T. Suzuki
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DISTRIBUTIONS WITH AUTOMORPHY AND DIRICHLET SERIES

TOSHIAKI SUZUKI

Introduction

In [1], Maass proved that the Dirichlet series associated with Siegel modular form satisfies a function equation. In this paper, we try to generalize the above fact to the indefinite case.

Denote by $V_{\boldsymbol{R}}$ the vector space of real symmetric matrices of size $n \geqq 3$ and let V_i be the subset of $V_{\boldsymbol{R}}$ consisting of matrices with i positive and $n-i$ negative eigenvalues $(0 \leqq i \leqq n)$. Let L (resp. L^*) be lattices in $V_{\boldsymbol{R}}$ consisting of integral (resp. half-integral) symmetric matrices, Set $L_i^* = L^* \cap V_i$ $(0 \leqq i \leqq n)$ and denote by $L_i^*/\sim$ the set of $SL(n, \boldsymbol{Z})$-equivalence classes in L_i^* under the action $T \mapsto gT{}^tg$ $(T \in L_i^*$, $g \in SL(n, \boldsymbol{Z}))$. In [3], T. Shintani introduced "zeta functions associated with the vector space of quadratic forms"

$$\sum_{T \in L_i^*/\sim} \mu(T)\,|\det T|^{-s} \qquad (0 \leqq i \leqq n)\,,$$

and proved that they have functional equations.

We call $F(X)$ a distribution with automorphy of weight k, if $F(X)$ is a distributions on $V_{\boldsymbol{R}} - S = \{X \in V_{\boldsymbol{R}}\,;\det X \neq 0\}$ and satisfies the following conditions:

(1) If we take $dv = |\det X|^{-(n+1)/2}dx$ $(dx = \prod_{1 \leqq i \leqq j \leqq n} dx_{ij})$ as a volume element of $V_{\boldsymbol{R}} - S$, then we can write

$$F(X) = \sum_{T \in L^*} a(T) \exp\,(2\pi\sqrt{-1}\,\mathrm{Tr}\,XT)\,,$$

i.e., $F(X)$ is the mapping

$$f \mapsto \int F(X)f(X)dv = \sum_{T \in L^*} a(T) \int \exp\,(2\pi\sqrt{-1}\,\mathrm{Tr}\,XT)f(X)dV\,,$$

where f is a smooth function with compact support on $V_{\boldsymbol{R}} - S$.

(2) $F(-X^{-1}) = \langle X \rangle F(X)$, where $\langle X \rangle$ is given by

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$$\langle X\rangle = b_i |\det X|^k \qquad \text{if } X \in V_i \ (k, b_i \in \mathbf{C},\ 0 \leqq i \leqq n)\,.$$

Here, the distribution $F(-X^{-1})$ is the mapping

$$f \mapsto \int F(X) f(-X^{-1}) dv\,,$$

where f is a smooth function with compact support on $V_{\mathbf{R}} - S$.

(3) $F(gX^tg) = F(X)$ for any $g \in SL(n, \mathbf{Z})$,

i.e., $a(gT^tg) = a(T)$ for any $g \in SL(n, \mathbf{Z})$.

(4) There exist positive constants e_1 and e_2 such that

$$|a(T)| \leqq e_1 |\det T|^{e_2}\,.$$

We associate with $F(X)$ the Dirichlet series

$$\xi_i(s) = (2\pi)^{-ns}\Gamma(s)\Gamma\left(s - \frac{1}{2}\right)\cdots\Gamma\left(s - \frac{n-1}{2}\right)\sum_{T \in L_i^*/\sim} a(T)\mu(T)|\det T|^{-s} \qquad (0 \leqq i \leqq n)\,.$$

Then, we can prove the following theorem (§ 1).

THEOREM 1. *Dirichlet series $\xi_i(s)$ $(0 \leqq i \leqq n)$ have analytic continuations to meromorphic functions in the whole complex plane satisfying the following functional equations*:

$$\begin{aligned}
&\exp \frac{\pi\sqrt{-1}n}{2} s \cdot \sum_{i=0}^{n} \xi_i(s) u_{i,n-j}(s) \\
&\quad = b_j \exp \frac{\pi\sqrt{-1}n}{2}(k - s) \cdot \sum_{i=0}^{n} \xi_i(k - s) u_{i,j}(k - s)\,,
\end{aligned} \qquad (0 \leqq j \leqq n)\,.$$

(For $u_{i,j}(s)$, see § 1 or [3].)

We show an example of $F(X)$ related to (non-holomorphic) Eisenstein series on the Siegel upper half plane (§ 2). By Theorem 1, we obtain the corresponding Dirichlet series with functional equations, which are previously obtained by T. Arakawa ([4]) in a different way.

We can also show examples of $F(X)$ by taking boundary values of Siegel modular forms, which correspond to the results of Maass [1]. It is easy to see that, if we can take boundary values of (non-holomorphic) Siegel modular forms, then the boundary values are Distributions with automorphy. In § 3, we consider a class of functions on the Siegel upper half plane with the above property.

