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(0, 2) – INTERPOLATION OF ENTIRE FUNCTIONS

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1. Introduction. Given a triangular matrix A whose n^{th} row consists of the n points

$$(1.1) \qquad 1 \geqq x_{n,1} > x_{n,2} > \ldots > x_{n,n} \geqq -1,$$

Turán et al. ([**12**], [**1**], [**2**], [**3**]) considered the problem of existence, uniqueness, representation, convergence, etc. of polynomials f_{2n-1} of degree $\leqq 2n - 1$ where the values of f_{2n-1} and those of its second derivative are prescribed at the points (1.1), i.e.,

$$(1.2) \qquad \left.\begin{array}{l} f_{2n-1}(x_{n,\nu}) = y_\nu \\ f''_{2n-1}(x_{n,\nu}) = y^*_\nu \end{array}\right\} (\nu = 1, 2, \ldots, n).$$

The choice of the points (1.1) is important. They found the zeros

$$(1.3) \qquad 1 = \xi_{n,1} > \xi_{n,2} > \ldots > \xi_{n,n-1} > \xi_{n,n} = -1$$

of the polynomial

$$\pi_n(x) := (1 - x^2)P'_{n-1}(x),$$

where P_{n-1} is the $(n - 1)^{\text{th}}$ Legendre polynomial with the normalization $P_{n-1}(1) = 1$ to be the most convenient. If

$$x_{n,\nu} = \xi_{n,\nu}, \quad (\nu = 1, 2, \ldots, n)$$

then for even n there is a uniquely determined polynomial f_{2n-1} of degree $\leqq 2n - 1$ satisfying (1.2). This means, of course that in the case

$$y_\nu = y^*_\nu = 0 \quad (\nu = 1, 2, \ldots, n;\ n \text{ even})$$

the only solution of (1.2) is $f(x) \equiv 0$. Always for even n we may write

$$f_{2n-1}(x) = \sum_{\nu=1}^{n} y_\nu r_\nu(x) + \sum_{\nu=1}^{n} y^*_\nu \rho_\nu(x)$$

where the fundamental polynomials $r_\nu(x)$ and $\rho_\nu(x)$ are defined by

$$(1.4) \qquad r_\nu(\xi_{n,j}) = \begin{cases} 1 \text{ for } j = \nu \\ 0 \text{ for } j \neq \nu \end{cases} \text{ and } r''_\nu(\xi_{n,j}) = 0 \text{ for all } j\text{'s}$$

and

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$$(1.5) \quad \rho_\nu(\xi_{n,j}) = 0 \quad \text{for all } j\text{'s} \quad \text{and } \rho_\nu''(\xi_{n,j}) = \begin{cases} 1 & \text{for } j = \nu \\ 0 & \text{for } j \neq \nu, \end{cases}$$

respectively. In particular, if π_{2n-1} is an arbitrary polynomial of degree $\leqq 2n - 1$, then

$$(1.6) \quad \pi_{2n-1}(x) \equiv \sum_{\nu=1}^{n} \pi_{2n-1}(\xi_{n,\nu})r_\nu(x) + \sum_{\nu=1}^{n} \pi_{2n-1}''(\xi_{n,\nu})\rho_\nu(x).$$

Based on this Balázs and Turán [**3**] proved the following

THEOREM A. *Let π_{2n-1} be a polynomial of degree $\leqq 2n - 1$ such that*

$$(1.7) \quad |\pi_{2n-1}(\xi_{n,\nu})| \leqq M_1, \quad |\pi_{2n-1}''(\xi_{n,\nu})| \leq M_2,$$

for $\nu = 1, 2, \dots, n$. Then for $-1 \leqq x \leqq 1$ we have

$$|\pi_{2n-1}(x)| \leqq \pi^6 n M_1 + \frac{\pi^5}{n} M_2.$$

This kind of interpolation has been studied under the heading of (0, 2)-interpolation.

For (0, 2)-interpolation to periodic functions by trigonometric polynomials ([**10**], [**11**]) the equally spaced nodes are the most convenient to work with. It was shown by Kiš [**10**] that given a periodic function f with period 2π there exists for odd n a unique trigonometric polynomial $T_n(f; x)$ of the form

$$(1.8) \quad a_0 + \sum_{j=1}^{n-1} (a_j \cos jx + b_j \sin jx) + a_n \cos nx,$$

which interpolates the function f in the points

$$\frac{2k\pi}{n}, \quad (k = 0, 1, \dots, n - 1)$$

and whose second derivative assumes prescribed values $\beta_{n,k}$ at these points. Explicit formulae for the fundamental trigonometric polynomials of (0, 2)-interpolation have been worked out in [**11**, Theorem 1] and a representation analogous to (1.6) holds for trigonometric polynomials of the form (1.8) where n is odd.

