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EXPANSIONS WITH POISSON KERNELS AND RELATED TOPICS

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Abstract Let $P(r,\theta)$ be the two-dimensional Poisson kernel in the unit disc D. It is proved that there exists a special sequence $\{\mathfrak{a}_k\}$ of points of D which is non-tangentially dense for ∂D and such that any function on ∂D can be expanded in series of $P(|\mathfrak{a}_k|,(\cdot)-\arg\mathfrak{a}_k)$ with coefficients depending continuously on f in various classes of functions. The result is used to solve a Cauchy-type problem for $\Delta u=\mu$, where μ is a measure supported on $\{\mathfrak{a}_k\}$.

Keywords: Poisson kernels; interpolation; Cauchy problem

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Secondary 42–99

1. Introduction

Let $P(r,\theta)$ be the two-dimensional Poisson kernel in the disc $D=\{|z|<1\}$:

$$P(r,\theta)=\frac{1}{2}\frac{1-r^2}{1-2r\cos\theta+r^2},\quad 0\leqslant r<1,\ \theta\in\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}.$$

The function $P(r,\cdot)$ is a 2π-periodic oscillatory function and it is natural to ask if superpositions of functions of the form $P(r_\mu,(\cdot)-\theta_\mu)$, for suitable values of r_μ and θ_μ, might approximate functions on $\mathbb{T}$.

This problem and related ones have been studied by Bonsall [**2**–**4**], Bonsall and Walsh [**5**] and Hayman and Lyons [**10**]; it turns out that, if the sequence of points $\mathfrak{b}_\mu=r_\mu e^{\theta_\mu i}$ is non-tangentially dense for ∂D (see § 4 for the definition), then every $f\in L^1(\partial D)$ can be written as

$$\sum_{\mu=1}^{\infty}\lambda_\mu P(r_\mu,(\cdot)-\theta_\mu)\quad\text{with } \{\lambda_\mu\}\in\ell^1.$$

The solution is non-unique and the series converges in $L^1(\partial D)$.

Our approach is somewhat different. We choose, once and for all, points $\mathfrak{a}_\mu$ in the following way.

Let $\sigma \in (0,1)$, suitably chosen; for any $n \in \mathbb{N}$, let us denote by $\zeta^{(1)}_{2n,l}$, $\zeta^{(2)}_{2n,l}$, with $0 \leqslant l \leqslant 2n-1$, the $2n$th roots of σ^2 and $-\sigma^2$, respectively, ordered as follows:

$$\zeta^{(1)}_{2n,l} = \sigma^{1/n} \exp\left\{-\frac{\pi}{n} l \mathrm{i}\right\}, \qquad l = 0, \dots, 2n-1, \tag{1.1}$$

$$\zeta^{(2)}_{2n,l} = \sigma^{1/n} \exp\left\{\left(\frac{\pi}{2n} - \frac{\pi}{n} l\right)\mathrm{i}\right\}, \quad l = 0, \dots, 2n-1. \tag{1.2}$$

Our choice for the points in D is $\mathfrak{a}_0 = 0$, $\mathfrak{a}_\mu = \zeta^{(j)}_{2n,l}$, where $\mu = 1+2(n-1)n+2(j-1)n+l$. It turns out that $\mathcal{N} := \cup\{\mathfrak{a}_\mu : \mu \in \mathbb{N} \cup \{0\}\}$ has no limit points in D and it is non-tangentially dense for ∂D.

Let

$$\mathcal{P}^j_n(\theta) := \frac{1}{2n\sigma} \sum_{h=0}^{n-1} [P(\sigma^{1/n}, \theta - \arg \zeta^{(j)}_{2n,2h}) - P(\sigma^{1/n}, \theta - \arg \zeta^{(j)}_{2n,2h+1})].$$

It will be proved that the functions $\mathcal{P}^j_n$ are uniformly bounded and periodic of period $2\pi/n$. Our main goal is to represent functions f on ∂D as sums of the form

$$a_0 P(0,(\cdot)) + \sum_{n=1}^{\infty} (\alpha_n \mathcal{P}^1_n + \beta_n \mathcal{P}^2_n),$$

so that there is a one-to-one mapping between f and the expansion above in several classes of functions.