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Notation

$M(n, \boldsymbol{Q})$: the set of rational matrices of size n

$\mathscr{S}(V_{\boldsymbol{R}})$: the space of rapidly decreasing functions on a vector space $V_{\boldsymbol{R}}$

$\mathscr{C}_0^\infty(M)$: the space of smooth functions with compact support on a smooth manifold M

$\mathrm{Re}\,(x)$: the real part of a complex number x

For real symmetric matrix T, $T > 0$ and $T \geqq 0$ mean T is positive-definite and semi-positive-definite, respectively.

$e[x] = \exp(2\pi\sqrt{-1}x)$ $(x \in \boldsymbol{C})$

§ 1. Dirichlet series and distributions with automorphy

Let G be $GL(n, \boldsymbol{C})$, let V be the vector space of complex symmetric matrices of size $n \geqq 3$, and let ρ be the representation of G on V defined by $\rho(g)X = gX{}^tg$ $(g \in G,\ X \in V)$. Then it is easy to see that the triple (G, ρ, V) is a prehomogeneous vector space (see [2], [3]) whose set of singular points S is the irreducible hypersurface given by $S = \{X \in V;\ \det X = 0\}$. Put $\chi(g) = (\det g)^2$, then $\det(\rho(g)X) = \chi(g)\det X$. In the following, we identify V with its dual via the symmetric bilinear form $\mathrm{Tr}\, XY$. The triple (G, ρ, V) has a natural $\boldsymbol{R}$-structure such that $G_{\boldsymbol{R}}$ is $GL(n, \boldsymbol{R})$ and $V_{\boldsymbol{R}}$ is the set of real symmetric matrices of size n. Let V_i be the set of real symmetric matrices with i positive and $n - i$ negative eigenvalues $(0 \leqq i \leqq n)$. Then $V_{\boldsymbol{R}} - S = V_0 \cup \cdots \cup V_n$ is the decomposition of $V_{\boldsymbol{R}} - S$ into its connected components. Let dv be the $G_{\boldsymbol{R}}$-invariant volume element of $V_{\boldsymbol{R}}$ defined by $dv = |\det X|^{-(n+1)/2}dX$, where dX is given by $dX = \prod_{1\leqq i\leqq j\leqq n} dx_{ij}$.

For any $f \in \mathscr{S}(V_{\boldsymbol{R}})$, set

$$f^*(Y) = \int_{V_{\boldsymbol{R}}} f(X)e[\mathrm{Tr}\, XY]dv$$

and

$$\Phi_i(f, s) = \int_{V_i} f(X)|\det X|^s\, dv\ , \qquad \left(\mathrm{Re}\,(s) > \frac{n+1}{2},\ 0 \leqq i \leqq n\right).$$

Let $M_n = \det X \det(\partial/\partial X)$ where $\det(\partial/\partial X)$ is the differential operator with constant coefficients on V_R which satisfies the following equality:

$$\det\left(\frac{\partial}{\partial x}\right)\exp \operatorname{Tr} XY = \det Y \cdot \exp \operatorname{Tr} XY .$$

It is shown that M_n is a G_R-invariant differential operator on $V_R - S$ and

$$M_n |\det X|^s = \mathrm{b}(s) |\det X|^s , \qquad (X \in V_R - S,\ s \in C) ,$$

where $b(s) = s(s + \frac{1}{2}) \cdots (s + (n-1)/2))$ (see [3]).

Set

$$\gamma(s) = \prod_{i=0}^{n-1} \Gamma\left(s + 1 - \frac{i}{2}\right)$$

and

$$u_{ij}(s) = \sqrt{-1}^{(n+1)(i+j-n)/2}(-1)^{(n-j)(n-j+1)/2} \times \sum_{r=\mathrm{Max}(0,i+j-n)}^{\mathrm{Min}(i,j)} (-1)^{r(n+1)} \alpha_{jr}\alpha_{n-j\ i-r} \exp \pi\sqrt{-1}s(2r - i - j) ,$$

$$\sqrt{-1}^{-\ell(\ell+m)/2}\alpha_{\ell m} = \begin{cases} (-1)^{(\ell-m)/2}\binom{\ell/2}{m/2} & (\ell, m \text{ even}) , \\ 0 & (\ell \text{ even}, m \text{ odd}) , \\ (-1)^{(\ell+1-m)/2}\binom{(\ell-1)/2}{m/2} & (\ell \text{ odd}, m \text{ even}) , \\ (-1)^{(\ell-m)/2}\binom{(\ell-1)/2}{(m-1)/2} & (\ell, m \text{ odd}) . \end{cases}$$

LEMMA 1–1. *As analytic functions of $s, \Phi_i(f, s)$ $(0 \leqq i \leqq n, f \in \mathscr{S}(V_R))$ have analytic continuations to meromorphic functions in the whole complex plane satisfying the following functional equations:*

$$\Phi_i(f^*, s) = \pi^{n(n-1)/4}(2\pi)^{-sn}\gamma\left(s - \frac{n+1}{2}\right)e[ns/4] \sum_{j=0}^{n} u_{ij}(s)\Phi_j(f, -s) .$$

Proof. See Lemma 15 of [3].

