# INTEGRAL CLOSURES OF IDEALS RELATIVE TO ARTINIAN MODULES, AND EXACT SEQUENCES

*by* R. Y. SHARP and Y. TIRAŞ

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**Introduction.** In [3], Sharp and Taherizadeh introduced concepts of reduction and integral closure of an ideal $I$ of a commutative ring $R$ (with identity) relative to an Artinian $R$-module $A$, and they showed that these concepts have properties which reflect some of those of the classical concepts of reduction and integral closure introduced by Northcott and Rees in [2].

We say that the ideal $I$ of $R$ is a *reduction* of the ideal $J$ of $R$ *relative to* $A$ if $I \subseteq J$ and there exists $s \in \mathbb{N}$ such that $(0 :_A IJ^s) = (0 :_A J^{s+1})$. (We use $\mathbb{N}$ (respectively $\mathbb{N}_0$) to denote the set of positive (respectively non-negative) integers.) An element $x$ of $R$ is said to be *integrally dependent on I relative to A* if there exists $n \in \mathbb{N}$ such that

$$\left(0 :_A \sum_{i=1}^{n} x^{n-i} I^i\right) \subseteq (0 :_A x^n).$$

In fact, this is the case if and only if $I$ is a reduction of $I + Rx$ relative to $A$ [3, Lemma (2.2)]; moreover,

$$I^* := \{y \in R : y \text{ is integrally dependent on } I \text{ relative to } A\}$$

is an ideal of $R$, called the *integral closure of I relative to A*, and is the largest ideal of $R$ which has $I$ as a reduction relative to $A$. In this paper, we shall indicate the dependence of $I^*$ on the Artinian $R$-module $A$ by means of the extended notation $I^{*(A)}$.

In fact, this notation is very relevant to this paper, because its purpose is to investigate how the ideal $I^{*(A)}$ depends on the Artinian module $A$ in the context of exact sequences: our main result is that, if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of Artinian $R$-modules, then $I^{*(B)} = I^{*(A)} \cap I^{*(C)}$.

Although this result does not appear to have any counterpart in the classical theory of integral closure, where the underlying ring tended to be fixed at the outset, there is a dual result which is perhaps worthy of mention. In [4, Section 1], dual concepts of reduction and integral closure of the ideal $I$ relative to a Noetherian $R$-module $N$ were introduced; we shall denote the integral closure of $I$ relative to $N$ by $I^{-(N)}$. However, it turns out that this integral closure is related to the classical integral closure of Northcott and Rees: the ring $R/(0 : N)$ is Noetherian, and, by [4, 1.6], $I^{-(N)} \supseteq (0 : N)$ and

$$I^{-(N)}/(0 : N) = ((I + (0 : N))/(0 : N))^{-},$$

where the ordinary, classical integral closure of an ideal $J$ of a commutative Noetherian ring $R'$ is denoted by $J^{-}$. This fact means that we can quickly deduce the following.

1.1. Proposition. *Let*

$$0 \to L \to M \to N \to 0$$

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*be an exact sequence of Noetherian modules over the commutative ring R (with identity), and let I be an ideal of R. Then* $I^{-(M)} = I^{-(L)} \cap I^{-(N)}$.

*Proof.* Let $x \in I^{-(M)}$, so that, in view of the comment immediately preceding the statement of this proposition, there exist $n \in \mathbb{N}$ and elements $c_1, \ldots, c_n \in R$ with $c_i \in I^i$ for $i = 1, \ldots, n$ such that

$$x^n + c_1x^{n-1} + \ldots + c_{n-1}x + c_n \in (0:M).$$

But $(0:M) \subseteq (0:L) \cap (0:N)$, and so it again follows from [**4**, 1.6] that

$$x + (0:N) \in ((I + (0:N))/(0:N))^- = I^{-(N)}/(0:N)$$

and $x \in I^{-(N)}$. Similarly $x \in I^{-(L)}$, and so we have proved that $I^{-(M)} \subseteq I^{-(L)} \cap I^{-(N)}$.

