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ANNIHILATING POLYNOMIALS FOR GROUP RINGS AND WITT RINGS

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ABSTRACT. Natural annihilating polynomials for group rings are produced; this yields, as a special case, the annihilating polynomials for Witt rings that have been discovered only recently.

0. **Introduction.** For a positive interger n let the polynomial $L_n \in \mathbf{Z}[t]$ be given by

$$L_n(t) = (t-n)(t-n+2)\cdot \ldots \cdot (t+n-2)(t+n).$$

Only recently it has been discovered that, *for any field F of characteristic $\neq 2$, every n-dimensional quadratic form q over F satisfies*

$$L_n(q) = 0$$

in the Witt ring $W(F)$ of F; [3].

We will see in our note that this result by D. W. Lewis does not depend heavily on quadratic forms and Witt theory, but will follow directly from elementary Fourier Analysis on groups. In fact, for any abelian torsion group G, we obtain a theorem on annihilating polynomials for the group ring $\mathbf{C}[G]$, see (1.3), which in the special case of $G = F^*/F^{*2}$ has the formal consequence, via the natural map $\mathbf{Z}[G] \longrightarrow W(F)$, that every n-dimensional quadratic form over F is annihilated by L_n, see (2.4).

For a positive integer n let the polynomial $P_n \in \mathbf{Z}[t]$ be given by

$$\begin{aligned} P_n(t) &= (t-n)(t-n+2)\cdot \ldots \cdot (t-3)(t-1) \quad &&\text{if } n \text{ is odd,}\\ P_n(t) &= (t-n)(t-n+2)\cdot \ldots \cdot (t-2)t &&\text{if } n \text{ is even.}\end{aligned}$$

At about the same time it has been proved via Burnside rings that, *for any field F of characteristic $\neq 2$, every n-dimensional sum q of trace forms over F satisfies already*

$$P_n(q) = 0$$

in the Witt ring $W(F)$ of F; [2].

We do not expect this result by P. E. Conner to be as well a formal consequence of our theorem on group rings. However, theorem (1.3) yields immediately that, *for*

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any field F of characteristic $\neq 2$, every n-dimensional sum of Pfister forms over F is already annihilated by P_n, see (3.3). Now, for any F, every sum of Pfister forms over F is Witt equivalent to a sum of trace forms over F. Moreover, for example for all algebraic number fields F, every sum of trace forms over F is Witt equivalent to a sum of Pfister forms over F.

1. **Annihilating polynomials for group rings.** Let G be any abelian group. We denote by T the group of complex numbers $z \in \mathbf{C}$ of absolute value $|z| = 1$, and put $\hat{G} = \mathrm{Hom}(G, T)$, the dual group of G.

The complex group ring $\mathbf{C}[G]$ consists of all sums $f = \Sigma n_g g$, the summation taken over all $g \in G$, with $n_g \in \mathbf{C}$, $n_g = 0$ for almost all $g \in G$. Clearly, every group homomorphism $\chi : G \longrightarrow T$ in $\hat{G}$ extends to a $\mathbf{C}$-algebra homomorphism $\underline{\chi}: \mathbf{C}[G] \longrightarrow \mathbf{C}$ via $\underline{\chi}(\Sigma n_g g) = \Sigma n_g \chi(g)$. As a direct consequence of the Fourier Inversion Theorem for locally compact abelian groups, e.g. [4], we obtain that the intersection of the kernels of those $\mathbf{C}$-algebra homomorphisms $\underline{\chi}$ is the zero ideal in $\mathbf{C}[G]$; namely:

LEMMA 1.1. *Let G be an abelian group. Then $\bigcap_{\chi \in \hat{G}} \ker \underline{\chi} = \{0\}$.*

PROOF. Any $f = \Sigma n_g g \in \mathbf{C}[G]$ can be considered to be a (continuous) function $f : G \longrightarrow \mathbf{C}$ on the discrete group G with (finite) compact support; just put $f(g) = n_g$. Let $f \in \bigcap_{\chi \in \hat{G}} \ker \underline{\chi}$; that is, $\Sigma n_g \chi(g) = 0$ for all $\chi \in \hat{G}$; this means in terms of the Fourier transform $\hat{f}$ of f that $\hat{f}(\chi) = \int_G f(g)\chi(g) d\mu_G(g) = 0$ for all χ in the compact group $\hat{G}$. By the Inversion Theorem we have $f(g) = \int_{\hat{G}} \hat{f}(\chi)\overline{\chi(g)} d\mu_{\hat{G}}(\chi)$ for all $g \in G$, where the Haar measure on $\hat{G}$ has been normalized appropriately. Thus $f(g) = 0$ for all $g \in G$; that is, $f = 0$ in $\mathbf{C}[G]$. $\square$

We would like to thank J. W. Hoffman for suggesting to prove our lemma in this way.