The prehomogeneous vector space (G, ρ, V) has a natural $\boldsymbol{Q}$-structure such that $G_Q = GL(n, \boldsymbol{Q})$ and $V_Q = M(n, \boldsymbol{Q}) \cap V$. Set $G_+ = \{g \in GL(n, \boldsymbol{R});\ \det g > 0\}$. For any $X \in V$, denote by G_X the isotropy subgroup of X in G. Further, let dg be a Haar measure on G_+ defined by

$$dg = (\det g)^{-n} \prod_{1 \leqq i,\ j \leqq n} dg_{ij} .$$

For every $X \in V_Q - S$ and any bounded domain U_X such that $X \in U_X \subset \overline{U}_x \subset V_R - S$, let $W_X = \{g \in G_+ ; \rho(g)X \in U_X\}$ and let $(W_X)_0$ be a fundamental domain of W_X with respect to $SL(n, \boldsymbol{Z}) \cap G_X$. Then the ratio

$$\mu(X) = \int_{(W_X)_0} dg \Big/ \int_{U_X} dv$$

is independent of the choice of U_X and finite (see [3]).

We call $X, Y \in V_R$ equivalent if they lie on the same $SL(n, Z)$-orbit. For any $SL(n, \boldsymbol{Z})$-invariant set M in V_R, denote $M/\sim$ the complete system of $SL(n, \boldsymbol{Z})$-equivalence classes in M. Let L (resp. L^*) be the set of integral (resp. half-integral) symmetric matrices.

Let $F(X)$ be a mapping defined by the following (in formal):

$$F(X): f \mapsto \sum_{T \in L^*} a(T) f^*(T) = \sum_{T \in L^*} a(T) \int f(X) e[\operatorname{Tr} XT] dv ,$$

where $f \in \mathscr{C}_0^\infty(V_R - S)$ and $a(T) \in \boldsymbol{C}$. Under some conditions on $\{a(T)\}_{T \in L^*}$, $F(X)$ may be considered a distribution on $V_R - S$. We may write $F(X) = \sum_{T \in L^*} a(T) e[\operatorname{Tr} XT]$. In particular, using $\hat{X}$ instead of $-X^{-1}$, we call $F(X)$ the distribution with automorphy of weight k, if $F(X)$ satisfies the following conditions:

(1–1) $F(\hat{X}) = F(-X^{-1}) = \langle X \rangle F(X)$, where $\langle X \rangle$ is given by

$$\langle X \rangle = b_i |\det X|^k \qquad \text{if } X \in V_i \ (b_i, k \in \boldsymbol{C},\ 0 \leqq i \leqq n) .$$

(1–2) $F(\rho(g)X) = F(X)$ for any $g \in SL(n, \boldsymbol{Z})$.

(1–3) There exist positive constants e_1 and e_2 such that

$$|a(T)| \leqq e_1 |\det T|^{e_2} .$$

We associate with $F(X)$ the Dirichlet series

$$D_i(s) = \sum_{T \in L_i^*/\sim} a(T) \mu(T) |\det T|^{-s} , \qquad (0 \leqq i \leqq n) ,$$

where $L_i^* = V_i \cap L^*$. It is shown that, if $\operatorname{Re}(s) > (n+1)/2 + e_2$, $D_i(s)$ $(0 \leqq i \leqq n)$ are absolutely convergent (see [3]).

Set $Z(f, \{a(T)\}, s) = \int_{G_+/SL(n,\boldsymbol{Z})} \chi(g)^s \sum_{T \in (L^*)'} a(T) f(\rho(g)T) dg$, $(f \in \mathscr{S}(V_R)$, $(L^*)' = L^* \cap (V_R - S))$.

LEMMA 1–2. *The integral* $Z(f, \{a(T)\}, s)$ *converges absolutely, when* $\operatorname{Re}(s)$ *is sufficiently large, and is equal to*

$$\sum_{i=0}^{n} D_i(s)\Phi_i(f, s) .$$

Proof. This is a similar calculation to Lemma 16 of [3]. (Q.E.D.)

The condition (1–1) of $F(X)$ implies that, for any $f \in \mathscr{C}_0^\infty(V_{\boldsymbol{R}} - S)$,

$$(1\text{–}4) \quad \sum_{T \in L^*} a(T)(\hat{f})^*(T) = \sum_{T \in L^*} a(T)(\langle X \rangle f)^*(T)$$

where $\hat{f}(X) = f(\hat{X})$.

