigns to each 2-cell in $((S * u_3) \times [\langle u_1\rangle \cup \langle u_2\rangle]) \cup (S \times s)$ either ± 1, then z bounds in $P(A^*) \cup (A[u_1, u_2, u_3] \times B)$.*

Proof. There is a 3-chain associated with the subset

$$[S \times B] \cup [d \times (s_1 \cup s_2)] \cup \left[S' \times \left((u_3 * R) \cup \bigcup_{\alpha=1}^{q} c_\alpha\right)\right]$$
$$\cup \left[(d \cup (S * u_3)) \times \left(\bigcup_{\alpha=1}^{q} R_\alpha' \cup \bigcup \left\{r \,\middle|\, r \text{ is a 1-simplex of } R - \bigcup_{\alpha=1}^{q} R_\alpha\right\}\right)\right]$$

of $P(A^*) \cup (A[u_1, u_2, u_3] \times B)$ whose boundary is z.

It may happen that A does not have property P at u_3 with respect to (S, u_1, u_2) but the 2-cycle z which assigns to each 2-cell in

$$((S * u_3) \times [\langle u_1\rangle \cup \langle u_2\rangle]) \cup (S \times s)$$

either ± 1 still bounds in $P(A^*) \cup (A[u_1, u_2, u_3] \times B)$.

4. Some results from a previous paper. If we examine [**1**, the proofs of Theorems 9 and 14], we observe that we have proved the following theorem.

THEOREM 2. *Let A be a finite, contractible, 2-dimensional polyhedron such that A^* is connected, and let X be the polyhedron obtained from A by adding a 2-simplex $B = \langle u_1, u_2, u_3\rangle$.*

(i) *If $A \cap B = \langle u_1, u_2\rangle$ and $\partial(\mathrm{St}(\langle u_1, u_2\rangle, A))$ is contractible, then $X^* \simeq A^*$.*

(ii) *If $A \cap B = \langle u_1, u_3\rangle \cup \langle u_2, u_3\rangle$ and $\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)$ is contractible, then $X^* \simeq A^*$.*

5. The cone over the houses-and-wells figure. If

$$Y = \bigcup_{2 \leqq i, j \leqq 4} \langle v_1, v_i, v_{j+3}\rangle,$$

then Y is the cone over the houses-and-wells figure. In this section, we calculate the homotopy type of the deleted product of this cone, and we examine the effect on the homology groups of the deleted product when the 2-simplex B is added to A and $\partial(\mathrm{St}(u_3, A)) \cup s$ contains a houses-and-wells figure which contains s, where s is the 1-face of B which is not in A.

Let $X_0 = \bigcup_{i=2}^{4} \langle v_1, v_i, v_5\rangle$, $B_i = \langle v_1, v_{i+1}, v_6\rangle$, $1 \leqq i \leqq 3$, and

$$B_j = \langle v_1, v_{j-2}, v_7\rangle, \; 4 \leqq j \leqq 6.$$

Inductively, we define $X_i = X_{i-1} \cup B_i$, $1 \leqq i \leqq 6$. We note that $Y = X_6$. Then we shall prove the following result.

THEOREM 3. *$X_i^* \simeq S^2$ $(i \neq 4, 6)$, $X_4^* \simeq S^2 \vee S^2 \vee S^2$, and $X_6^* = Y^* \simeq S^3$.*

Proof. That $X_i^* \simeq S^2$, $0 \leqq i \leqq 3$, follows easily from Theorem 2. To determine X_4^*, let $C_1 = X_3[v_1, v_2] \times B_4$, $C_2 = X_3[v_2|v_1] \times \langle v_1, v_7\rangle$, $C_3 = X_3[v_1|v_2] \times \langle v_2, v_7\rangle$, $C_4 = \mathrm{Cl}(\mathrm{St}(\langle v_1, v_2\rangle, X_3)) \times \langle v_7\rangle$, and $C = C_1 \cup \ldots \cup C_4$. Then $P(X_4^*) = P(X_3^*) \cup C \cup \rho(C)$. It is easy to see that