OBSERVATION 1.2. *Let G be an abelian torsion group and $f \in \mathbf{C}[G]$. Then $S_f := \{\underline{\chi}(f) : \chi \in \hat{G}\}$ is finite.*

PROOF. Let $f = \Sigma n_g g \in \mathbf{C}[G]$. There are only finitely many coefficients $n_g \neq 0$. Since G is a torsion group, $\chi(g)$ can take on only finitely many values for each $g \in G$. Thus, for each $f \in \mathbf{C}[G]$, there are only finitely many values $\underline{\chi}(f) = \Sigma n_g \chi(g)$, where χ ranges over all of $\hat{G}$. $\square$

The finiteness of the set S_f allows us to associate with each $f \in \mathbf{C}[G]$, where G is torsion, a natural polynomial $L_f \in \mathbf{C}[t]$, namely:

$$L_f(t) = \prod_{\underline{\chi}(f) \in S_f} (t - \underline{\chi}(f)).$$

THEOREM 1.3. *Let G be an abelian torsion group and $f \in \mathbf{C}[G]$. Then f is annihilated by L_f; that is, $L_f(f) = 0$ in $\mathbf{C}[G]$.*

PROOF. In view of Lemma 1.1, it suffices to show that $L_f(f)$ is in ker $\underline{\chi}$ for any $\chi \in \hat{G}$. Consider any $\chi_0 \in \hat{G}$; then

$$\begin{aligned}\underline{\chi}_0(L_f(f)) &= \underline{\chi}_0\left(\prod_{\underline{\chi}(f)\in S_f}(f-\underline{\chi}(f))\right)\\ &= \prod_{\underline{\chi}(f)\in S_f}(\underline{\chi}_0(f)-\underline{\chi}(f)) = 0\end{aligned}$$

in $\mathbf{C}[G]$, since $\underline{\chi}_0$ is a $\mathbf{C}$-algebra homomorphism and $\underline{\chi}_0(f) \in S_f$. This being true for any $\chi_0 \in \hat{G}$, we conclude that $L_f(f) = 0$. □

2. **Application to Witt rings.** Let F be any field of characteristic $\neq 2$; any n-dimensional regular quadratic form q over F can be written as $q = \langle g_1, g_2, \ldots, g_n\rangle$ with $g_i \in F^*/F^{*2}$. We identify q with its Witt class in $W(F)$, the Witt ring of F.

The square class group F^*/F^{*2} is a 2-torsion group, possibly trivial. There is a natural surjective ring homomorphism h from the integral group ring $\mathbf{Z}[F^*/F^{*2}]$ onto the Witt ring $W(F)$,

$$h: \mathbf{Z}[F^*/F^{*2}] \longrightarrow W(F)$$

induced by $g \longrightarrow \langle g\rangle$ for $g \in F^*/F^{*2}$.

DEFINITION 2.1. *Let G be an abelian group. An element $f = \Sigma n_g g \in \mathbf{C}[G]$ is called an n-dimensional form over G if $n_g \in \mathbf{Z}$, $n_g \geqq 0$, $\Sigma n_g = n$; that is, if $f \in \mathbf{Z}[G]$ with non-negative coefficients and augmentation n.*

Clearly, every Witt class of an n-dimensional regular quadratic form q over F is the image, under the ring homomorphism $h: \mathbf{Z}[G] \longrightarrow W(F)$, of an n-dimensional form f over $G = F^*/F^{*2}$. By Theorem 1.3, we know how to annihilate those $f \in \mathbf{Z}[G] \subseteq \mathbf{C}[G]$.

OBSERVATION 2.2. *Let G be a 2-torsion group, $f \in \mathbf{Z}[G]$ an n-dimensional form over G. Then the finite set $S_f = \{\underline{\chi}(f): \chi \in \hat{G}\}$ is a subset of $\{n, n-2, \ldots, -n+2, -n\}$.*

PROOF. The group G being 2-torsion is automatically abelian; for each $g \in G$ and each $\chi \in \hat{G}$ we have $\chi(g) = \pm 1$. Hence, if $f = \Sigma n_g g$ is an n-dimensional form over G, then $\underline{\chi}(f) = \Sigma \pm n_g \in \mathbf{Z}$, $|\underline{\chi}(f)| \leqq \Sigma n_g = n$ and $\underline{\chi}(f) \equiv n \bmod 2$. Hence $S_f \subseteq \{n, n-2, \ldots, -n+2, -n\}$. □