Let $\hat{M}_n$ be the $G_{\boldsymbol{R}}$-invariant differential operator on $V_{\boldsymbol{R}} - S$ defined by $\hat{M}_n f(X) = \{M_n f(\hat{X})\}_{X \to \hat{X}}$. Since the differential operators $\langle X \rangle M_n \langle \hat{X} \rangle$ and $\hat{M}_n$ are both $G_{\boldsymbol{R}}$-invariant, $\langle X \rangle M_n \langle \hat{X} \rangle$ commutes with $\hat{M}_n$ (see § 5 of [1]). Further we can show

$$\hat{M}_n = (-1)^n |\det X|^{(n+1)/2} \det\left(\frac{\partial}{\partial X}\right) \det X \,|\det X|^{-(n+1)/2} ,$$

i.e., $\hat{M}_n$ is the adjoint operator of M_n with respect to dv (see § 6 of [1]).

Let ρ^* be the representation of G on V defined by $\rho^*(g)X = {}^t g^{-1} X g^{-1}$ ($g \in G$, $X \in V$).

LEMMA 1–3. *For any* $f \in \mathscr{C}_0^\infty(V_j)$ $(0 \leqq j \leqq n)$, *we have*

$$\sum_{T \in (L^*)'} a(T) \det T \cdot \{\det X \langle X \rangle M_n \langle \hat{X} \rangle \hat{f}\}^*(\rho^*(g)T)$$
$$= b_j\, b_{n-j} \chi(g)^{k+2} \sum_{T \in (L^*)'} a(T) \det T \cdot \{\det X \langle X \rangle M_n f\}^*(\rho(g)T) .$$

Proof. In (1–4), we put $\langle \hat{X} \rangle \hat{M}_n \langle X \rangle M_n f(\rho^*(g)X)$ instead of $f(X)$. By the facts mentioned above, we have

$$\begin{aligned}
&\{\widehat{\langle \hat{X} \rangle \hat{M}_n \langle X \rangle M_n f(\rho^*(g)X)}\}^*(T) \\
&= \{\langle X \rangle M_n \langle \hat{X} \rangle \hat{M}_n f(\rho^*(g)\hat{X})\}^*(T) \\
&= \int_{V_{\boldsymbol{R}}} \langle X \rangle M_n \langle \hat{X} \rangle \hat{M}_n f(\rho^*(g)\hat{X}) e[\mathrm{Tr}\, XT] dv \\
&= \int_{V_{\boldsymbol{R}}} \hat{M}_n \langle X \rangle M_n \langle \hat{X} \rangle f(\rho^*(g)\hat{X}) e[\mathrm{Tr}\, XT] dv \\
&= (2\pi\sqrt{-1})^n \det T \int_{V_{\boldsymbol{R}}} \det X \langle X \rangle M_n \langle \hat{X} \rangle f(\rho^*(g)\hat{X}) e[\mathrm{Tr}\, XT] dv \\
&= (2\pi\sqrt{-1})^n \det T \chi(g)^{-1} \int_{V_{\boldsymbol{R}}} \det X \langle X \rangle M_n \langle \hat{X} \rangle \hat{f}(X) e[\mathrm{Tr}\, X\rho^*(g)T] dv \\
&= (2\pi\sqrt{-1})^n \det T \cdot \chi(g)^{-1} \{\det X \langle X \rangle M_n \langle \hat{X} \rangle \hat{f}\}^*(\rho^*(g)T)
\end{aligned}$$

and

$$\begin{aligned}
&\{\langle X\rangle\langle\hat{X}\rangle\hat{M}_n\langle X\rangle M_n f(\rho^*(g)X)\}^*(T)\\
&\quad = b_j b_{n-j}\int_{V_R}\hat{M}_n\langle X\rangle M_n f(\rho^*(g)X)e[\mathrm{Tr}\,XT]dv\\
&\quad = (2\pi\sqrt{-1})^n \det T b_j b_{n-j}\chi(g)^{k+1}\int_{V_R}\det X\langle X\rangle M_n f(X)e[\mathrm{Tr}\,X\rho(g)T]dv\\
&\quad = (2\pi\sqrt{-1})^n b_j b_{n-j}\det T\cdot\chi(g)^{k+1}\{\det X\langle X\rangle M_n f\}^*(\rho(g)T)\,. \qquad \text{(Q.E.D.)}
\end{aligned}$$

Set $\xi_i(s) = (2\pi)^{-ns}\gamma(s-(n+1)/2))D_i(s)$ $(0 \leqq i \leqq n)$.