Neither polynomials nor trigonometric polynomials are suitable for interpolating a function

$$f:\mathbf{R} \to \mathbf{R},$$

in an infinite set of points x_n, $n = 0, \pm 1, \pm 2, \dots$ such that

$$\lim_{n\to\pm\infty} x_n = \pm\infty.$$

However, entire functions of exponential type can be and indeed have been used for this purpose ([**5**], [**6**], [**7**], [**8**], [**9**]). In the case of (0, 2)-interpolation only the equidistant nodes have been used as interpolation points. For sake of simplicity but without any loss of generality we will take the equidistant points to be

(1.9) $\quad 0, \pm 1, \pm 2, \ldots .$

The entire functions

$$(1.10)\quad A_\nu(z):=\begin{cases} \dfrac{\sin(\pi z)}{\pi z}+\dfrac{\sin(\pi z)}{\pi}\displaystyle\int_0^z \frac{1}{\zeta^2}\left(1-\frac{\sin(\pi\zeta)}{\pi\zeta}\right)d\zeta & \text{if } \nu=0 \\ (-1)^\nu \dfrac{z}{\pi\nu}\dfrac{\sin(\pi z)}{z-\nu} \\ \quad +(-1)^\nu\dfrac{\sin(\pi z)}{\pi}\displaystyle\int_{-\nu}^{-\nu+z}\frac{1}{\zeta^2}\left(1-\frac{\sin(\pi\zeta)}{\pi\zeta}\right)d\zeta \\ \quad -\dfrac{\sin(\pi z)}{(\pi\nu)^3}(1-\cos(\pi z)) \text{ if } \nu\neq 0 \end{cases}$$

and

$$(1.11)\quad B_\nu(z):=\begin{cases} \dfrac{\sin(\pi z)}{2\pi}\displaystyle\int_0^z \frac{\sin(\pi\zeta)}{\pi\zeta}d\zeta \text{ if } \nu=0 \\ (-1)^\nu\dfrac{\sin(\pi z)}{2\pi^2}\displaystyle\int_{-\nu}^{-\nu+z}\left(\frac{1}{\nu}+\frac{1}{\zeta}\right)\sin(\pi\zeta)d\zeta \text{ if } \nu\neq 0 \end{cases}$$

which have been determined so as to have the properties

(i) A_ν, B_ν are of exponential type 2π and are bounded on the real axis,

(ii) A_ν, B''_ν assume the value 1 at the point ν but vanish at the other points of (1.9),

(iii) A''_ν, B_ν vanish at all the points of (1.9),

(iv) $A'_\nu(0) = A'''_\nu(0) = B'_\nu(0) = B'''_\nu(0) = 0$,

are called the fundamental functions of (0, 2)-interpolation by entire functions of exponential type. They are unique.

If $f:\mathbf{R}\to\mathbf{R}$ is twice differentiable with

$$(1.12)\quad \sum_{\nu=-\infty}^{\infty} |f(\nu)| < \infty$$

and

$$(1.13) \quad \sum_{\nu=-\infty}^{\infty} |f''(\nu)| < \infty,$$

then

$$(1.14) \quad R(f; z) := \sum_{\nu=-\infty}^{\infty} (f(\nu)A_\nu(z) + f''(\nu)B_\nu(z))$$

is an entire function of exponential type 2π such that

$$\left.\begin{array}{l} R(f; \nu) = f(\nu) \\ R''(f; \nu) = f''(\nu) \end{array}\right\} \quad (\nu = 0, \pm 1, \pm 2, \dots).$$

But even if f happens to be an entire function of exponential type 2π it is by no means clear that analogously to (1.6) a representation formula of the form

$$(1.15) \quad f(z) = \sum_{\nu=-\infty}^{\infty} (f(\nu)A_\nu(z) + f''(\nu)B_\nu(z))$$

does indeed hold. In fact, it does not hold except under certain additional hypotheses. A result like Theorem A for entire functions of exponential type is therefore not a matter of imitating the argument in [**3**]. Much less, we cannot even claim that if f is an entire function of exponential type 2π such that

$$(1.16) \quad \sup_{\nu \in \mathbf{Z}} |f(\nu)| < \infty$$

and

$$(1.17) \quad \sup_{\nu \in \mathbf{Z}} |f''(\nu)| < \infty$$

then

$$(1.18) \quad \sup_{x \in \mathbf{R}} |f(x)| < \infty.$$

The purpose of the present investigation is to look for appropriate conditions under which (1.15) and (1.18) would hold.

The following uniqueness theorem constitutes a major step in this direction. In Section 2 we present some lemmas needed to establish this result which we shall refer to as "the uniqueness theorem" or Theorem 1. Subsequently we return to the original problem.

THEOREM 1. *If f is an entire function of exponential type 2π such that*

(i) $f(\nu) = f''(\nu) = 0, \quad \nu = 0, \pm 1, \pm 2, \dots$

and

(ii) $f(x) = o(|x|)$ as $x \to \pm\infty$,

then

$$f(z) = C_1 \sin(\pi z) + C_2 \sin(2\pi z),$$

where C_1 and C_2 are constants.