$$M = P(X_3^*) \cup C_1 \cup C_2 \cup C_3 \simeq X_3^*.$$

Since $M \cap C_4$ is a circle which bounds a disk in M, we see that $M \cup C_4 \simeq S^2 \vee S^2$. By repeating essentially the same argument, we can prove that $X_4^* \simeq S^2 \vee S^2 \vee S^2$. Next, to determine X_5^*, let $D_1 = X_4[v_1, v_3, v_7] \times B_5$, $D_2 = X_4[v_1|v_3, v_7] \times \langle v_3, v_7\rangle$, and $D = D_1 \cup D_2$. Then

$$P(X_5^*) = P(X_4^*) \cup D \cup \rho(D).$$

It is easy to see that $N = P(X_4^*) \cup D_1 \simeq X_4^*$. Since $N \cap D_2$ is a 2-sphere which bounds in D_2 and $[\langle v_2, v_5\rangle \cup \langle v_2, v_6\rangle \cup \langle v_4, v_5\rangle \cup \langle v_4, v_6\rangle] \times \langle v_1, v_3, v_7\rangle$ is a 3-dimensional set which fills up the space between the above 2-sphere and the 2-sphere

$$\begin{aligned}(&[\langle v_1, v_2, v_5\rangle \cup \langle v_1, v_2, v_6\rangle \cup \langle v_1, v_4, v_5\rangle \cup \langle v_1, v_4, v_6\rangle] \times [\langle v_3\rangle \cup \langle v_7\rangle])\\ &\cup ([\langle v_2, v_5\rangle \cup \langle v_2, v_6\rangle \cup \langle v_4, v_5\rangle \cup \langle v_4, v_6\rangle] \times [\langle v_1, v_3\rangle \cup \langle v_1, v_7\rangle]),\end{aligned}$$

which is "one of the three 2-spheres comprising X_4^*", we see that $N \cup D_2 \simeq S^2 \vee S^2$. By repeating essentially the same argument, we can prove that $X_5^* \simeq S^2$. Next, to determine X_6^*, let $E_1 = X_5[v_1, v_4, v_7] \times B_6$, $E_2 = X_5[v_1|v_4, v_7] \times \langle v_4, v_7\rangle$, and $E = E_1 \cup E_2$. Then $P(X_6^*) = P(X_5^*) \cup E \cup \rho(E)$. It is easy to see that $Q_1 = P(X_5^*) \cup E_1 \simeq X_5^*$. Now $Q_1 \cap E_2$ is a 2-sphere. It is clear that this 2-sphere bounds in E_2, and this 2-sphere is essentially "the 2-sphere in X_5^*". Therefore $Q_2 = Q_1 \cup E_2$ is contractible. It is easy to see that $Q_3 = Q_2 \cup \rho(E_1)$ is contractible. Since $Q_3 \cap \rho(E_2)$ is a 2-sphere and both of the spaces that we have put together in order to get this 2-sphere are contractible, the resulting space, namely Y^*, has the homotopy type of S^3.

THEOREM 4. *If $\partial(\mathrm{St}(u_3, A)) \cup \langle u_1, u_2\rangle$ contains a houses-and-wells figure F and S is the unique simple closed curve in F that does not meet $\langle u_1, u_2\rangle$, then the*

https://doi.org/10.4153/CJM-1970-140-5 Published online by Cambridge University Press

2-cycle z which assigns to each 2-cell in $([\langle u_1\rangle \cup \langle u_2\rangle] \times (S * u_3)) \cup (s \times S)$ *either* ± 1 *bounds in*

$$L = P(A^*) \cup (A[u_1, u_2, u_3] \times B) \cup (A[u_3|u_1, u_2] \times s) \cup (B \times A[u_1, u_2, u_3]).$$