LEMMA 1-4. *For any* $f \in \mathscr{C}_0^\infty(V_j)$ $(0 \leqq j \leqq n)$, *we have*

$$\begin{aligned}
&Z(\{\det X\langle X\rangle M_n\langle\hat{X}\rangle\hat{f}\}^*, \{a(T)\det T\}, s+1)\\
&\quad = (2\pi)^{-1}b_j b_{n-j}(-\sqrt{-1})^n\pi^{n(n-1)/4}e[ns/4]b(-s)b(s-k)\\
&\qquad\qquad \times\sum_{i=0}^n \xi_i(s)u_{i,n-j}(s)\cdot\Phi_j(f,s)
\end{aligned}$$

and

$$\begin{aligned}
&Z(\{\det X\langle X\rangle M_n f\}^*, \{a(T)\det T\}, s+1)\\
&\quad = (2\pi)^{-1}b_j(-\sqrt{-1})^n\pi^{n(n-1)/4}e[ns/4]b(-s)b(s-k)\\
&\qquad\qquad \times\sum_{i=0}^n \xi_i(s)u_{ij}(s)\cdot\Phi_j(f,k-s)\,.
\end{aligned}$$

Proof. By Lemma 1-1 and 1-2, we have

$$\begin{aligned}
&Z(\{\det X\langle X\rangle M_n\langle\hat{X}\rangle\hat{f}\}^*, \{a(T)\det T\}, s+1)\\
&\quad = \sum_{i=0}^n D_i(s)(-1)^{n-i}\cdot\Phi_i(\{\det X\langle X\rangle M_n\langle\hat{X}\rangle\hat{f}\}^*, s+1)\\
&\quad = \pi^{n(n-1)/4}(2\pi)^{-(s+1)n}\gamma\left(s+1-\frac{n+1}{2}\right)e[n(s+1)/4]\\
&\qquad \cdot\sum_{i=0}^n D_i(s)(-1)^{n-i}u_{i,n-j}(s+1)\cdot\Phi_{n-j}(\det X\langle X\rangle M_n\langle\hat{X}\rangle\hat{f}, -s-1)\\
&\quad = \pi^{n(n-1)/4}(2\pi)^{-1}(-\sqrt{-1})^n b(-s)\sum_{i=0}^n\xi_i(s)u_{i,n-j}(s)(-1)^j\\
&\qquad\qquad \times\Phi_{n-j}(\det X\langle X\rangle M_n\langle\hat{X}\rangle\hat{f}, -s-1)
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_{n-j}(\det X\langle X\rangle M_n\langle\hat{X}\rangle\hat{f}, -s-1)\\
&\quad = \Phi_j(\det\hat{X}\langle\hat{X}\rangle\hat{M}_n\langle X\rangle f, s+1)\\
&\quad = (-1)^j b_j b_{n-j}\cdot\Phi_j(\hat{M}_n|\det X|^k f, s-k)\\
&\quad = (-1)^j b_{n-j}\cdot b(s-k)\Phi_j(f,s)\,.
\end{aligned}$$

Similarly, by Lemma 1–1 and 1–2, we have

$$
\begin{aligned}
&Z(\{\det X\langle X\rangle M_n f\}^*, \{a(T)\det T\}, s+1)\\
&\quad = \sum_{i=0}^{n} D_i(s)(-1)^{n-i}\Phi_i(\{\det X\langle X\rangle M_n f\}^*, s+1)\\
&\quad = \pi^{n(n-1)/4}(2\pi)^{-1}(-\sqrt{-1})^n b(-s)\sum_{i=0}^{n}\xi_i(s)(-1)^{n-j}u_{ij}(s)\\
&\qquad\qquad \times\Phi_j(\det X\langle X\rangle M_n f, -s-1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Phi_j(\det X\langle X\rangle M_n f, -s-1)\\
&\quad = (-1)^{n-j}\Phi_j(\langle X\rangle M_n f, -s)\\
&\quad = (-1)^{n-j} b_j \Phi_j(M_n f, -s+k)\\
&\quad = (-1)^{n-j} b_j b(s-k)\Phi_j(f, -s+k)\,. \qquad\qquad \text{(Q.E.D.)}
\end{aligned}
$$

THEOREM 1. *Dirichlet series $\xi_i(s)$ $(0 \leqq i \leqq n)$ have analytic continuations to meromorphic functions in the whole complex plane satisfying the following functional equations:*

$$
e[ns/4]\sum_{i=0}^{n}\xi_i(s)u_{i,n-j}(s) = b_j e[n(k-s)/4]\sum_{i=0}^{n}\xi_i(k-s)u_{ij}(k-s)\,, \qquad (0 \leqq j \leqq n)\,.
$$

Proof. For $f \in \mathscr{C}_0^\infty(V_j)$ and sufficiently large $\mathrm{Re}(s)$, we have

$$
\begin{aligned}
&\int_{G_+/SL(n,\mathbf{Z})}\chi(g)^{s+1}\sum_{T\in(L^*)'}a(T)\det T\{\det X\langle X\rangle M_n\langle\hat X\rangle\hat f\}^*(\rho(g)T)dg\\
&\quad = \int_{G_+/SL(n,\mathbf{Z})\chi(g)\geqq 1} dg + \int_{G_+/SL(n,\mathbf{Z})\chi(g)\leqq 1} dg\,.
\end{aligned}
$$