2. Some lemmas. We shall need a number of auxiliary results but the most important of them all is

THE PRINCIPAL LEMMA. *Let F be holomorphic and of exponential type in the sector*

$$S(\alpha): = \{z \in \mathbf{C}:|\arg z| \leqq \alpha\} \cup \{0\}, \quad \alpha \in (0, \pi/2]$$

such that

$$h_F(\pm\alpha) < 2\pi \sin \alpha$$

where

$$h_F(\theta): = \limsup_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r}, \quad |\theta| \leqq \alpha$$

denotes the Phragmén-Lindelöf indicator function. If G is an entire function of exponential type 2π such that

$$G(\nu) = F(\nu), G''(\nu) = F''(\nu) \quad (\nu = 0, 1, 2, \ldots),$$

then

$$F(z) - G(z) = \left(a + \int_0^z \psi(t)\sin(\pi t)dt\right)\sin(\pi z), \tag{2.1}$$

where a is a constant and ψ is holomorphic and of exponential type in $S(\alpha)$. Furthermore,

(i) *if $|G(x)| = O(1)$ as $x \to \pm\infty$, then*

$$|\psi(re^{i\theta})| = O(1) \quad \textit{as } r \to +\infty \tag{2.2}$$

uniformly in θ on every compact subset of $(-\alpha, \alpha)$;

(ii) *if $G(x) = o(|x|)$ as $x \to \pm\infty$, then*

$$\psi(re^{i\theta}) = o(r) \quad \textit{as } x \to +\infty \tag{2.3}$$

uniformly in θ on every compact subset of $(-\alpha, \alpha)$.

The next five lemmas will help the proof of the Principal Lemma to flow smoothly. Except possibly for the fact that the exponential type of the function ϕ in Lemma 1 is τ they are "essentially" known.

LEMMA 1. *Let f be holomorphic and of exponential type τ in $S(\alpha)$ such that $f(n) = 0$ for $n = 0, 1, 2, \ldots$. Then*

$$f(z) = \phi(z)\sin(\pi z),$$

where ϕ is holomorphic and of exponential type τ in $S(\alpha)$.

Proof. The function

$$\phi(z) := f(z)/\sin(\pi z)$$

is clearly holomorphic in $S(\alpha)$. We only have to show that it is of exponential type τ.

The function f being of exponential type τ, for every given $\epsilon > 0$ there exists a number A such that

$$|f(z)| \leqq Ae^{(\tau+\epsilon)|z|} \quad \text{for all } z \in S(\alpha).$$

Now let us denote by y_0 the only positive root of the equation

$$\sinh(\pi y) = 1.$$

Choose an integer n_0 in $\left[\frac{1}{2} + \frac{y_0}{\sin\alpha}, \infty\right)$ and consider the subsets

$$D_1 := \{z \in S(\alpha) : |\text{Im } z| > y_0\}$$

and

$$D_2 := \left\{z \in S(\alpha) : \text{Re } z > n_0 - \frac{1}{2}, |\text{Im } z| \leqq y_0\right\}$$

of $S(\alpha)$. For $x, y \in \mathbf{R}$

(2.4) $$|\sin(x + iy)| \geqq \max\{|\sin x|, |\sinh y|\}$$

and so

(2.5) $$|\phi(z)| \leqq Ae^{(\tau+\epsilon)|z|} \quad \text{for all } z \in D_1.$$

In order to estimate $|\phi(z)|$ at an arbitrary point $z \in D_2$ we choose $n \in \mathbf{N}$ such that

$$n - \frac{1}{2} < \text{Re } z \leqq n + \frac{1}{2}.$$

Then z belongs to the closed rectangle R_n with corners at the points $n \pm \frac{1}{2} \pm iy_0$. Using the maximum modulus principle in conjunction with (2.4) we obtain

(2.6) $$|\phi(z)| \leqq \max_{\zeta \in R_n} |\phi(\zeta)| \leqq \max_{\zeta \in R_n} |f(\zeta)|$$
$$\leqq \max_{\zeta \in R_n} Ae^{(\tau+\epsilon)|\zeta|} \leqq Ae^{(\tau+\epsilon)|1+|z|+iy_0|}.$$

Since $S(\alpha)\backslash(D_1 \cup D_2)$ is compact the desired result follows from (2.5) and (2.6).

LEMMA 2. *Let F be holomorphic and of exponential type in $S(\alpha)$. If $h_F(\theta) \not\equiv -\infty$, then*

$$h_{F'}(\theta) \leqq h_F(\theta) \quad \textit{for all } |\theta| < \alpha.$$

Using the continuity of the indicator function [**4**, Theorem 5.1.4.] the result is easily deduced from Cauchy's integral formula for $F'(re^{i\theta})$.

