APPLICATION OF A METHOD OF SZEMEREDI

by H. HALBERSTAM

To Robert Rankin on his 70th Birthday

1. Let $\mathcal{B} = \{b_i : b_1 < b_2 < \ldots \}$ be an infinite sequence of positive integers that exceed 1 and are pairwise coprime, so that

$$ (b_i, b_j) = 1, \quad i \neq j. $$

Assume also that

$$ \sum_{i=1}^{\infty} \frac{1}{b_i} < \infty. \quad (1.2) $$

Let $\mathcal{A} = \mathcal{A}_\mathcal{B}$ denote the sequence of $\mathcal{B}$-free numbers, that is, of positive integers divisible by no element of $\mathcal{B}$. This concept, generalizing square-free and $k$-free numbers, derives from Erdős [2] who proved in 1966 that there exists a constant $c, 0 < c < 1$, independent of $\mathcal{B}$, such that the interval $(x, x + x^c)$ contains elements of $\mathcal{A}$ provided only that $x$ is large enough. This result of Erdős was shown by Szemeredi [7] in 1973 to hold with $c = \frac{1}{2} + \varepsilon$, if $x \geq x_0(\varepsilon, \mathcal{B})$, and quite recently Bantle and Grupp [1] have sharpened Szemeredi's result to $c = 9/20 + \varepsilon$.

The purpose of this note is to show how the method of Szemeredi can be used to derive, virtually without change, the following result.

**Theorem.** Let $k$ be a positive integer and let $h$ satisfy $1 \leq h < k, (h, k) = 1$. Given $\delta > 0$, there exists a $\mathcal{B}$-free number $a$ such that

$$ a \equiv h \mod k, \quad a \leq k^{2+\delta} $$

provided only that $k \geq k_0(\varepsilon, \mathcal{B})$.

Define, as (1.2) permits us to do,

$$ \beta = \prod_{i=1}^{\infty} \left(1 - \frac{1}{b_i}\right), \quad (1.3) $$

and denote by $s$ the least positive integer so that

$$ \sum_{i=s+1}^{\infty} \frac{1}{b_i} < \frac{1}{100} \varepsilon \beta. \quad (1.4) $$

It is easy to see that, without any loss of generality, one may assume $b_1, \ldots, b_s$ to be prime. We shall use the letters $p$ and $q$, with or without suffices, exclusively to denote primes.

We shall prove the theorem by showing that, actually, there exist at least $(1/20)\varepsilon \beta k^{1+\varepsilon}$ $\mathcal{B}$-free numbers $a \leq k^{2+\varepsilon}$ in the arithmetic progression $h \mod k$, provided that $k$ is large enough.

2. Proof of the theorem. The natural approach would be to develop an argument to show that the set \( \{ n : 1 \leq n \leq k^{2+\varepsilon}, n \equiv h \mod k \} \) contains elements of \( A \) if \( k \) is large enough. Instead, following Szemerédi, we narrow attention at the outset to a subset of these integers each having a large prime factor. It turns out that this surrender of advantage is more than compensated by the increased difficulty of having such integers divisible by large elements of \( B \).

Accordingly, let
\[
P = \{ p : 2k^{1+\varepsilon/2} < p < k^{1+\varepsilon}, \ p \not\in B, \ p \nmid k \},
\]
and focus on the set of integers
\[
\mathcal{C} = \{ n : 1 \leq n \leq k^{2+\varepsilon}, n \equiv h \mod k, \ n \text{ divisible by a prime of } P \}.
\]
If an integer \( n \) in \( \mathcal{C} \) were to have two prime divisors from \( P \) we should have \( k^{2+\varepsilon} > n > (2k^{1+\varepsilon/2})^2 \), a contradiction. Hence \( \mathcal{P} \) induces the partition
\[
\mathcal{C} = \bigcup_{p \in \mathcal{P}} \mathcal{C}^{(p)},
\]
where
\[
\mathcal{C}^{(p)} = \{ n : 1 \leq n \leq k^{2+\varepsilon}, n \equiv h \mod k, n \equiv 0 \mod p \}.
\]
Moreover, if \( \mathcal{C}_1 \) now denotes the number of elements of \( \mathcal{C} \) divisible by none of \( b_1, \ldots, b_s \), then the cardinality \( |\mathcal{C}_1| \) of \( \mathcal{C}_1 \) is given by
\[
|\mathcal{C}_1| = \sum_{p \in \mathcal{P}} |\mathcal{C}_1^{(p)}| = \sum_{p \in \mathcal{P}} \left| \left\{ m : 1 \leq m \leq k^{2+\varepsilon}p^{-1}, m \equiv hp' \mod k, (m, b_1 \ldots b_s) = 1 \right\} \right| \quad (2.4)
\]
where the interpretation of \( \mathcal{C}_1^{(p)} \) is obvious and \( p' = p'(k) \) is the inverse of \( p \) modulo \( k \), i.e. \( pp' \equiv 1 \mod k \).