It is easy to see that, in the latter member, the first integral is continued to holomorphic function on s. For the second integral, using Lemma 1–3, we calculate

$$
\begin{aligned}
&\int_{G_+/SL(n,\mathbf{Z})\chi(g)\leqq 1}\chi(g)^{s+1}\sum_T a(T)\det T\{\det X\langle X\rangle M_n\langle\hat X\rangle\hat f\}^*(\rho(g)T)dg\\
&\quad = \int_{G_+/SL(n,\mathbf{Z})\chi(g)\geqq 1}\chi(g)^{-s-1}\sum_T a(T)\det T\{\det X\langle X\rangle M_n\langle\hat X\rangle\hat f\}^*(\rho^*(g)T)dg\\
&\quad = b_j b_{n-j}\int_{G_+/SL(n,\mathbf{Z})\chi(g)\geqq 1}\chi(g)^{k-s+1}\sum_T a(T)\det T\{\det X\langle X\rangle M_n f\}^*(\rho(g)T)dg\,.
\end{aligned}
$$

So, the second integral is also continued to meromorphic function on s. Furthermore, we have

$$Z(\{\det X\langle X\rangle M_n\langle \hat{X}\rangle \hat{f}\}^*, \{\alpha(T)\det T\}, s+1)$$
$$= b_j b_{n-j} Z(\{\det X\langle X\rangle M_n f\}^*, \{\alpha(T)\det T\}, k-s+1)$$

which is holomorphic on s.

Here, we apply Lemma 1–4, then we have

$$b(-s)b(s-k)e[ns/4]\sum_{i=0}^{n}\xi_i(s)u_{i,n-j}(s)\cdot\Phi_j(f,s)$$
$$= b_j b(-s)b(s-k)e[n(k-s)/4]\sum_{i=0}^{n}\xi_i(k-s)u_{ij}(k-s)\cdot\Phi_j(f,s)\,.$$

For any $s\in\boldsymbol{C}$, we can find $f\in\mathscr{C}_0^\infty(V_j)$ such that $\Phi_j(f,s)\neq 0$. So, we obtain our desired results.

Remark. By the above arguments, $b(s-k)\sum_{i=0}^{n}(-1)^i D_i(s)u_{i,n-j}(s+1)\gamma(s+1-(n+1)/2)$ $(0\leqq j\leqq n)$ are holomorphic on s. Then, we see that $b(s-k)D_i(s)/\gamma(1-s)$ $(0\leqq i\leqq n)$ are holomorphic on s. On the other hand, $D_i(s)$ $(0\leqq i\leqq n)$ are holomorphic on s if $\mathrm{Re}(s)$ is sufficiently large. So, $D_i(s)$ $(0\leqq i\leqq n)$ have only finite poles.

§ 2. Fourier coefficients of Eisenstein series

In this section, we construct distributions with automorphy which are related to Eisenstein series on the Siegel upper half plane.

For any $X\in V_Q$, let $\nu(X)$ be the product of the denominators of the elementary divisors of the symmetric X. It is known that, for any $X\in V_Q$, there uniquely exists a coprime symmetric pair $\{C,D\}$, up to left-multiplications by unimodular matrices, such that $X=C^{-1}D$ (see [1]). Then, we have $\nu(X)=|\det C|$. We define the mapping F by

$$f\mapsto\sum_{R\in V_Q}\nu(R)^{-k}f(R)\qquad(f\in\mathscr{S}(V_{\boldsymbol{R}}),\ k\in\boldsymbol{C})\,.$$

In other words, $F(X)=\sum_{R\in V_Q}\nu(R)^{-k}\delta(X-R)$ where $\delta(X)$ is the Dirac function on $V_{\boldsymbol{R}}$, i.e., $\int_{V_{\boldsymbol{R}}}\delta(X)f(X)dX=f(0)$ $(f\in\mathscr{S}(V_{\boldsymbol{R}}))$.

LEMMA 2–1. *When* $\mathrm{Re}(k)>n+1$, F *is a tempered distribution on* $V_{\boldsymbol{R}}$.

Proof. We show the convergence of $\sum_{R\in V_Q}\nu(R)^{-k}f(R)$. Since $f\in\mathscr{S}(V_{\boldsymbol{R}})$, there exists a constant K such that

$$|f(X)|\leqq K|\det(\sqrt{-1}I+X)|^{-\mathrm{Re}(k)}\qquad(I:\text{ the unite matrix})\,.$$

Then, using the fact $R = C^{-1}D$, we have

$$\begin{aligned}|\nu(R)^{-k}f(R)| &\leqq \nu(R)^{-k}|\det(\sqrt{-1}I + R)|^{-\mathrm{Re}\,(k)} \\ &= |\det C|^{-\mathrm{Re}\,(k)}|\det(\sqrt{-1}I + C^{-1}D)|^{-\mathrm{Re}\,(k)} \\ &= |\det(\sqrt{-1}C + D)|^{-\mathrm{Re}\,(k)} .\end{aligned}$$

Other parts follow from the convergence of Eisenstein series on the Siegel upper half plane. (Q.E.D.)