LEMMA 3. *Let G be holomorphic and of exponential type τ in the closed upper half-plane.*

(i) *If $|G(x)| = O(1)$ as $x \to \pm\infty$, then*

$$(2.7) \qquad |G(re^{i\theta})| = O(e^{\tau r|\sin\theta|}) \quad \textit{as } r \to \infty$$

uniformly in θ for $\theta \in [0, \pi]$.

(ii) *If $G(x) = o(|x|)$ as $x \to \pm\infty$, then*

$$(2.8) \qquad |G(re^{i\theta})| = o(re^{\tau r|\sin\theta|}) \quad \textit{as } r \to \infty$$

uniformly in θ for $\theta \in [0, \pi]$.

Proof. For (i) see [**4**, Theorem 6.2.4.]. For (ii) apply [**4**, Theorem 6.2.8] to the function

$$f{:}z \to \frac{G(z)}{z + i}.$$

This way we obtain the desired asymptotic growth on compact subsets of $[0, \pi/2)$ and analogously of $(\pi/2, \pi]$. The proof may then be completed by considering the function

$$g{:}z \to \frac{G(z)}{z + i} e^{i\tau z}$$

and using [**4**, Theorems 1.4.2 and 1.4.4].

LEMMA 4. *Let G be an entire function of exponential type.*

(i) *If $|G(x)| = O(1)$ as $x \to \pm\infty$, then*

$$|G'(x)| = O(1) \quad \textit{as } x \to \pm\infty.$$

(ii) *If $G(x) = o(|x|)$ as $x \to \pm\infty$, then*

$$G'(x) = o(|x|) \quad \textit{as } x \to \pm\infty.$$

Proof. Statement (i) is a crude version of the Bernstein's inequality [**4**, Theorem 11.1.2] for entire functions of exponential type. As regards (ii) the conclusion for $x \to +\infty$ may be obtained by applying [**4**, Theorem 11.3.4*] to the function

$$f{:}z \to \frac{G(z)}{z+1}.$$

The case $x \to -\infty$ may be handled by considering the function $G(-z)$.

LEMMA 5. *Let ψ be holomorphic and of exponential type in $S(\alpha)$ and let $\gamma \in (0, \alpha)$.*

(i) *If $|\psi(re^{\pm i\gamma})| = O(1)$ as $r \to \infty$, then $|\psi|$ is bounded in $S(\gamma)$.*

(ii) *If $\psi(re^{\pm i\gamma}) = o(r)$ as $r \to \infty$, then $\psi(re^{i\theta}) = o(r)$ as $r \to \infty$ uniformly for $\theta \in [-\gamma, \gamma]$.*

Proof. Statement (i) is a consequence of the Phragmén-Lindelöf principle (see [**4**, Theorem 1.4.3]). For (ii) we may apply [**4**, Theorem 1.4.4] to the function

$$f{:}z \to \frac{G(z)}{z+1}.$$

Now we are in a position to present the

Proof of the Principal Lemma. Put

$$\phi(z){:} = F(z) - G(z). \tag{2.9}$$

Then ϕ is holomorphic and of exponential type in $S(\alpha)$ such that

$$\phi(\nu) = 0,\ \phi''(\nu) = 0 \quad (\nu = 0, 1, 2, \dots).$$

Applying Lemma 1 to ϕ and then to ϕ'' we see that

$$\phi(z) = \varphi(z)\sin(\pi z) \tag{2.10}$$

and in turn

$$\varphi'(z) = \psi(z)\sin(\pi z) \tag{2.11}$$

where φ and ψ are holomorphic and of exponential type in $S(\alpha)$. This readily gives us the representation

$$\phi(z) = \left(\varphi(0) + \int_0^z \psi(t)\sin(\pi t)dt\right)\sin(\pi z),$$

which proves (2.1).

Using (2.9)-(2.11) we may also write ψ in the form

$$\psi(z) = \psi_1(z) - \psi_2(z), \tag{2.12}$$

where

$$\psi_1(z) = \frac{F'(z) - \pi\cos(\pi z)F(z)/\sin(\pi z)}{\sin^2(\pi z)} \tag{2.13}$$

and

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$$(2.14)\quad \psi_2(z) = \frac{G'(z) - \pi\cos(\pi z)G(z)/\sin(\pi z)}{\sin^2(\pi z)}.$$

Now let us choose β arbitrarily in $(0, \alpha)$. We shall show that (2.2) and (2.3) hold uniformly for $\theta \in [-\beta, \beta]$ according as "$|G(x)| = O(1)$ as $x \to \pm\infty$" or "$G(x) = o(|x|)$ as $x \to \pm\infty$", respectively. Using the well-known continuity properties of the indicator function and then Lemma 2 we can find a $\gamma \in [\beta, \alpha)$ such that

$$h_F(\pm\gamma) < 2\pi \sin\gamma \quad \text{and} \quad h_{F'}(\pm\gamma) < 2\pi\sin\gamma.$$

Hence, in view of (2.13)

$$\lim_{r\to\infty} \psi_1(re^{\pm i\gamma}) = 0,$$

from which it follows (see [**4**, Theorem 1.4.4]) that

$$(2.15)\quad \psi_1(re^{i\theta}) = o(1) \quad \text{as } r \to \infty$$

uniformly for $\theta \in [-\gamma, \gamma]$.