The reverse inclusion is almost as easy to prove. Let $x \in I^{-(L)} \cap I^{-(N)}$; thus there exist $n \in \mathbb{N}$ and elements $c_1, \ldots, c_n \in R$ with $c_i \in I^i$ for $i = 1, \ldots, n$ such that

$$x^n + c_1x^{n-1} + \ldots + c_{n-1}x + c_n \in (0:L),$$

and there exist $h \in \mathbb{N}$ and elements $d_1, \ldots, d_h \in R$ with $d_i \in I^i$ for $i = 1, \ldots, h$ such that

$$x^h + d_1x^{h-1} + \ldots + d_{h-1}x + d_h \in (0:N).$$

But then

$$\left(x^n + \sum_{i=1}^{n} c_ix^{n-i}\right)\left(x^h + \sum_{i=1}^{h} d_ix^{h-i}\right) \in (0:L)(0:N) \subseteq (0:M),$$

and it follows from this that $x \in I^{-(M)}$. This completes the proof.

We have not found it so easy to prove the corresponding result for a short exact sequence of Artinian modules, and our proof of this dual result forms the content of the next section.

**2. The result.** Throughout, $R$ denotes a commutative ring (with identity). We begin with some preparatory comments which will be helpful in the proof of the main result.

2.1. Remark. Let $A, B$ and $C$ be Artinian $R$-modules, and let $I$ be an ideal of $R$. Then it follows from [**3**, (1.5) and (1.6)(i)] that there exists a finitely generated ideal $I'$ of $R$ such that $I' \subseteq I$ and, for all $n \in \mathbb{N}$,

$$(0:_A I'^n) = (0:_A I^n), \qquad (0:_B I'^n) = (0:_B I^n), \qquad (0:_C I'^n) = (0:_C I^n).$$

Note that such an $I'$ must be a reduction of $I$ relative to each of $A, B, C$, so that, by [**3**, (2.4)(iv)],

$$I'^{*(A)} = I^{*(A)}, \qquad I'^{*(B)} = I^{*(B)}, \qquad I'^{*(C)} = I^{*(C)}.$$

We shall make use of the following fact from [**3**].

2.2. Proposition [**3**, (1.7)]. *Let A be an Artinian R-module, and let I, J be ideals of R such that* $I \subseteq J$ *and I is finitely generated by* $r_1, \ldots, r_s$. *Let* $X_1, \ldots, X_s$ *be independent indeterminates over R, and consider* $R[X_1, \ldots, X_s]$ *as a graded ring in the usual way. Let*

*$G_A(J) = G = \bigoplus_{n\in\mathbb{Z}} G_n$ be the graded $R[X_1, \ldots, X_s]$-module defined as follows: for $n \in \mathbb{Z}$,*

$$G_n = \begin{cases} 0 & \text{for } n > 0, \\ A/(0:_A J^{-n}) & \text{for } n \le 0; \end{cases}$$

*the ideas of [**1**, p. 55] are used to turn $G$ into a graded $R[X_1, \ldots, X_s]$-module in such a way that, for a negative integer $n$ and $a \in A$,*

$$X_i(a + (0:_A J^{-n})) = r_i a + (0:_A J^{-n-1})$$

*for $i = 1, \ldots, s$.*

*Then $G_A(J)$ is an Artinian $R[X_1, \ldots, X_s]$-module if and only if $I$ is a reduction of $J$ relative to $A$.*

2.3. Corollary. *Let $B$ be an Artinian $R$-module, and let $I, J$ be ideals of $R$ such that $I \subseteq J$ and $I$ is finitely generated by $r_1, \ldots, r_s$. Let $A$ be a submodule of $B$ such that $I$ is a reduction of $J$ relative to $B/A$.*

*Let $H_{B,A}(J) = H = \bigoplus_{n\in\mathbb{Z}} H_n$ be defined as follows: for $n \in \mathbb{Z}$, set*

$$H_n = \begin{cases} 0 & \text{for } n > 0, \\ B/(A + (0:_B J^{-n})) & \text{for } n \le 0; \end{cases}$$

