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CIRCUMRADIUS-DIAMETER AND WIDTH-INRADIUS RELATIONS FOR LATTICE CONSTRAINED CONVEX SETS

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Let K be a planar, compact, convex set with circumradius R, diameter d, width w and inradius r, and containing no points of the integer lattice. We generalise inequalities concerning the 'dual' quantities $(2R-d)$ and $(w-2r)$ to rectangular lattices. We then use these results to obtain corresponding inequalities for a planar convex set with two interior lattice points. Finally, we conjecture corresponding results for sets containing one interior lattice point.

1. INTRODUCTION

Let $\mathcal{K}^2$ denote the set of all planar, compact, convex sets. Let K be a set in $\mathcal{K}^2$ with circumradius $R(K) = R$, diameter $d(K) = d$, inradius $r(K) = r$, and width $w(K) = w$. Let K^o denote the interior of K and let $\Lambda_R(\mathbf{u}, \mathbf{v})$ be a rectangular lattice generated by the vectors $\mathbf{u} = (u, 0)$ and $\mathbf{v} = (0, v)$, $u \leqslant v$. In the case where $u = v = 1$, we have the integral lattice, denoted by Γ. Let $G(K, \Lambda)$ denote the number of points of lattice Λ in K. A number of results concerning the 'dual' quantities $(2R-d)$ and $(w-2r)$ have been obtained by Scott [**2, 3, 4**] and Awyong [**1**]. In particular, Awyong [**1**] proves

THEOREM 1. *Let K be a set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 0$. Then*

$$2R - d \leqslant \frac{1}{3},$$
$$w - 2r \leqslant \frac{1}{6}\left(2 + \sqrt{3}\right),$$

with equality when and only when $K \cong E_0$ (Figure 1).

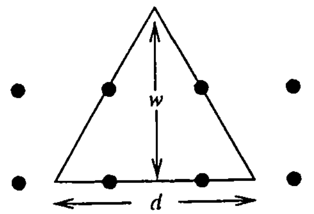

Figure 1: The equilateral triangle E_0.

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The purpose of this paper is to generalise Theorem 1 to rectangular lattices and to use the result to obtain the corresponding inequalities for a set $K \in \mathcal{K}^2$ having $G(K^o, \Gamma) = 2$. We prove the following results:

Theorem 2. *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$. Then*

$$2R - d \leqslant \frac{2}{3}\left(2 - \sqrt{3}\right)\left(\frac{\sqrt{3}}{2}u + v\right) \tag{1}$$

$$w - 2r \leqslant \frac{1}{3}\left(\frac{\sqrt{3}}{2}u + v\right), \tag{2}$$

with equality when and only when $K \cong E_R$ (Figure 2).

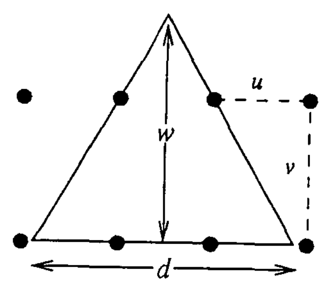

Figure 2: The equilateral triangle E_R.

Corollary 1. *Let K be a set in $\mathcal{K}^2$ with $G(K^o, \Gamma) = 2$. Then*

$$2R - d \leqslant \frac{1}{3}\left(5 - 2\sqrt{3}\right) \approx 0.512,$$

$$w - 2r \leqslant \frac{1}{3}\left(2 + \frac{\sqrt{3}}{2}\right) \approx 0.955,$$

with equality when and only when $K \cong E_2$ (Figure 3).

2. Proof of Theorem 2

In [1], it was proved that for a set $K \in \mathcal{K}^2$,

$$2R - d \leqslant \frac{2}{3}\left(2 - \sqrt{3}\right)w, \tag{3}$$

$$w - 2r \leqslant \frac{w}{3}, \tag{4}$$

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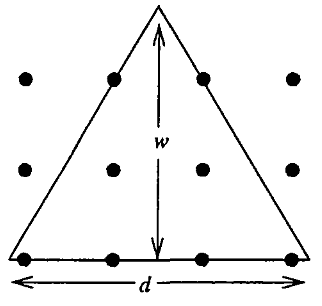

Figure 3: The equilateral triangle E_2.

with equality when and only when K is an equilateral triangle.

By applying a result by Vassallo [**6**] to rectangular lattices, we have the result that if K is a set in $\mathcal{K}^2$ with $G(K^o, \Lambda_R) = 0$, then

$$w \leqslant \frac{\sqrt{3}}{2}u + v. \tag{5}$$

Theorem 2 follows immediately by combining inequality (5) with (3) and (4).

