i.e. \[2S_n = n^2(n + 1) - S_{n-1} - \sum_{k=1}^{n-1} k.\]

Using the fact that \(S_{n-1} = S_n - n^2\): we have an equation for \(S_n\),
\[3S_n = n^2(n + 1) + n^2 - \frac{1}{2}(n - 1)n,\]
i.e. \[6S_n = 2n^2(n + 1) + 2n^2 - n^2 + n = n(n + 1)(2n + 1).\]
Hence \[S_n = \frac{1}{6}n(n + 1)(2n + 1).\]

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96.33 A solution to the quartic equation

Introduction

In this paper, we propose a new, simple, easy-to-handle solution to the quartic equation using radicals. The basic idea behind the proposed solution is to transform a general quartic into a bi-quadratic equation, (i.e. a quadratic in \(x^2\)), whose solution is well known. In order to accomplish this, a transformation of variables is used to convert a quartic into self-reciprocal (symmetrical or palindromic) form from which, through another transformation, a bi-quadratic is encountered. Such a methodology is not new and can be found in the literature (e.g. [1, 2, 3, 4]). The procedure used here, however, leads to a resolvent (subsidiary) cubic, whose four coefficients, referred to as test parameters, are used to identify three possible cases, two of which encompass special cases whereas the remaining one comprises the general case. For the special cases, the solution of the subsidiary cubic is not required, and the solutions of the quartic equation are easily and straightforwardly written in terms of its coefficients. The general case calls for the solution of the subsidiary cubic, but the solution of the quartic equation is compactly written in terms of its coefficients and of a
root of the subsidiary cubic. As opposed to the other available methods, in the proposed solution tests of signs are totally dispensable. Another attractive feature of our results is the way the exceptional cases are dealt with in terms of the test parameters and how the solution of the general case is written in a manner that can be easily implemented.

**Proposed solution of the quartic equation**

Given a general quartic equation

\[ ax^4 + bx^3 + cx^2 + dx + e = 0 \]  

(1)

where \( a \neq 0 \), \( b, c, d, e \in \mathbb{C} \), four test parameters, namely \( t_3, t_2, t_1 \) and \( t_0 \), are determined by

\[
\begin{align*}
    t_3 &= b^3 - 4abc + 8a^2d \\
    t_2 &= b^2c - 4ac^2 + 2abd + 16a^2e \\
    t_1 &= b^2d - 4acd + 8abe \\
    t_0 &= -ad^2 + b^2e.
\end{align*}
\]

(2)

Three different situations are then identified as follows*

- If \( t_3 = t_2 = 0 \) and \( t_2 \neq 0 \), then the roots of (1) are given by

\[
x_i = \frac{-b^\pm \sqrt{3b^2 - 8ac}}{4a}, \quad i = 1, \ldots, 4.
\]

(3)

In this case, \( x_i \) has multiplicity four (four identical roots) if \( 3b^2 - 8ac = 0 \), or multiplicity two, otherwise.

- If \( t_3 = 0 \) and \( t_2 \neq 0 \), then the roots of (1) are given by

\[
x_i = -\frac{b^\pm \sqrt{2b^4 - 8b^2ac + 16a^2c^2 - 64a^3e + 3b^2 - 8ac}}{4a}, \quad i = 1, \ldots, 4.
\]

(4)

- Otherwise, the roots of (1) are given by

\[
x_i = \frac{-b^\pm A^\pm \sqrt{B^\pm A(b + 4aw)}}{4a}, \quad i = 1, \ldots, 4,
\]

(5)

where

\[
\begin{align*}
    A &= \sqrt{b^3 + 8a^2d - 4abc} \\
    B &= \frac{b^3 - 4a^2d - 2abc + 6ab^2w - 16a^2cw}{b + 4aw}
\end{align*}
\]

and \( w \) is any of the three possible roots of

\[
t_3w^3 + t_2w^2 + t_1w + t_0 = 0.
\]

(6)

* In (3), (4) and (5), the signs are taken row-wise for each one of the four roots.
The situations given previously cover all possible scenarios for the solution of the quartic equation as proposed here. There is, however, another condition that leads to a simpler solution, although (5) can also be used for this case. This is given as follows.

• If \( t_3 \neq 0 \), and \( \left( \frac{t_2}{t_3} \right)^2 = \frac{t_1}{t_3} \), and \( \left( \frac{t_2}{t_1} \right)^3 = \frac{t_0}{t_3} \), then the roots of (1) are given by

\[
x_i = -\frac{b^2c - 4ac^2 + 2abd + 16a^2e}{3(b^3 - 4abc + 8a^2d)}, \quad i = 1, 2, 3 \quad (7)
\]

\[
x_4 = -\frac{b^4 - 5ab^2c + 6a^2bd + 4ac^2e - 16a^3e}{a(b^3 - 4abc + 8a^2d)}, \quad (8)
\]

i.e. one of the roots has multiplicity three.