(i) If $|G(x)| = O(1)$ as $x \to \pm\infty$, then from (2.14) in conjunction with the first parts of Lemmas 3-5 it follows that $|\psi_2|$ is bounded in $S(\gamma)$. This together with (2.15) shows that (2.2) holds uniformly for $\theta \in [-\gamma, \gamma]$ and so in particular for $\theta \in [-\beta, \beta]$.

(ii) If $G(x) = o(|x|)$ as $x \to \pm\infty$ then we may use the second parts of Lemmas 3-5 and obtain the desired conclusion in an analogous way.

For the proof of Theorem 1 we shall need two additional lemmas.

LEMMA 6 [**9**, p. 187]. *For $x \in \mathbf{R}$ let n_x be the larger of the possibly two integers closest to x and denote by $N(x)$ the set of all integers between* 0 *and n_x (including both* 0 *and n_x). Then there exist constants $c_1, c_2, \ldots, c_6$ such that for every $z = x + iy (x, y \in \mathbf{R})$*

$$|A_n(z)| \leqq \begin{cases} c_1\dfrac{|z|e^{\pi|y|}}{|n(n-n_x)|} + c_2\left(\dfrac{1}{|n-n_x|^2} + \dfrac{1}{|n|^3}\right)e^{2\pi|y|} & \text{if } n \notin N(x), \\[2ex] c_3\dfrac{(1+|z|)e^{\pi|y|}}{1+|n(n-n_x)|} + c_4 e^{2\pi|y|} & \text{if } n \in N(x), \end{cases}$$

$$|B_n(z)| \leqq \begin{cases} c_5\dfrac{|z|e^{2\pi|y|}}{|n(n-n_x)|} & \text{if } n \notin N(x), \\[2ex] c_6 e^{2\pi|y|} & \text{if } n \in N(x). \end{cases}$$

These estimates for $|A_n(z)|$ and $|B_n(z)|$ readily imply:

LEMMA 7. *Let $(a_n)_{n\in\mathbf{Z}}$ and $(b_n)_{n\in\mathbf{Z}}$ be sequences of complex numbers such that*

$$(2.16) \quad \sum_{\nu=-\infty}^{\infty} |a_\nu| < \infty$$

and

$$(2.17) \quad \sum_{\nu=-\infty}^{\infty} |b_\nu| < \infty$$

then the series

$$(2.18) \quad H(z) := \sum_{\nu=-\infty}^{\infty} (a_\nu A_\nu(z) + b_\nu B_\nu(z))$$

converges absolutely and uniformly on every compact subset of **C** *and represents an entire function of exponential type* 2π *such that* $|H(x)|$ *is bounded on* **R**.

3. Proof of the uniqueness theorem and some applications.

Proof of Theorem 1. Let $\alpha := \pi/2$, $F \equiv 0$ and $G(z) := f(z)$. By the Principal Lemma the representation (2.1) holds in the closed right half-plane where by statement (ii) of the same lemma ψ is holomorphic and of exponential type such that for every $\beta \in (0, \pi/2)$

$$\psi(re^{i\theta}) = o(r) \quad \text{as } r \to \infty$$

uniformly on $[-\beta, \beta]$.

Clearly, the same observations hold also for the left half-plane, where the function ψ may a priori be different; call it $\tilde{\psi}$. Since the two representations must coincide on the imaginary axis so must ψ and $\tilde{\psi}$. It follows that ψ and $\tilde{\psi}$ are restrictions of the same entire function, which we will also denote by ψ, to the right and the left half-planes respectively. Applying Lemma 5 (ii) four times appropriately we deduce that

$$\psi(re^{i\theta}) = o(r) \quad \text{as } r \to \infty$$

uniformly for $\theta \in [0, 2\pi]$. According to a trivial generalization of Liouville's theorem the function ψ must be a constant. With this Theorem 1 is proved.

As a consequence of Theorem 1 we prove

THEOREM 2. *Let f be an entire function of exponential type 2π satisfying* (1.12) *and* (1.13). *If $f(x) = o(|x|)$ as $x \to \pm\infty$ then the series* (1.14) *converges absolutely and uniformly on every compact subset of* **C** *and*

$$(3.1) \quad f(z) = R(f; z) + C_1 \sin(\pi z) + C_2 \sin(2\pi z),$$

where

$$(3.2)\qquad \begin{aligned} C_1 &= \frac{1}{3}\left(\frac{4}{\pi}f'(0) + \frac{1}{\pi^3}f'''(0)\right), \\ C_2 &= -\frac{1}{6}\left(\frac{1}{\pi}f'(0) + \frac{1}{\pi^3}f'''(0)\right). \end{aligned}$$

Proof. By Lemma 7, $R(f;\ z)$ is an entire function of exponential type 2π such that $|R(f;\ x)|$ is bounded on the real axis. Thus the function $f(z) - R(f;\ z)$ satisfies the conditions of Theorem 1 and hence the representation (3.1) holds. Since for all ν

$$A'_\nu(0) = B'_\nu(0) = A'''_\nu(0) = B'''_\nu(0) = 0$$

we must have

$$\pi C_1 + 2\pi C_2 = f'(0),$$

$$-\pi^3 C_1 - (2\pi)^3 C_2 = f'''(0),$$

from which the desired values of C_1 and C_2 are deduced.