**Lemma 1.** If \( k \) is sufficiently large,
\[
|\mathcal{C}_1| \geq \frac{\varepsilon}{10} \beta k^{1+\varepsilon}.
\]

**Proof.** By (2.4) and the definition of \( \mathcal{P} \), which guarantees that \( (p, b_1 \ldots b_s) = 1 \) when \( p \in \mathcal{P} \), we have
\[
|\mathcal{C}_1^{(p)}| = \sum_{d \mid b_1, \ldots, b_s} \mu(d) \sum_{\substack{1 \leq m \leq k^{2+\varepsilon}/p \\ m \equiv hp' \mod k \\ m \equiv 0 \mod d}} 1
\]
\[
= \sum_{d \mid p_1, \ldots, p_s} \mu(d) \left( \frac{k^{1+\varepsilon}}{pd} + \theta_{p,d} \right), \quad |\theta_{p,d}| < 1,
\]
\[
\geq \frac{k^{1+\varepsilon}}{p} \sum_{d \mid p_1, \ldots, p_s} \mu(d) \left( \frac{1}{d} - 2^s \right)
\]
\[
= \frac{k^{1+\varepsilon}}{p} \left( 1 - \prod_{i=1}^{s} \left( 1 - \frac{1}{b_i} \right) - 2^s \right)
\]
\[
\geq \frac{k^{1+\varepsilon}}{p} \beta - 2^s.
\]
Hence, by (2.4),

\[ |\mathcal{C}_1| \geq \beta k^{1+\varepsilon} \sum_{p \in \mathcal{P}} \frac{1}{p} - 2^s \pi(k^{1+\varepsilon}) \]

\[ \geq k^{1+\varepsilon} \left\{ \beta \sum_{p \in \mathcal{P}} \frac{1}{p} - \frac{2^{s+1}}{\log k} \right\}. \]

But

\[
\sum_{p \in \mathcal{P}} \frac{1}{p} = \sum_{2k^{1+\varepsilon/2} < p < k^{1+\varepsilon}} \frac{1}{p} - \sum_{p \in \mathcal{P}} \frac{1}{p} - \sum_{p \in \mathcal{P}} \frac{1}{p} \\
\geq \varepsilon + O\left( \frac{1}{\log k} \right) - \sum_{b_i} \frac{1}{b_i} - k^{-1-\varepsilon/2} \omega(k)
\]

if \( k \) is large enough (to ensure that \( k^{1+\varepsilon/2} \geq b_{i+1} \)), where \( \omega(k) \) is the number of distinct prime factors of \( k \); so that, by (1.4),

\[
\sum_{p \in \mathcal{P}} \frac{1}{p} \geq \frac{\varepsilon}{5} - \frac{\varepsilon \beta}{100} + O\left( \frac{1}{\log k} \right)
\]

and

\[ |\mathcal{C}_1| \geq \beta k^{1+\varepsilon} \left\{ \frac{\varepsilon}{5} - \frac{\varepsilon \beta}{100} + O\left( \frac{1}{\log k} \right) \right\} \geq \frac{\varepsilon \beta}{10} k^{1+\varepsilon}
\]

for all sufficiently large \( k \), as required.