**Rationale for the proposed solution**

The idea behind the proposed solution is delineated as follows. Given a general quartic equation as in (1), the aim is to convert it into a bi-quadratic equation whose solution is well known. In order to accomplish this, two transformations are applied. Firstly, a linear transformation of the kind \( x = w + ky \) is used, where \( w \) and \( k \) are constants to be determined and \( y \) is the new variable. The resulting equation in \( y \) is then forced to be self-reciprocal by equating the coefficient of \( y^4 \) to that of \( y^0 \) and the coefficient of \( y^3 \) to that of \( y^1 \). The values of \( w \) and \( k \) are then determined from these two equations (two equations and two unknowns). Now, another transformation, a bilinear one, of the kind \( y = \frac{1}{1 + t} \) is applied to the self-reciprocal equation. Such a transformation reduces the self-reciprocal equation straight into a bi-quadratic equation whose solution follows directly. The solution then is that presented in (5).

On the other hand, such a procedure may be impaired by particular relations among the coefficients \( a, b, c, d \) and \( e \) of the quartic equation (1) leading to singularities. In such cases, the solutions of the quartic equation are always substantially simpler than those presented in (5). These singularities are identified as follows.

• When \( t_3 = t_2 = 0 \), then \( t_1 = t_0 = 0 \), and the solution of (6) is not possible. Therefore (5) cannot be used.

• When \( t_1 = 0 \) and \( t_2 \neq 0 \), then the solution of (6) is \( w = -\frac{1}{4t_0} \), and the use of (5) is not possible.

• When \( t_3 \neq 0 \), and \( \left( \frac{t_2}{t_3} \right)^2 = \frac{t_1}{t_3} \), and \( \left( \frac{t_2}{t_1} \right)^3 = \frac{t_0}{t_3} \), then (5) can be used, although the simpler solution is obviously preferred.

Owing to lack of space, the detailed demonstrations of the general as well as the special cases are not given here, but can be found in [5]. It must be emphasised, however, that the proofs arise as simple algebraic, but
heavy, ‘brute-force’ manipulations. In the general case, which is the most involved one, the key points are the two transformations used. In the special cases, the relations between the test parameters $t_3, t_2, t_1$ and $t_0$ lead to special relations between the coefficients $a, b, c, d$ and $e$ that, replaced back into the general equation, simplify the formulations.

Some examples

In this section, some illustrative examples are given to show the method in action. We implemented our formulas using the Maple and Mathematica software packages.

• Example 1. Consider the quartic equation

$$x^4 - 11x^3 + 7x^2 + 111x - 108 = 0.$$  \(9\)

From (2) and (9), $t_3 = -135$, $t_2 = -3519$, $t_1 = 19827$, $t_0 = -25389$, and no singularities are found. The solution of (6) yields $w_1 = -31$, $w_2 = \frac{7}{3}$, $w_3 = \frac{13}{3}$. (For a solution to the cubic equation the reader is referred to, e.g., [5]) Using any one of these three roots and the coefficients of the given quartic in (5), we obtain $x_1 = 1, x_2 = -3, x_3 = 4, x_4 = 9$.

• Example 2. Consider the quartic equation

$$x^4 - 4x^3 - 66x^2 + 140x + 1225 = 0.$$  \(10\)

From (2) and (10), $t_3 = t_2 = t_1 = t_0 = 0$. Therefore, by means of (3), $x_1 = x_2 = -5, x_3 = x_4 = 7$.

• Example 3. Consider the quartic equation

$$x^4 + 3x^3 + \frac{1}{3}x^2 - 3x - 8 = 0.$$  \(11\)

From (2) and (11), $t_3 = 0, t_2 = -144, t_1 = -216, t_0 = -81$. Therefore, by means of (4), $x_1 = -\frac{3}{4} + \frac{i}{4}\sqrt{23}, x_2 = -\frac{3}{4} - \frac{i}{4}\sqrt{23}, x_3 = -\frac{3}{4} + \frac{i}{4}\sqrt{23}, x_4 = -\frac{3}{4} - \frac{i}{4}\sqrt{23}$.

• Example 4. Consider the quartic equation

$$x^4 - 5x^3 + 6x^2 + 4x - 8 = 0.$$  \(12\)

From (2) and (12), $t_3 = 27, t_2 = -162, t_1 = 324, t_0 = -216$ which satisfy the conditions \(\left(\frac{t_1}{3t_3}\right)^2 = \frac{1}{3t_2}\) and \(\left(\frac{t_3}{3t_1}\right)^3 = \frac{t_2}{t_1}\). Therefore, by means of (7) and (8), $x_1 = x_2 = x_3 = 2, x_4 = -1$. Note, however, that this Example 4 can also be solved by means of (5).
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96.34 Determining the prime factorisation of a matrix

The authors of a previous Gazette, articles [1] and [2] applied the concept of prime factorisation to a specific set of matrices, and proved that each nonidentity matrix in this set can be written uniquely as a product of positive integer powers of two ‘prime’ matrices. This note will expand on this work by giving a method to easily form the prime factorisation and then demonstrate a connection to a continued fraction expansion of specific entries in a matrix.

Let $M_2$ be the set of $2 \times 2$ matrices $X$, each with non-negative integer entries and determinant 1, denoted $|X| = 1$. Let $I \in M_2$ be the identity matrix. If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2$ and $X \neq I$ then one of the following is true:

(a) $a \geq b$ and $c \geq d$, or
(b) $a \leq b$ and $c \leq d$.

In each case, at least one of the inequalities is strict. For (a), we say that the left column of $X$ dominates the right column (or is the dominant column) and for (b), the right column of $X$ is the dominant column [2].

If $a$ is the largest value in $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the left column is dominant. Exchanging $b$ and $c$ gives matrix $Y \in M_2$ where the left column