Theorem 2 readily implies:

COROLLARY 1. *Let f be an entire function of exponential type 2π satisfying* (1.12) *and* (1.13). *Then* (1.18) *holds provided $f(x) = o(|x|)$ as $x \to \pm\infty$.*

4. Alternative sets of conditions. From the proof of Theorem 2 it is clear that the representation (3.1) would remain valid if the conditions (1.12) and (1.13) were replaced by alternative conditions which would ensure simply that

$$(4.1)\qquad R(f;\ x) = o(|x|) \quad \text{as } x \to \pm\infty.$$

We are thus led to

LEMMA 7′. *Let $(a_n)_{n\in\mathbf{Z}}$ and $(b_n)_{n\in\mathbf{Z}}$ be sequences of complex numbers such that*

$$(4.2)\qquad \sum_{\nu=-n}^{n} |a_\nu| = o(n)$$

and

$$(4.3)\qquad \sum_{\nu=-n}^{n} |b_\nu| = o(n)$$

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as $n \to \infty$, *then the series* (2.18) *converges absolutely and uniformly on every compact subset of* $\mathbf{C}$ *and represents an entire function of exponential type* 2π *such that*

$$(4.4) \qquad H(x) = o(|x|) \quad \text{as } x \to \pm\infty.$$

Proof. In what follows $c_7, c_8, \dots$ will denote appropriate constants. Let $\mathscr{E}$ be any compact subset of $\mathbf{C}$ so that there exists an integer k with

$$\mathscr{E} \subset \{z \in \mathbf{C}: |z| \leqq k\}.$$

In view of Lemma 6 we have for all $z \in \mathscr{E}$

$$\left.\begin{array}{l} |A_\nu(z)| \leqq c_7, \\ |B_\nu(z)| \leqq c_7 \end{array}\right\} \quad \text{if } |\nu| \leqq 2k$$

and

$$\left.\begin{array}{l} |A_\nu(z)| \leqq \dfrac{c_8}{|\nu|(|\nu| - k)}, \\ |B_\nu(z)| \leqq \dfrac{c_8}{|\nu|(|\nu| - k)}, \end{array}\right\} \quad \text{if } |\nu| \geqq 2k.$$

In order to establish the absolute and uniform convergence of (2.18) under the conditions (4.2) and (4.3) it is therefore enough to show that the series

$$\sum_{|\nu| \geqq 2k} \frac{|a_\nu| + |b_\nu|}{|n|(|\nu| - k)}$$

converges. For this we may restrict ourselves to

$$S: = \sum_{\nu=2k}^{\infty} \frac{|a_\nu| + |b_\nu|}{\nu(\nu - k)}.$$

Clearly

$$S = \sum_{\nu=k}^{\infty} \frac{|a_{\nu+k}| + |b_{\nu+k}|}{(\nu + k)\nu} \leqq \sum_{\nu=k}^{\infty} \frac{|a_{\nu+k}| + |b_{\nu+k}|}{\nu^2}$$

where Abel's summation readily shows that the latter series converges. Hence (2.18) represents an entire function which must be of exponential type 2π as is clear from Lemma 6.

Now let us verify (4.4). Without loss of generality we may assume x to be positive. Then, using the fact that

$$(4.5) \qquad \left.\begin{array}{l} |A_\nu(x)| \leqq c_9 \\ |B_\nu(x)| \leqq c_9 \end{array}\right\} \quad \text{for all } x \in \mathbf{R} \text{ and } \nu \in \mathbf{Z}$$

we obtain

$$\sum_{\nu=0}^{n_x} |a_\nu A_\nu(x) + b_\nu B_\nu(x)| = o(n_x) \quad \text{as } x \to +\infty.$$

It remains to estimate

$$S_1 := \sum_{\nu \notin N(x)} \frac{|a_\nu|}{(\nu - n_x)^2}$$

and

$$S_2 := \sum_{\nu \notin N(x)} \frac{x}{|\nu(\nu - n_x)|}(|a_\nu| + |b_\nu|).$$

For this we split each of the sums S_1 and S_2 into four sums each, say $S_{1,j}$, $S_{2,j}$, ($j = 1, 2, 3, 4$). The summation index ν varies from

1. $-2n_x + 1$ to -1 if $j = 1$,
2. $n_x + 1$ to $2n_x - 1$ if $j = 2$,
3. $-2n_x$ to $-\infty$ if $j = 3$,
4. $2n_x$ to ∞ if $j = 4$.