Proof. We may assume that S consists of four arcs $r_1, \ldots, r_4$ and four vertices $u_4, \ldots, u_7$ such that $r_i \cap r_{i+1} = u_{i+4}$, $1 \leqq i \leqq 3$, and $r_4 \cap r_1 = u_4$. Further, we may assume that there are four arcs $s_1, \ldots, s_4$ such that $F = \bigcup_{i=1}^4 (r_i \cup s_i) \cup \langle u_1, u_2\rangle$, $s_i \cap S = u_{i+3}$, $1 \leqq i \leqq 4$, and $s_i \cap s_{i+2} = u_i$, $i = 1, 2$, and $s_i \cap s_j = \emptyset$ for $|i - j| \neq 2$. Let $R_i = [r_{i+2} \cup s_{i+2} \cup s_{i+3}] \times (r_i * u_3)$, $1 \leqq i \leqq 4$, and $S_i = [r_{i+1} \cup r_{i+2} \cup s_{i+1} \cup s_{i+3}] \times (s_i * u_3)$, $1 \leqq i \leqq 4$, where indices are taken modulo 4. Then it is clear that there is a 3-chain associated with the subset

$$\bigcup_{i=1}^{4} [(R_i \cup S_i) \cup \rho(R_i \cup S_i) \cup ((r_i * u_3) \times s) \cup (r_i \times B) \cup \rho(r_i \times B)]$$

of L whose boundary is z.

6. The cone over the complete graph on five vertices.

If

$$Y = \bigcup_{2 \leqq i < j \leqq 6} \langle v_1, v_i, v_j\rangle,$$

then Y is the cone over the complete graph on five vertices. In this section, we calculate the homotopy type of the deleted product of this cone, and we examine the effect on the homology groups of the deleted product when the 2-simplex B is added to A and $\partial(\mathrm{St}(u_3, A)) \cup s$ contains a complete graph on five vertices.

Let $X_0 = \bigcup_{i=3}^5 \langle v_1, v_2, v_i\rangle$, $B_1 = \langle v_1, v_2, v_6\rangle$, $B_i = \langle v_1, v_3, v_{i+2}\rangle$, $2 \leqq i \leqq 4$, $B_j = \langle v_1, v_4, v_j\rangle$, $j = 5, 6$, and $B_7 = \langle v_1, v_5, v_6\rangle$. Inductively, we define $X_i = X_{i-1} \cup B_i$, $1 \leqq i \leqq 7$. We note that $Y = X_7$. Then we shall prove the following result.

THEOREM 5. $X_0^* \simeq S^2$, $X_i^* \simeq S^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2$, $1 \leqq i \leqq 4$, $X_5^* \simeq S^2 \vee S^2 \vee S^2$, $X_6^* \simeq S^2$, *and* $X_7^* = Y^* \simeq S^3$.

Proof. By [**2**, Theorem 3], $X_0^* \simeq S^2$. To determine X_1^*, let

$$C_1 = X_0[v_1, v_2] \times B_1, \qquad C_2 = X_0[v_2|v_1] \times \langle v_1, v_6\rangle,$$
$$C_3 = X_0[v_1|v_2] \times \langle v_2, v_6\rangle, \qquad C_4 = \mathrm{Cl}(\mathrm{St}(\langle v_1, v_2\rangle, X_0)) \times \langle v_6\rangle,$$

and $C = C_1 \cup \ldots \cup C_4$. Then $P(X_1^*) = P(X_0^*) \cup C \cup \rho(C)$. It is easy to see that $M = P(X_0^*) \cup C_1 \cup C_2 \cup C_3 \simeq S^2$. Since $M \cap C_4$ is the union of two circles, both of which bound in M, it is easy to see that $M \cup C_4 \simeq S^2 \vee S^2 \vee S^2$. By essentially repeating the same argument, we can prove that

$$X_1^* \simeq S^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2.$$

That $X_i^* \simeq S^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2$, $2 \leqq i \leqq 4$, follows easily from

