CIRCUMRADIUS-DIAMETER AND WIDTH-INRADIUS RELATIONS FOR LATTICE CONSTRAINED CONVEX SETS

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Let $K$ be a planar, compact, convex set with circumradius $R$, diameter $d$, width $w$, and inradius $r$, and containing no points of the integer lattice. We generalise inequalities concerning the 'dual' quantities $(2R - d)$ and $(w - 2r)$ to rectangular lattices. We then use these results to obtain corresponding inequalities for a planar convex set with two interior lattice points. Finally, we conjecture corresponding results for sets containing one interior lattice point.

1. INTRODUCTION

Let $\mathcal{K}^2$ denote the set of all planar, compact, convex sets. Let $K$ be a set in $\mathcal{K}^2$ with circumradius $R(K) = R$, diameter $d(K) = d$, inradius $r(K) = r$, and width $w(K) = w$. Let $K^o$ denote the interior of $K$ and let $\Lambda_R(u, v)$ be a rectangular lattice generated by the vectors $u = (u, 0)$ and $v = (0, v)$, $u \leq v$. In the case where $u = v = 1$, we have the integral lattice, denoted by $\Gamma$. Let $G(K, \Lambda)$ denote the number of points of lattice $\Lambda$ in $K$. A number of results concerning the 'dual' quantities $(2R - d)$ and $(w - 2r)$ have been obtained by Scott [2, 3, 4] and Awyong [1]. In particular, Awyong [1] proves

**THEOREM 1.** Let $K$ be a set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 0$. Then

\[
2R - d \leq \frac{1}{3},
\]

\[
w - 2r \leq \frac{1}{6} \left(2 + \sqrt{3}\right),
\]

with equality when and only when $K \cong E_0$ (Figure 1).

![Figure 1: The equilateral triangle $E_0$.](image)
The purpose of this paper is to generalise Theorem 1 to rectangular lattices and to use the result to obtain the corresponding inequalities for a set \( K \in \mathcal{K}^2 \) having \( G(K^\circ, \Gamma) = 2 \). We prove the following results:

**Theorem 2.** Let \( K \) be a set in \( \mathcal{K}^2 \) with \( G(K^\circ, \Lambda_R) = 0 \). Then

\[
\begin{align*}
(1) & \quad 2R - d \leq \frac{2}{3} \left( 2 - \sqrt{3} \right) \left( \frac{\sqrt{3}}{2} u + v \right) \\
(2) & \quad w - 2r \leq \frac{1}{3} \left( \frac{\sqrt{3}}{2} u + v \right),
\end{align*}
\]

with equality when and only when \( K \cong E_R \) (Figure 2).

![Figure 2: The equilateral triangle \( E_R \).](image)

**Corollary 1.** Let \( K \) be a set in \( \mathcal{K}^2 \) with \( G(K^\circ, \Gamma) = 2 \). Then

\[
\begin{align*}
2R - d & \leq \frac{1}{3} \left( 5 - 2\sqrt{3} \right) \approx 0.512, \\
w - 2r & \leq \frac{1}{3} \left( 2 + \frac{\sqrt{3}}{2} \right) \approx 0.955,
\end{align*}
\]

with equality when and only when \( K \cong E_2 \) (Figure 3).

**2. Proof of Theorem 2**

In [1], it was proved that for a set \( K \in \mathcal{K}^2 \),

\[
\begin{align*}
(3) & \quad 2R - d \leq \frac{2}{3} \left( 2 - \sqrt{3} \right) w, \\
(4) & \quad w - 2r \leq \frac{w}{3},
\end{align*}
\]
with equality when and only when $K$ is an equilateral triangle.

By applying a result by Vassallo [6] to rectangular lattices, we have the result that if $K$ is a set in $K^2$ with $G(K^0, \Lambda_R) = 0$, then

$$w \leq \frac{\sqrt{3}}{2} u + v.$$  

(5)

Theorem 2 follows immediately by combining inequality (5) with (3) and (4).

3. PROOF OF COROLLARY 1

Let $K$ now be a set satisfying the conditions of Corollary 1. Without loss of generality, we may assume that the origin $O$ is one of the lattice points. Let $L$ denote the other lattice point contained in $K^0$ and let the coordinates of $L$ be $(z_1, z_2)$, where without loss of generality, $z_1 \geq 0$, $z_2 \geq 0$. By a reflection about $y = x$ if necessary, it suffices to consider those cases for which $z_1 \neq z_2$. Since $K^0$ contains no other lattice points, the open line segment $OL$ contains no lattice points. Hence we may assume that either $z_1 = 1$ and $z_2 = 0$ or else $z_1$ and $z_2$ are relatively prime.