Our main result is the following. Let $\mathbb{A}$ be the space of the sums of absolutely convergent Fourier series in $\mathbb{T}$. Then every $f \in \mathbb{A}$ can be written as either

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{1.3}$$

or

$$f(\theta) = a_0 P(0,\theta) + \sum_{n=1}^{\infty} (\alpha_n \mathcal{P}^1_n(\theta) + \beta_n \mathcal{P}^2_n(\theta)), \tag{1.4}$$

$\theta \in \mathbb{T}$. There is a one-to-one continuous mapping in ℓ^1 between $\{\alpha_n\}$ and $\{a_n\}$, $\{\beta_n\}$ and $\{b_n\}$; both (1.3) and (1.4) satisfy the Weierstrass M-test and are absolutely and uniformly convergent.

In other words, every $f \in \mathbb{A}$ can be approximated by suitable linear combination of Poisson kernels, with continuous dependence upon the coefficients.

Sharper results are proved if derivatives of f are in $\mathbb{A}$.

If $1 < p < \infty$, it is proved that there is a one-to-one continuous mapping $I + X$ in $L^p(\mathbb{T})$ with the following property. Let $f \in L^p(\mathbb{T})$,

$$g(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta) = (I + X)f;$$

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one can formally write the expansion (1.4) using the Fourier coefficients α_n and β_n of g; then, the partial sums of the series in the right-hand side of (1.4) tend to f in $L^p(\mathbb{T})$.

The approximation theorems are used to solve the following problems.

Let $f^{(0)}, \mathrm{d}f^{(0)}/\mathrm{d}\theta \in \mathbb{A}$, $f^{(1)} \in \mathbb{A}$. Then, there exists a Radon complex measure μ, supported on $\mathcal{N}$, with the following property. The Cauchy-type problem:

$$\left.\begin{aligned} \Delta u = \mu \quad \text{in } D,\\ u|_{\partial D} = f^{(0)},\\ \partial_{\boldsymbol{n}} u|_{\partial D} = f^{(1)}, \end{aligned}\right\} \tag{1.5}$$

has a (distribution) solution $u \in W^{1,p}(D)$, $1 \leqslant p < 2$; the outer normal derivative $\partial_{\boldsymbol{n}} u$ is defined in a generalized sense. Our solution is different from the classical harmonic solutions, which assume that the boundary data have radial limits in a set of first category (see, for example, [**12**, p. 76] or [**6**, Theorem 8.11]). Problem (1.5) can be solved using the approach in [**5**]; however, the solution is not unique and does not depend continuously upon the data. Our solution, instead, continuously depends upon the data. In [**9**] we use this solution for solving a Cauchy-type problem for homogeneous two-dimensional elliptic equations.

Our final application is an interpolation-type theorem for harmonic functions in D. Notice that the points in $\mathcal{N}$ are not uniformly separated (in a Carleson sense; see, for example, [**11**]). It turns out that (in some sense) the points in $\mathcal{N}$ are too numerous: a uniqueness result can be proved, but complicated compatibility conditions on the function's values need to be assumed, for the existence result.

The paper is organized as follows. In § 2 some contractions in spaces of sequences and in $L^p(\mathbb{T})$, $1 \leqslant p < \infty$, are studied. These results are needed to prove the expansion theorems. In § 3 preliminary results on Poisson kernels are considered and the expansion theorems are proved. In § 4 a more detailed comparison with previous results is made. In § 5 the Cauchy problem is studied. In § 6 the interpolation result is proved.

2. On some contractions in ℓ^p and $L^p(\mathbb{T})$

Let ℓ^1 be the Banach space of the complex sequences $x = \{x_j\}$ such that $\|x\|_{\ell^1} = \sum_j |x_j|$ is finite. Recall that the dual space $(\ell^1)'$ of ℓ^1 may be identified with the space ℓ^∞ of the bounded sequences $x = \{x_j\}$ with norm $\|x\|_{\ell^\infty} = \sup_j |x_j|$.