In the following, we assume that $\mathrm{Re}\,(k) > n + 1$. Let F be considered a distribution on $V_R - S$. In other words, we assume that $f \in \mathscr{C}_0^\infty(V_R - S)$.

LEMMA 2–2. *As a distribution on $V_R - S$, F satisfies the following equation*:

$$F(\hat{X}) = |\det X|^{n+1-k}F(X) .$$

Proof. Since we have

$$\int F(\hat{X})f(X)dX = \int F(X)f(\hat{X})|\det X|^{-n-1}\,dX \qquad (f \in \mathscr{C}_0^\infty(V_R - S)) ,$$

we only show that

$$\sum_{R \in V_Q} \nu(R)^{-k}f(\hat{R})|\det R|^{-n-1} = \sum_{R \in V_Q} |\det R|^{n+1-k}\nu(R)^{-k}f(R) .$$

This follow from the fact $\nu(R^{-1})^k\nu(R)^k = |\det R|^{-k}$. (Q.E.D.)

Remark. It is easy to see that, even if we consider $F(X)$ the mapping

$$f \mapsto \int F(X)f(X)dv , \qquad (f \in \mathscr{C}_0^\infty(V_R - S) ,$$

the above equation $F(\hat{X}) = |\det X|^{n+1-k}F(X)$ holds, too.

For any $T \in L^*$, set

$$S(k, T) = \sum_{R \bmod 1} \nu(R)^{-k}e[\mathrm{Tr}\,RT] .$$

We can show that there exists a constant e, depending only on k, such that $|S(k, T)| < e$ (see [4]).

LEMMA 2–3. *We have*

$$F(X) = \sum_{T \in L^*} S(k, T)e[\mathrm{Tr}\,XT] .$$

Proof. By Poisson's summation formula

$$\sum_{R\in L} \delta(X-R) = \sum_{T\in L^*} e[\mathrm{Tr}\, XT] ,$$

we have

$$\begin{aligned}\sum_{R\in V_Q} \nu(R)^{-k}\delta(X-R) &= \sum_{R \bmod 1} \nu(R)^{-k} \sum_{R'\in L} \delta(X-R-R') \\ &= \sum_{R \bmod 1} \nu(R)^{-k} \sum_{T\in L^*} e[\mathrm{Tr}\,(X-R)T] \\ &= \sum_{T\in L^*} S(k,T)e[\mathrm{Tr}\, XT] . \qquad \text{(Q.E.D.)}\end{aligned}$$

Here, we regard F as the mapping

$$f \mapsto \int F(X)f(X)dv = \sum_{T\in L^*} S(k,T)f^*(T) , \qquad (f\in \mathscr{C}_0^\infty(V_R - S)) ,$$

then the distribution F satisfies the conditions (1–1), (1–2) and (1–3) ($b_i = 1$ for all i). So, $F(X)$ is a distribution with automorphy of weight $n+1-k$.

Set

$$\Xi_i(s) = (2\pi)^{-ns}\gamma\left(s - \frac{n+1}{2}\right) \sum_{T\in L_i^*/\sim} S(k,T)\mu(T)\,|\det T|^{-s} .$$

THEOREM 2. *Dirichlet series $\Xi_i(s)$ $(0 \leqq i \leqq n)$ have analytic continuations to meromorphic functions in the whole complex plane satisfying the following functional equations:*

$$\begin{aligned} &e[ns/4] \sum_{i=0}^{n} \Xi_i(s)u_{i,n-j}(s) \\ &\quad = e[n(n+1-k-s)/4] \sum_{i=0}^{n} \Xi_i(n+1-k-s)u_{ij}(n+1-k-s) , \\ &\hfill (0 \leqq j \leqq n) .\end{aligned}$$

Set

$$\eta_i(s) = (2\pi)^{-ns}\gamma\left(s - \frac{n+1}{2}\right) \sum_{T\in L_i^*/\sim} S(k,T)\,|\det T|^{k-(n+1)/2}\,\mu(T)\,|\det T|^{-s}$$

$(0 \leqq i \leqq n)$, then, using the facts $(u_{i,j}(s))(u_{i,n-j}((n+1)/2 - s)) = I$ and $u_{i,n-j}(s) = u_{n-i,j}(s)$, Theorem 2 is changed into the following:

THEOREM 2′.

$$\begin{aligned} &e[ns/4](\eta_0(s), \cdots, \eta_n(s))(u_{ij}(k-s))^{-1} \\ &\quad = e[n(k-s)/4](\eta_0(k-s), \cdots, \eta_n(k-s))(u_{ij}(s))^{-1} .\end{aligned}$$

The above result is consistent with that of [4].

§3. Boundary values of Siegel modular forms

In this section, we introduce functions on the Siegel upper half plane satisfying several conditions, and consider their boundary values which is distributions with automorphy. Specially, by the distributions with automorphy derived from Siegel modular forms, we obtain the Dirichlet series corresponding to the case of Maass.