If $z_1$ and $z_2$ are both odd, we consider the sublattice

$$\Gamma' = \{(x, y) : x + y \equiv 1 \pmod{2}\}.$$  

Clearly, $O \notin \Gamma'$, $L \notin \Gamma'$ and $G(K^0, \Gamma') = 0$. Here we have $u = v = \sqrt{2}$ and by Theorem 2

$$2R - d \leq \frac{1}{3} \sqrt{2} \approx 0.4714 < \frac{1}{3} \left( 5 - 2\sqrt{3} \right) \approx 0.512,$$

$$w - 2r \leq \frac{\sqrt{2}}{3} \left( 1 + \frac{\sqrt{3}}{2} \right) \approx 0.879 < \frac{1}{3} \left( 2 + \frac{\sqrt{3}}{2} \right) \approx 0.955.$$
If $z_1$ is odd and $z_2$ is even, we consider the sublattice.

$$\Gamma'' = \{(x, y) : x = m, y = 2n + 1, \ m, n, \in \mathbb{Z}\}.$$ 

Clearly $O \notin \Gamma''$, $L \notin \Gamma''$ and $G(K^o, \Gamma'') = 0$. In the case where $z_1$ is even and $z_2$ is odd, we consider the lattice

$$\Gamma''' = \{(x, y) : x = 2m + 1, y = n, m, n, \in \mathbb{Z}\}.$$ 

Here, we have $G(K^0, \Gamma''') = 0$. By an appropriate transformation, this is equivalent to the case where $z_1$ is odd and $z_2$ is even. In this case $u = 1$ and $v = 2$ and by Theorem 2, we have

$$2R - d \leq \frac{1}{3} \left(5 - 2\sqrt{3}\right) \approx 0.512,$$

$$w - 2r \leq \frac{1}{3} \left(2 + \sqrt{3}\right) \approx 0.955.$$ 

Equality is attained when and only when $K \cong E_2$ (Figure 3).

4. A Conjecture

We now conjecture the corresponding inequalities for a set $K$ having $G(K^o, \Gamma) = 1$.

Conjecture. Let $K$ be a set in $\mathcal{K}^2$ having $G(K^o, \Gamma) = 1$. Then

$$2R - d \leq \sqrt{2} \left(\frac{7}{6} - \frac{\sqrt{3}}{2}\right) \approx 0.425,$$

$$w - 2r \leq \sqrt{\frac{2}{12}} \left(5 + \sqrt{3}\right) \approx 0.793,$$

with equality when and only when $K \cong E_1$ (Figure 4).

![Figure 4: The equilateral triangle $E_1$.](https://www.camos.org/core/terms)
The difficulty which occurs here is that for a set $K$ having $G(K^o, \Gamma) = 1$, $w \leq 1 + \sqrt{2}$, with equality when and only when $K$ is congruent to the isosceles triangle shown in Figure 5 [5]. As this set of largest width is not an equilateral triangle, (3) and (4) do not give sharp inequalities.

Figure 5: The isosceles triangle $I_1$.

A simple calculation shows that the width of $E_1$ is $\left(\sqrt{2}(5 + \sqrt{3})\right)/4 \approx 2.38$. Hence if $0 < w \leq \left(\sqrt{2}(5 + \sqrt{3})\right)/4$, it follows from (3) and (4) that for this given range of $w$,

$$2R - d \leq \sqrt{2} \left(\frac{7}{6} - \frac{\sqrt{3}}{2}\right) \approx 0.425,$$
$$w - 2r \leq \frac{\sqrt{2}}{12} (5 + \sqrt{3}) \approx 0.793,$$

with equality when and only when $K \cong E_1$ (Figure 4).

This leaves unresolved those cases for which $\left(\sqrt{2}(5 + \sqrt{3})\right)/4 < w \leq 1 + \sqrt{2}$. We believe that the set for which $(2R - d)$ and $(w - 2r)$ are maximal is congruent to the equilateral triangle $E_1$ (Figure 4).

REFERENCES


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