Thus for any n-dimensional form f over any 2-torsion group G, the polynomial $L_f(t) = \prod_{\underline{\chi}(f)\in S_f}(t - \underline{\chi}(f))$ is in $\mathbf{Z}[t]$ and divides the polynomial $L_n \in \mathbf{Z}[t]$ given by

$$L_n(t) = (t-n)(t-n+2)\cdot \ldots \cdot (t+n-2)(t+n).$$

By Theorem 1.3 we conclude

COROLLARY 2.3. *Let G be a 2-torsion group and $f \in \mathbf{Z}[G]$ an n-dimensional form over G. Then f is annihilated by L_n; that is, $L_n(f) = 0$ in $\mathbf{Z}[G]$.*

In particular, this implies the Lewis result stated in the introduction:

COROLLARY 2.4. *Let F be a field of characteristic $\neq 2$ and q a regular n-dimensional quadratic form over F. Then q is annihilated by L_n; that is, $L_n(q) = 0$ in $W(F)$.*

PROOF. Use Corollary 2.3 for $G = F^*/F^{*2}$ and apply the surjective ring homomorphism $h: \mathbf{Z}[G] \rightarrow W(F)$. $\square$

The last corollary has also been proved directly by A. Wadsworth and W. Scharlau, using induction on the dimension n, [5].

3. **Positive forms.** Let G be a 2-torsion group. For special classes of n-dimensional forms f over G the annihilating polynomial $L_n \in \mathbf{Z}[t]$ in 2.3 can be replaced by a polynomial of smaller degree. Here is an example.

DEFINITION 3.1. *Let G be a 2-torsion group, $f \in \mathbf{Z}[G]$ an n-dimensional form over G. Then f is called positive if $\chi(f) \geqq 0$ for all $\chi \in \hat{G}$; that is, if $S_f \subseteq \{n, n-2, \ldots, 3, 1\}$ for n odd and $S_f \subseteq \{n, n-2, \ldots, 2, 0\}$ for n even.*

Accordingly, we consider the polynomial $P_n \in \mathbf{Z}[t]$ given by

$$\begin{aligned} P_n(t) &= (t-n)(t-n+2)\cdot\ldots\cdot(t-3)(t-1) && \text{if } n \text{ is odd,} \\ P_n(t) &= (t-n)(t-n+2)\cdot\ldots\cdot(t-2)t && \text{if } n \text{ is even.} \end{aligned}$$

Clearly, P_n is a divisor of L_n in $\mathbf{Z}[t]$.

PROPOSITION 3.2. *Let G be a 2-torsion group, $f \in \mathbf{Z}[G]$ a positive n-dimensional form over G. Then f is annihilated by P_n; that is, $P_n(f) = 0$ in $\mathbf{Z}[G]$.*

PROOF. This is clear from Theorem 1.3, since L_f divides P_n for each positive n-dimensional form f over G. $\square$

Over any 2-torsion group G, every sum of forms $\prod(1+g_i) \in \mathbf{Z}[G]$ with $g_i \in G$ is obviously a positive form over G. In particular, we notice the formal consequence:

COROLLARY 3.3. *Let F be a field of characteristic $\neq 2$ and q an n-dimensional sum of Pfister forms over F. Then q is annihilated by P_n; that is, $P_n(q) = 0$ in $W(F)$.*

How does this relate to the result on trace forms stated in the introduction? Over any F, every sum of Pfister forms over F is Witt equivalent to a sum of trace forms over F, as one sees by using towers of relative quadratic extensions over F. Conversely, every sum of trace forms over F is known to be Witt equivalent to a sum of Pfister forms over F, for example, for all fields F of stability index $\leqq 2$; those fields include all algebraic number fields and function fields in one variable over algebraic number fields as well as all function fields in at most two variables over real closed fields, [1].

Whereas Corollary 3.3 is known to be true, over any F, for all n-dimensional sums of trace forms over F, by no means it will generalize to all n-dimensional positive forms over F; that is, n-dimensional quadratic forms over F all of whose signatures are non-negative; just check $n = 1$.

References

1. L. Bröcker, *Zur Theorie der quadratischen Formen über formalreellen Körpern*, Math. Ann. **210** (1974), 233–256.

2. P. E. Conner, *A proof of the conjecture concerning algebraic Witt classes*, Preprint, 1987.

3. D. W. Lewis, *Witt rings as integral rings*, Invent. Math. **90**, 631–633, 1987.

4. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math. **12**, Wiley, New York-London, 1962.

5. W. Scharlau, Private Communication, 1987.

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