It follows from Lemma 1 that if \( \mathcal{C}_0 \) denotes the set of elements of \( \mathcal{C} \) having no divisors from \( \mathcal{B} \), then

\[ |\mathcal{C}_0| \geq |\mathcal{C}_1| - |\mathcal{C}_2| - |\mathcal{C}_3| \geq \frac{\varepsilon \beta}{10} k^{1+\varepsilon} - |\mathcal{C}_2| - |\mathcal{C}_3|,
\]

where \( \mathcal{C}_2 \) is the set of all those integers in \( \mathcal{C} \) that are divisible by an element \( b \) in \( \mathcal{B} \) of 'intermediate' size, i.e. one satisfying

\[ b_{i+1} \leq b \leq k^{1+\varepsilon}, \]

and \( \mathcal{C}_3 \) is the set of all integers belonging to \( \mathcal{C}_1 \) that have a large factor \( b \) from \( \mathcal{B} \), i.e. satisfying \( b > k^{1+\varepsilon} \). We shall prove that \( |\mathcal{C}_2| + |\mathcal{C}_3| \) is relatively small.

**Lemma 2.** We have \( |\mathcal{C}_2| \leq \frac{\varepsilon \beta}{50} k^{1+\varepsilon} \).
Proof. We argue quite crudely. We have that
\[
(b,k) = n^h \mod b
\]
by (1.4), and this completes the proof of the lemma.

Lemma 3. We have \(|\mathcal{C}_3| \leq \frac{1}{2} k^{1+\varepsilon/2}\).

Proof. Suppose \(n\) is counted in \(\mathcal{C}_3\). Then \(n\) is divisible by a (unique) prime \(p\) from \(\mathcal{P}\), and \(n\) is divisible also by an element \(b > k^{1+\varepsilon}\) from \(\mathcal{B}\). This cannot happen if \((b, p) = 1\), for then \(np > k^{1+\varepsilon} \cdot 2k^{1+\varepsilon/2} = 2k^{2+3\varepsilon/2}\), a contradiction. Hence the \(b\) dividing \(n\) is composite and divisible by a \(p > 2k^{1+\varepsilon/2}\). Writing \(b = lp\), we have \(1 < l < \frac{1}{2} k^{1+\varepsilon/2}\); and given such an integer \(l\), there is, by (1.1), at most one \(b \in \mathcal{B}\) divisible by \(l\). Hence there are at most \(\frac{1}{2} k^{1+\varepsilon/2}\) available choices of \(b\). Finally, given such a \(b\), if \(1 \leq n \leq k^{2+\varepsilon}\), \(n \equiv h \mod k\) and \(n \equiv 0 \mod b\) with \(b > k^{1+\varepsilon}\), there is at most one such \(n\). This proves the lemma.

We are now able to complete the proof of the theorem. By (2.5) and Lemmas 2 and 3 we have
\[
|\mathcal{C}_0| \geq \frac{e}{10} \beta k^{1+\varepsilon} - \frac{e}{50} \beta k^{1+\varepsilon} - \frac{1}{2} k^{1+\varepsilon/2} \geq \frac{e}{20} \beta k^{1+\varepsilon}
\]
if \(k\) is large enough. Thus the theorem is proved, in a quantitative form.

3. Some concluding remarks. If we replace condition (1.2) by the more demanding
\[
B(x) := |\{b \in \mathcal{B} : b \leq x\}| \ll x^\theta,
\]
where \(0 < \theta < 1\), we can, with only a little more trouble, replace the exponent \(2+\varepsilon\) in the theorem by \(1+\theta+\varepsilon\). Thus when \(\mathcal{B}\) is the sequence of squares of primes and \(\mathcal{A}_\theta\) is the sequence of squarefree numbers, we obtain the exponent \((3/2) + \varepsilon\) which is very close to the best that was known until the recent work of Heath-Brown [3]. While it is unlikely that one can emulate Heath-Brown’s delicate argument in the more general situation, I do believe that the theorem itself can be improved a little, in the spirit of [1].

The condition (1.1) can be relaxed somewhat in the theorem. For such and other variations of Szemeredi’s theorem see Narlikar and Ramachandra [4].

REFERENCES


University of Illinois at Urbana-Champaign
Department of Mathematics
1409 West Green Street
Urbana
Illinois 61801
U.S.A.