The sums $S_{1,1}$ and $S_{1,2}$ are obviously of order $o(n_x)$. As regards $S_{1,3}$, $S_{1,4}$ we may use the estimate

$$\frac{|a_\nu|}{(\nu - n_x)^2} \leqq c_{10}\frac{|a_\nu|}{\nu^2}.$$

Then Abel's summation shows that they exist and remain bounded as $x \to +\infty$. In the case of S_2 it is sufficient to estimate $S_{2,2}$ and $S_{2,4}$. But clearly

$$S_{2,2,} \leqq \sum_{\nu=1}^{n_x - 1} \frac{1}{\nu}(|a_{\nu+n_x}| + |b_{\nu+n_x}|) = o(n_x) \quad \text{as } x \to +\infty$$

and

$$S_{2,4} \leqq \sum_{\nu=n_x}^{\infty} \frac{x}{\nu^2}(|a_{\nu+n_x}| + |b_{\nu+n_x}|) = o(x) \quad \text{as } x \to +\infty,$$

where the last conclusion again follows by Abel's summation. With this the proof of (4.4) is complete.

Going through the proof of Theorem 2 we see that in view of Lemma 7′ the following representation theorem also holds.

THEOREM 3. *Let f be an entire function of exponential type 2π such that*

$$(4.6) \qquad \sum_{\nu=-n}^{n} |f(\nu)| = o(n)$$

and

$$(4.7) \qquad \sum_{\nu=-n}^{n} |f''(\nu)| = o(n)$$

as $n \to \infty$. *If* $f(x) = o(|x|)$ *as* $x \to \pm\infty$ *then the conclusions of Theorem* 2 *remain valid.*

In case f is of exponential type $< 2\pi$ the hypothesis "$f(x) = o(|x|)$ as $x \to \pm\infty$" in Theorem 3 is superfluous as is shown by

LEMMA 8. *Let f be holomorphic and of exponential type in $S(\alpha)$. If*

$$(4.8) \qquad h_f(\pm\alpha) < 2\pi \sin \alpha,$$

$$(4.9) \qquad \sum_{\nu=0}^{n} |f(\nu)| = o(n) \quad \text{as } n \to \infty,$$

and

$$(4.10) \qquad \sum_{\nu=0}^{n} |f''(\nu)| = o(n) \quad \text{as } n \to \infty,$$

then

$$(4.11) \qquad f(x) = o(x) \quad \text{as } x \to \infty.$$

Proof. Let us take $F: = f$ and

$$G(z): = \sum_{\nu=0}^{\infty} (f(\nu)A_\nu(z) + f''(\nu)B_\nu(z))$$

in the Principal Lemma. The statement (ii) of that lemma applies since in view of (4.8) and (4.9), Lemma 7′ implies that $G(x) = o(|x|)$ as $x \to \pm\infty$. Hence (2.1) holds where the behaviour of the function ψ is given by (2.3). By Cauchy's integral formula

$$|\psi^{(k)}(x)| = \left| \frac{k!}{2\pi i} \int_{|\zeta - x| = x \sin \alpha/2} \frac{\psi(\zeta)}{(\zeta - x)^{k+1}} d\zeta \right| = o(|x|^{1-k})$$

because of (2.3). This fact easily leads us to the desired conclusion since using integration by parts we obtain

$$\int_0^x \psi(t)\sin(\pi t)dt = \frac{1}{\pi}(\psi(0) - \psi(x)\cos(\pi x))$$

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$$+ \frac{1}{\pi^2}(\psi'(x)\sin(\pi x) - \int_0^x \psi''(t)\sin(\pi t)dt).$$

Theorem 3 combined with Lemma 8 gives:

THEOREM 4. *The conclusions of Theorem* 2 *hold for all entire functions f of exponential type* $< 2\pi$ *satisfying* (4.6) *and* (4.7) (*and so a fortiori for all entire functions of exponential type* $< 2\pi$ *satisfying* (1.12) *and* (1.13)).

In Corollary 1 as well, the condition "$f(x) = o(|x|)$ as $x \to \pm\infty$" is superfluous if f happens to be of exponential type $< 2\pi$. This is because of the following result which can be proved in much the same way as Lemma 8.

THEOREM 5. *Let f be holomorphic and of exponential type in* $S(\alpha)$ *satisfying* (4.8). *If*

$$(4.12) \qquad \sum_{\nu=0}^{\infty} |f(\nu)| < \infty$$

and

$$(4.13) \qquad \sum_{\nu=0}^{\infty} |f''(\nu)| < \infty,$$

then

$$(4.14) \qquad \sup_{x \geqq 0} |f(x)| < \infty.$$

5. Conclusion and remarks. The results which correspond to Theorem A are Theorem 5 and Corollary 1. The analogy is not complete in as much as we do not give explicit upper bounds for $\sup|f(x)|$. Besides, a couple of remarks are in order.