Theorem 2. Next, to determine X_5^*, let

$$D_1 = X_4[v_1, v_4, v_5] \times B_5, \qquad D_2 = X_4[v_1|v_4, v_5] \times \langle v_4, v_5\rangle,$$

and $D = D_1 \cup D_2$. Then $P(X_5^*) = P(X_4^*) \cup D \cup \rho(D)$. It is easy to see that $N = P(X_4^*) \cup D_1 \simeq X_4^*$. Since $N \cap D_2$ is a 2-sphere which bounds in D_2 and

$$([\langle v_3, v_6\rangle \cup \langle v_2, v_6\rangle \cup \langle v_2, v_3\rangle] \times B_5) \cup (\langle v_3, v_6\rangle \times [\langle v_1, v_2, v_4\rangle \cup \langle v_1, v_2, v_5\rangle]) \cup (\langle v_1, v_3, v_6\rangle \times [\langle v_2, v_4\rangle \cup \langle v_2, v_5\rangle])$$

is a 3-dimensional set which "fills up the space between the above 2-sphere and one of the five 2-spheres comprising X_4^*", we see that

$$N \cup D_2 \simeq S^2 \vee S^2 \vee S^2 \vee S^2.$$

By essentially repeating the same argument, we can prove that $X_5^* \simeq S^2 \vee S^2 \vee S^2$. The proof that $X_6^* \simeq S^2$ is essentially the same as the proof that $X_5^* \simeq S^2 \vee S^2 \vee S^2$. Finally, the proof that $X_7^* \simeq S^3$ is essentially the same as the proof that $X_6^* \simeq S^3$ in the proof of Theorem 3.

THEOREM 6. *If $\partial(\mathrm{St}(u_3, A)) \cup \langle u_1, u_2\rangle$ contains a complete graph F on five vertices and S is the unique simple closed curve in F that does not meet $\langle u_1, u_2\rangle$, then the 2-cycle z which assigns to each 2-cell in*

$$([\langle u_1\rangle \cup \langle u_2\rangle] \times (S * u_3)) \cup (s \times S)$$

either ± 1 bounds in

$$L = P(A^*) \cup (A[u_1, u_2, u_3] \times B) \cup (A[u_3|u_1, u_2] \times s) \cup (B \times A[u_1, u_2, u_3]).$$

Proof. We may assume that S consists of three arcs r_1, r_2, r_3 and three vertices u_4, u_5, u_6 such that $r_i \cap r_{i+1} = u_{i+3}$, $i = 1, 2$, and $r_3 \cap r_1 = u_6$. Further, we may assume that there are six arcs $s_1, s_2, s_3, t_1, t_2, t_3$ such that $F = \bigcup_{i=1}^3 (r_i \cup s_i \cup t_i) \cup \langle u_1, u_2\rangle$, $s_i \cap S = t_i \cap S = u_{i+3}$, $1 \leqq i \leqq 3$, $s_i \cap s_j = u_1$, $1 \leqq i < j \leqq 3$, $t_i \cap t_j = u_2$, $1 \leqq i < j \leqq 3$, $s_i \cap t_i = u_{i+3}$, $1 \leqq i \leqq 3$, and $s_i \cap t_j = \emptyset$ for $|i - j| > 0$. Let $R_i = [s_{i+1} \cup t_{i+1}] \times (r_i * u_3)$, $1 \leqq i \leqq 3$, $S_i = [r_{i+2} \cup t_{i+1} \cup t_{i+2}] \times (s_i * u_3)$, $1 \leqq i \leqq 3$, and

$$T_i = [r_{i+2} \cup s_{i+1} \cup s_{i+2}] \times (t_i * u_3), \qquad 1 \leqq i \leqq 3,$$

where indices are taken modulo 3. Then it is clear that there is a 3-chain associated with the subset

$$\bigcup_{i=1}^{3} [(R_i \cup S_i \cup T_i) \cup \rho(R_i \cup S_i \cup T_i) \cup ((r_i * u_3) \times s)] \cup (B \times S) \cup (S \times B)$$

of L whose boundary is z.

https://doi.org/10.4153/CJM-1970-140-5 Published online by Cambridge University Press

7. A lower bound for the 3-dimensional homology. Let $\mathscr{F}$ be the set consisting of the subsets of $\partial(\mathrm{St}(u_3, A)) \cup \langle u_1, u_2\rangle$ which contain $\langle u_1, u_2\rangle$ and are either PL-homeomorphs of the houses-and-wells figure or PL-homeomorphs of the complete graph on five vertices.