Set $Z = X + \sqrt{-1}Y$ and $\bar{Z} = X - \sqrt{-1}Y$, where $X, Y \in V_{\boldsymbol{R}}$ and $Y > 0$. Let $F(Z, \bar{Z})$ be a real-analytic function on the Siegel upper half plane H, satisfying the following conditions:

(3–1) $F(M\langle Z\rangle, M\langle\bar{Z}\rangle) = \det(CZ + D)^{\alpha}\det(C\bar{Z} + D)^{\beta}F(Z, \bar{Z})$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_p(n, \boldsymbol{Z})$, $M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$ and α, $\beta \in \boldsymbol{C}$ $(\alpha - \beta \in \boldsymbol{Z})$.

(3–2) There exist positive numbers e_1 and e_2 such that

$$|F(Z, \bar{Z})| \leqq e_1 |\det Y|^{-e_2} \qquad \text{when } Y \to 0$$

and

$$|F(Z, \bar{Z}) \leqq e_1 |\det Y|^{e_2} \qquad \text{when } Y \to \infty\,.$$

(3–3) $\lim_{Y \to 0} \int_{V_{\boldsymbol{R}}/L} F(Z, \bar{Z}) e[-\operatorname{Tr} XT] dX < \infty$ (for any $T \in L^*$).

By (3–1), $F(Z, \bar{Z})$ has the Fourier expansion

$$F(Z, \bar{Z}) = \sum_{T \in L^*} a(T, Y) e[\operatorname{Tr} XT]$$

where $a(T, Y) = \int_{V_{\boldsymbol{R}}/L} F(Z, \bar{Z}) e[\operatorname{Tr} XT] dX$.

We set $a(T) = \lim_{Y \to 0} a(T, Y)$, which exist owing to (3–3). It is easily shown that $a(T, Y)/a(T)$ depend only on ${}^t\sqrt{Y}T\sqrt{Y}$. We can set $a(T, Y) = a(T)W({}^t\sqrt{Y}T\sqrt{Y})$. Then, by (3–2), we have

$$|a(T)W({}^t\sqrt{Y}T\sqrt{Y})| \leqq e_1 |\det Y|^{-e_2} \qquad \text{when } Y \to 0\,.$$

So, there exists a constant e such that

$$|a(T)| \leqq e |\det T|^{e_2}\,.$$

Now, it is easy to see that $F(Z, \bar{Z})$ tends to the distribution

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$$F(X) = \sum_{T \in L^*} a(T)e[\operatorname{Tr} XT]$$

when $Y \to 0$, and $F(X)$ satisfies the equation

$$F(\hat{X}) = \det(X + \sqrt{-1}0)^{\alpha} \det(X - \sqrt{-1}0)^{\beta} F(X) \qquad \text{on } V_R - S\,,$$

where $\det(X + \sqrt{-1}0) = \lim_{Y\to 0} \det(X + \sqrt{-1}Y)$ and $\det(X - \sqrt{-1}0) = \lim_{Y\to 0} \det(X - \sqrt{-1}0)$. So, $F(X)$ is a distribution with automorphy of weight $\alpha + \beta$.

It is clear that a Siegel modular form $F(Z) = \sum_{T \geqq 0} a(T)e[\operatorname{Tr} ZT]$ of weight k (even integer) satisfies the above conditions (3–1), (3–2) and (3–3). Set $\xi(s) = (2\pi)^{-ns}\gamma(s - (n+1)/2)) \sum_{T \in L_n^*/\sim} a(T)\mu(T)\,|\det T|^{-s}$. Then, by Theorem 1, we have

$$e[ns/4]\xi(s)u_{n,n-j}(s) = e[n(k-s)/4]\xi(k-s)u_{n,j}(k-s) \qquad (0 \leqq j \leqq n)\,.$$

Using the fact $e[ns/4]u_{n,n-j}(s) = (-1)^{nk/2}e[n(k-s)/4]u_{n,j}(k-s)$, we have

$$\xi(k-s) = (-1)^{nk/2}\xi(s)$$

which is the result of Maass [1].

Added in proof. The formulation and the arguments in § 3 are not adequate. The author noticed that the Fourier transform of the boundary value for an automorphic form is a generalized Poisson's summation formula. For details, see the author's paper "Weil type representations and automorphic forms" (to appear in Nagoya Math. J.). (Feb. 1979)

References

[1] H. Maass, Siegel's Modular Forms and Dirichlet Series, Lecture note in Math., No. 216, Springer, Berlin-Heidelberg-New York, 1971.

[2] M. Sato and T. Shintani, On zeta functions associated with prehomogeneous vector space, Ann. of Math. **100** (1974), 131–170.

[3] T. Shintani, On zeta functions associated with the vector space of quadratic forms, J. Fac. Science, Univ. of Tokyo, Sec. IA. Vol. **22**, 25–65.

[4] T. Arakawa, Dirichlet series associated with Eisenstein series of higher degree, to appear in Comment. Univ. Sancti. Pauli.

Department of Mathematics
Nagoya University