Remark 1. One would like to know whether conditions (4.12) and (4.13) of Theorem 5 can be replaced by

$$(5.1) \qquad \sup_{\nu \geqq 0} |f(\nu)| < \infty$$

and

$$(5.2) \qquad \sup_{\nu \geqq 0} |f''(\nu)| < \infty$$

respectively. The answer is "no". Besides, the conclusion fails if the n-th partial sum of any of the two series in (4.12) and (4.13) is allowed to go to infinity even like $l_k n$ the k-th iterate of log n for any arbitrary k.

In order to construct the necessary examples we define inductively

$$e_1 : = e, \quad e_{k+1} = \exp(e_k) \quad (k = 1, 2, \dots)$$

and set

$$h(z) : = l_{k+1}(e_k + z).$$

Then, the function

$$(5.3) \qquad f : z \to h(z)\sin(\pi z) + \frac{2}{\pi} h'(z)\cos(\pi z),$$

which is holomorphic and of exponential type in the right half-plane satisfies (4.8), (5.1),

$$(5.4) \qquad \sum_{\nu=0}^{n} |f(\nu)| = o(l_k n) \quad \text{as } n \to \infty$$

and (4.13). But obviously (4.14) does not hold. Similarly, the function

$$(5.5) \qquad f : z \to h(z)\sin(\pi z)$$

satisfies (4.8), (4.12), (5.2) and

$$(5.6) \qquad \sum_{\nu=0}^{n} |f''(\nu)| = o(l_k n) \quad \text{as } n \to \infty$$

but again (4.14) does not hold.

Finally, we wish to mention that in Theorem 5, condition (4.8) cannot be replaced by

$$(4.8') \qquad h_f(\pm\alpha) \leqq 2\pi \sin \alpha.$$

This is shown by the example

$$(5.7) \qquad f(z) : = \pi z \sin(2\pi z) + \cos(2\pi z) - 1.$$

Remark 2. The conclusion of Corollary 1 does not hold if either the condition (1.12) is replaced by (1.16) or (1.17) is substituted for (1.13). Besides, any of the partial sums

$$\sum_{\nu=-n}^{n} |f(\nu)| \quad \text{and} \quad \sum_{\nu=-n}^{n} |f''(\nu)|$$

cannot be allowed to go to infinity even like $l_k n$.

In order to construct the necessary examples we choose δ in $(0, \pi)$ and define

$$(5.8)\qquad h(z) := \int_{-\infty}^{\infty}\left(\frac{\sin(\delta(z-t)/4)}{z-t}\right)^4 l_{k+1}\sqrt{e_k^2+t^2}\,dt \div \int_{-\infty}^{\infty}\left(\frac{\sin(\delta t/4)}{t}\right)^4 dt.$$

It is clear that h is an entire function of exponential type δ. Besides, it is easily checked that

$$\sup_{\nu\in\mathbf{Z}} |h'(\nu)| < \infty,$$

$$\sum_{\nu=-n}^{n} |h'(\nu)| = O(l_{k+1}n) \quad \text{as } n \to \infty,$$

$$\sum_{\nu=-\infty}^{\infty} |h'''(\nu)| < \infty,$$

whereas

$$\sup_{x\in\mathbf{R}} |h(x) - l_{k+1}\sqrt{e_k^2 + x^2}| < \infty.$$

Thus, substituting (5.8) in (5.3) we see that in Corollary 1, condition (1.12) can neither be replaced by (1.16) nor by

$$(5.9)\qquad \sum_{\nu=-n}^{n} |f(\nu)| = o(l_k n) \quad \text{as } n \to \infty.$$

Similarly, substituting (5.8) in (5.5) we see that condition (1.13) of Corollary 1 can neither be replaced by (1.17) nor by

$$(5.10)\qquad \sum_{\nu=-n}^{n} |f''(\nu)| = o(l_k n) \quad \text{as } n \to \infty.$$

We also wish to point out that the conclusion of Corollary 1 does not hold if the condition "$f(x) = o(|x|)$ as $x \to \pm\infty$" is replaced by "$f(x) = O(|x|)$ as $x \to \pm\infty$". Example (5.7) shows it.

Finally, we observe that Corollary 1 does not extend to functions of exponential type in a half-plane. Indeed, the function

$$f(z) := \left((\log(z+1))\cos(\pi z) - \int_0^z \frac{1}{t+1}\cos(\pi t)dt\right)\sin(\pi z)$$

is holomorphic and of exponential type 2π in the right half-plane such that $f(x) = O(\log x) = o(x)$ as $x \to \infty$,

$$f(\nu) = f''(\nu) = 0 \quad (\nu = 0, 1, 2, \dots),$$

while (4.14) does not hold.

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