The following simple closed curves in $\partial(\mathrm{St}(u_3, A)) - \{u_1, u_2\}$ play a crucial role in the addition of 3-dimensional homology:

(1) those simple closed curves S which have the property that there is a simple arc in $\partial(\mathrm{St}(u_3, A))$ joining u_1 to u_2 which does not meet S;

(2) those simple closed curves S such that A has property P at u_3 with respect to (S, u_1, u_2);

(3) those simple closed curves lying opposite $\langle u_1, u_2\rangle$ in some $F \in \mathscr{F}$.

We call a simple closed curve described in (1), (2), and (3) above, a simple closed curve of Types (1), (2), and (3), respectively.

Define $W(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle)$ to be the subgroup of

$$H_1(\partial(\mathrm{St}(u_3, A)) - \{u_1, u_2\})$$

generated by simple closed curves of Types (1), (2), and (3). Define $W_0(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle)$ to be the subgroup of $H_1(\partial(\mathrm{St}(u_3, A)) - \{u_1, u_2\})$ generated by simple closed curves of Types (1) and (2). Set

$$D(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle) = \operatorname{rank} W(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle)$$

and

$$E(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle) = \operatorname{rank} W_0(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle).$$

THEOREM 7.

$$\beta_3(X^*) \geqq \beta_3(A^*) + D(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle) + E(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle).$$

Proof. Let $E_1 = ((S * u_3) \times [\langle u_1\rangle \cup \langle u_2\rangle]) \cup (S \times s)$, $E_2 = A[u_1, u_2, u_3] \times B$, and $E_3 = A[u_3|u_1, u_2] \times s$. If S is a simple closed curve of Type (1), then, by [**1**, Theorem 11], every 2-cycle in E_1 bounds in $P(A^*) \cup E_2$. If S is a simple closed curve of Type (2), then, by Theorem 1, every 2-cycle in E_1 bounds in $P(A^*) \cup E_2$. In each case, it is clear that every 2-cycle in E_1 bounds in E_3. Therefore

$$\beta_3(P(A^*) \cup E_2 \cup E_3) \geqq \beta_3(A^*) + E(\partial(\mathrm{St}(u_3, A)), \langle u_1, u_2\rangle).$$

Again, by [**1**, Theorem 11], if S is a simple closed curve of Type (1), then every 2-cycle in $\rho(E_1)$ bounds in $\Gamma = P(A^*) \cup E_2 \cup E_3 \cup \rho(E_2)$, and, by Theorem 1, if S is a simple closed curve of Type (2), then every 2-cycle in $\rho(E_1)$ bounds in Γ. Now if S is a simple closed curve lying opposite $\langle u_1, u_2\rangle$ in $F \in \mathscr{F}$, where F is a PL-homeomorph of the houses-and-wells figure, then, by Theorem 4, every 2-cycle in $\rho(E_1)$ bounds in Γ. Finally, if S is a simple closed curve lying opposite $\langle u_1, u_2\rangle$ in $F \in \mathscr{F}$, where F is a PL-homeomorph of the complete graph on five vertices, then, by Theorem 6, every 2-cycle in $\rho(E_1)$ bounds in Γ. In each of these four cases, it is clear that every 2-cycle in $\rho(E_1)$ bounds in $\rho(E_3)$. This completes the proof of the theorem.

REFERENCES

1. C. W. Patty, *Homology of deleted products of contractible 2-dimensional polyhedra*. I, Can. J. Math. *20* (1968), 416–441.
2. ——— *Homology of deleted products of contractible 2-dimensional polyhedra*. II, Can. J. Math. *20* (1968), 842–854.

Virginia Polytechnic Institute and State University,
Blacksburg, Virginia