3. Proof of Corollary 1

Let K now be a set satisfying the conditions of Corollary 1. Without loss of generality, we may assume that the origin O is one of the lattice points. Let L denote the other lattice point contained in K^o and let the coordinates of L be (z_1, z_2), where without loss of generality, $z_1 \geqslant 0$, $z_2 \geqslant 0$. By a reflection about $y = x$ if necessary, it suffices to consider those cases for which $z_1 \geqslant z_2$. Since K^o contains no other lattice points, the open line segment OL contains no lattice points. Hence we may assume that either $z_1 = 1$ and $z_2 = 0$ or else z_1 and z_2 are relatively prime.

If z_1 and z_2 are both odd, we consider the sublattice

$$\Gamma' = \{(x, y) : x + y \equiv 1 (\mathrm{mod}\ 2)\}.$$

Clearly, $O \notin \Gamma'$, $L \notin \Gamma'$ and $G(K^o, \Gamma') = 0$. Here we have $u = v = \sqrt{2}$ and by Theorem 2

$$2R - d \leqslant \frac{1}{3}\sqrt{2} \approx 0.4714 < \frac{1}{3}\left(5 - 2\sqrt{3}\right) \approx 0.512,$$

$$w - 2r \leqslant \frac{\sqrt{2}}{3}\left(1 + \frac{\sqrt{3}}{2}\right) \approx 0.879 < \frac{1}{3}\left(2 + \frac{\sqrt{3}}{2}\right) \approx 0.955.$$

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If z_1 is odd and z_2 is even, we consider the sublattice.

$$\Gamma'' = \{(x, y) : x = m,\ y = 2n + 1,\ m, n, \in \mathbf{Z}\}.$$

Clearly $O \notin \Gamma''$, $L \notin \Gamma''$ and $G(K^o, \Gamma'') = 0$. In the case where z_1 is even and z_2 is odd, we consider the lattice

$$\Gamma''' = \{(x, y) : x = 2m + 1, y = n, m, n \in \mathbf{Z}\}.$$

Here, we have $G(K^o, \Gamma''') = 0$. By an appropriate transformation, this is equivalent to the case where z_1 is odd and z_2 is even. In this case $u = 1$ and $v = 2$ and by Theorem 2, we have

$$2R - d \leqslant \frac{1}{3}\left(5 - 2\sqrt{3}\right) \approx 0.512,$$

$$w - 2r \leqslant \frac{1}{3}\left(2 + \frac{\sqrt{3}}{2}\right) \approx 0.955.$$

Equality is attained when and only when $K \cong E_2$ (Figure 3).

4. A Conjecture

We now conjecture the corresponding inequalities for a set K having $G(K^o, \Gamma) = 1$.

Conjecture. Let K be a set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 1$. Then

$$2R - d \leqslant \sqrt{2}\left(\frac{7}{6} - \frac{\sqrt{3}}{2}\right) \approx 0.425,$$

$$w - 2r \leqslant \frac{\sqrt{2}}{12}\left(5 + \sqrt{3}\right) \approx 0.793,$$

with equality when and only when $K \cong E_1$ (Figure 4).

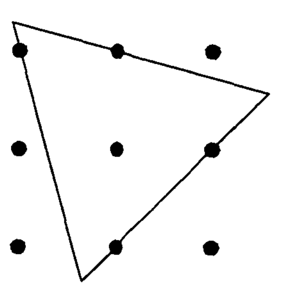

Figure 4: The equilateral triangle E_1.

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The difficulty which occurs here is that for a set K having $G(K^o, \Gamma) = 1$, $w \leqslant 1 + \sqrt{2}$, with equality when and only when K is congruent to the isosceles triangle shown in Figure 5 [**5**]. As this set of largest width is not an equilateral triangle, (3) and (4) do not give sharp inequalities.

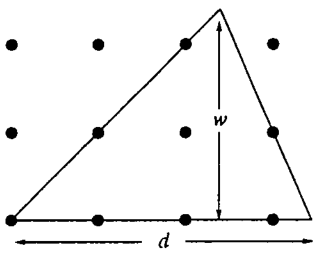

Figure 5: The isosceles triangle I_1.

A simple calculation shows that the width of E_1 is $\left(\sqrt{2}(5+\sqrt{3})\right)/4 \approx 2.38$. Hence if $0 < w \leqslant \left(\sqrt{2}(5+\sqrt{3})\right)/4$, it follows from (3) and (4) that for this given range of w,

$$2R - d \leqslant \sqrt{2}\left(\frac{7}{6} - \frac{\sqrt{3}}{2}\right) \approx 0.425,$$

$$w - 2r \leqslant \frac{\sqrt{2}}{12}\left(5+\sqrt{3}\right) \approx 0.793,$$

with equality when and only when $K \cong E_1$ (Figure 4).

This leaves unresolved those cases for which $\left(\sqrt{2}(5+\sqrt{3})\right)/4 < w \leqslant 1 + \sqrt{2}$. We believe that the set for which $(2R - d)$ and $(w - 2r)$ are maximal is congruent to the equilateral triangle E_1 (Figure 4).

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