*then the $R$-module $H_{B,A}(J)$ can be turned into a graded $R[X_1, \ldots, X_s]$-module in such a way that, for a negative integer $n$ and $b \in B$,*

$$X_i(b + (A + (0:_B J^{-n}))) = r_i b + (A + (0:_B J^{-n-1}))$$

*for $i = 1, \ldots, s$.*

*Then $H_{B,A}(J)$ is an Artinian $R[X_1, \ldots, X_s]$-module.*

*Proof.* Let $\tilde{G}_{B,A}(J) = \tilde{G} = \bigoplus_{n\in\mathbb{Z}} \tilde{G}_n$ be defined as follows: for $n \in \mathbb{Z}$, set

$$\tilde{G}_n = \begin{cases} 0 & \text{for } n > 0, \\ B/(A:_B J^{-n}) & \text{for } n \le 0; \end{cases}$$

turn the $R$-module $\tilde{G}$ into a graded $R[X_1, \ldots, X_s]$-module in such a way that, for a negative integer $n$ and $b \in B$,

$$X_i(b + (A:_B J^{-n})) = r_i b + (A:_B J^{-n-1})$$

for $i = 1, \ldots, s$. By 2.2 applied to $B/A$, the $R[X_1, \ldots, X_s]$-module $\tilde{G}_{B,A}(J)$ is Artinian.

By Kirby's 'Artin-Rees' Lemma for Artinian modules [**1**, Proposition 3], there exists $t \in \mathbb{N}_0$ such that

$$A + (0:_B J^n) = ((A + (0:_B J^t)):_B J^{n-t}) \quad \text{for all} \quad n \ge t.$$

Now let $\tilde{H} = \bigoplus_{n\in\mathbb{Z}} \tilde{H}_n$ be defined as follows: for $n \in \mathbb{Z}$, set

$$\tilde{H}_n = \begin{cases} 0 & \text{for } n > 0, \\ B/(A:_B J^{-n-t}) & \text{for } n \le 0, \end{cases}$$

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where $J^h$ is to be interpreted as $R$ when the integer $h$ is negative; turn the $R$-module $\tilde{H}$ into a graded $R[X_1, \ldots, X_s]$-module in such a way that, for a negative integer $n$ and $b \in B$,

$$X_i(b + (A :_B J^{-n-t})) = r_i b + (A :_B J^{-n-t-1})$$

for $i = 1, \ldots, s$. It follows easily from [**1**, Theorem 1(ii)], together with the above-mentioned fact that $\tilde{G}_{B,A}(J)$ is an Artinian $R[X_1, \ldots, X_s]$-module, that $\tilde{H}$ is an Artinian $R[X_1, \ldots, X_s]$-module.

Next note that, for all $n \in \mathbb{Z}$ with $n \leq -t$, we have

$$(A :_B J^{-n-t}) \subseteq ((A + (0 :_B J^t)) :_B J^{-n-t}) = A + (0 :_B J^{-n}),$$

while $A \subseteq A + (0 :_B J^{-n})$ for $-t < n \leq 0$. It is now easy to construct an $R[X_1, \ldots, X_s]$-epimorphism $\tilde{H} \to H_{B,A}(J)$, and so the result follows.

We are now in a position to prove the main result of this note.

2.4 Theorem. *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

*be an exact sequence of Artinian R-modules, and let I be an ideal of R. Then* $I^{*(B)} = I^{*(A)} \cap I^{*(C)}$.

*Proof.* We can assume that $A$ is a submodule of $B$, that $C = B/A$, that $f$ is the inclusion map, and that $g$ is the canonical epimorphism. By 2.1, we can, and do, assume that $I$ is finitely generated, say by $r_1, \ldots, r_s$.

Set $J = I^{*(A)} \cap I^{*(C)}$. By [**3**, (1.8) and (2.4)], the ideal $J$ has $I$ as a reduction relative to each of $A$ and $C$; our aim is to show that $I$ is a reduction of $J$ relative to $B$.

Consider the graded $R[X_1, \ldots, X_s]$-modules $G_A(J)$, $G_B(J)$ and $H_{B,A}(J)$, as in 2.2 and 2.3: by those results, $G_A(J)$ and $H_{B,A}(J)$ are Artinian $R[X_1, \ldots, X_s]$- modules. For each $n \in \mathbb{N}_0$, there is an exact sequence of $R$-modules

$$0 \to A/(0 :_A J^n) \to B/(0 :_B J^n) \to B/(A + (0 :_B J^n)) \to 0.$$

It is easy to obtain from this an exact sequence

$$0 \to G_A(J) \to G_B(J) \to H_{B,A}(J) \to 0$$

of $R[X_1, \ldots, X_s]$-modules and $R[X_1, \ldots, X_s]$-homomorphisms. Hence $G_B(J)$ is an Artinian $R[X_1, \ldots, X_s]$-module, and so it follows from 2.2 that $I$ is a reduction of $J$ relative to $B$. Hence, by [**3**, (2.4)],

$$J = I^{*(A)} \cap I^{*(C)} \subseteq I^{*(B)}.$$

The reverse inclusion can be proved more easily. Let $K = I^{*(B)}$; by 2.2, and with the notation thereof, $G_B(K)$ is an Artinian $R[X_1, \ldots, X_s]$-module; it is easy to see that there exist an $R[X_1, \ldots, X_s]$-monomorphism $G_A(K) \to G_B(K)$ and an $R[X_1, \ldots, X_s]$-epimorphism $G_B(K) \to G_{B/A}(K)$; hence $G_A(K)$ and $G_{B/A}(K)$ are Artinian $R[X_1, \ldots, X_s]$-modules; it therefore follows from 2.2 again that $I$ is a reduction of $K$ relative to each of $A$ and $B/A$; and so, by [**3**, (2.4)] again,

$$K = I^{*(B)} \subseteq I^{*(A)} \cap I^{*(C)}.$$

This completes the proof.

2.5. Concluding Remarks. It follows from 2.4 that if $A$ is either a submodule or a homomorphic image of the Artinian $R$-module $B$, and $I$ is an ideal of $R$, then $I^{*(B)} \subseteq I^{*(A)}$. Thus, speaking loosely, we see that the smaller the Artinian module, the larger the integral closure of $I$ relative to it; indeed, $I^{*(0)}$, the integral closure of $I$ relative to the (Artinian!) module 0, is $R$ itself. In the case when $R$ is semi-local and Noetherian and $E$ is an Artinian injective cogenerator (see [**5**, p. 46]) for $R$, it follows from [**4**, 2.1] that $I^{*(E)} = I^-$, the classical integral closure of $I$ studied by Northcott and Rees; moreover, in this situation, $I^- \subseteq I^{*(B)}$ for every Artinian $R$-module $B$. It is perhaps interesting to ask which ideals of $R$ that contain $I^-$ can arise as the integral closure of $I$ relative to some Artinian $R$-module. The above Theorem 2.4 might be of some assistance with such an investigation.

## REFERENCES

**1.** D. Kirby, Artinian modules and Hilbert polynomials, *Quart. J. Math. Oxford Ser.* (2) **24** (1973), 47–57.

**2.** D. G. Northcott and D. Rees, Reductions of ideals in local rings, *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.

**3.** R. Y. Sharp and A.-J. Taherizadeh, Reductions and integral closures of ideals relative to an Artinian module, *J. London Math. Soc.* (2) **37** (1988), 203–218.

**4.** R. Y. Sharp, Y. Tiraş and M. Yassi, Integral closures of ideals relative to local cohomology modules over quasi-unmixed local rings, *J. London Math. Soc.* (2) **42** (1990), 385–392.

**5.** D. W. Sharpe and P. Vámos, *Injective modules* (Cambridge University Press, 1972).

Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH

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