Let us now introduce four operators that will be used in the paper: for any given $k \in \mathbb{N}$, $\sigma \in (0,1)$ and $\gamma \geqslant 0$, let ψ_k, C_σ^γ, S_σ^γ and m_γ be the operators which act on $x = \{x_j\}$ as follows:

$$\psi_k(x) = \{y_j\}, \quad \text{where } y_j = \begin{cases} x_n & \text{if } j = kn,\\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

$$C_\sigma^\gamma(x) = \sum_{p=1}^{\infty} (2p+1)^\gamma \sigma^{2p} \psi_{2p+1}(x), \tag{2.2}$$

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$$S_\sigma^\gamma(x) = \sum_{p=1}^{\infty} (-1)^p (2p+1)^\gamma \sigma^{2p} \psi_{2p+1}(x), \tag{2.3}$$

$$m_\gamma(x) = \left\{ \frac{x_j}{j^\gamma} \right\}. \tag{2.4}$$

Basic properties of these operators are the following.

Lemma 2.1.

(a) *For any $k \in \mathbb{N}$, $\sigma \in (0,1)$ and $\gamma \geqslant 0$ the operators ψ_k, C_σ^γ, S_σ^γ and m_γ are bounded, linear operators from ℓ^1 to ℓ^1.*

(b) *For any $\gamma \geqslant 0$ there exists a constant $\sigma_\gamma \in (0,1)$ such that for any $0 < \sigma < \sigma_\gamma$, the operators C_σ^γ and S_σ^γ are contractions on ℓ^1. In particular, when $\gamma = 0$, the constant σ_0 is $1/\sqrt{2}$. It follows that, for any $\sigma \in (0, \sigma_\gamma)$, the operators $(I + C_\sigma^\gamma)$ and $(I + S_\sigma^\gamma)$ are invertible on ℓ^1.*

(c) *For any $\sigma \in (0, \sigma_\gamma)$ and for any $x \in \ell^1$,*

$$\left. \begin{aligned} m_\gamma((I + C_\sigma^\gamma)^{-1}(x)) &= (I + C_\sigma^0)^{-1}(m_\gamma(x)), \\ m_\gamma((I + S_\sigma^\gamma)^{-1}(x)) &= (I + S_\sigma^0)^{-1}(m_\gamma(x)). \end{aligned} \right\} \tag{2.5}$$

Proof. For any $x \in \ell^1$, we have that $\|\psi_k(x)\|_{\ell^1} = \|x\|_{\ell^1}$ and

$$\|C_\sigma^\gamma(x)\|_{\ell^1}, \|S_\sigma^\gamma(x)\|_{\ell^1} \leqslant \|x\|_{\ell^1} \sum_{p=1}^{\infty} (2p+1)^\gamma \sigma^{2p}.$$

Claim (a) follows.

To prove (b), observe that, by the uniform convergence of the previous series with respect to σ in any compact subset of $[0,1)$, it follows that there exists a constant σ_γ such that the sum is less than 1 for $\sigma < \sigma_\gamma$. If $\gamma = 0$, the sum of the series is $\sigma^2/(1-\sigma^2)$ and hence $\sigma_0 = 1/\sqrt{2}$.

Now, notice that

$$(m_\gamma(\psi_k(x)))_j = \begin{cases} \dfrac{x_n}{(kn)^\gamma} & \text{if } j = kn, \\ 0 & \text{otherwise,} \end{cases}$$

i.e.

$$m_\gamma(\psi_k(x)) = \frac{1}{k^\gamma} \psi_k(m_\gamma(x)).$$

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Therefore, for any $x \in \ell^1$ we have

$$\begin{aligned} m_\gamma(C_\sigma^\gamma(x)) &= m_\gamma\biggl(\sum_{p=1}^{\infty}(2p+1)^\gamma \sigma^{2p}\psi_{2p+1}(x)\biggr) \\ &= \sum_{p=1}^{\infty}\sigma^{2p}\psi_{2p+1}(m_\gamma(x)) \\ &= C_\sigma^0(m_\gamma(x)). \end{aligned}$$

From this the first equality in (c) follows. The other equality is proved similarly. □

We consider now the adjoint operators $m_\gamma^\star$, $\psi_k^\star$, $(C_\sigma^\gamma)^\star$, $(S_\sigma^\gamma)^\star : \ell^\infty \to \ell^\infty$. For such operators, the following lemma holds.

Lemma 2.2.

(a) *For any* $y = \{y_j\} \in \ell^\infty$,

$$m_\gamma^\star(y) = m_\gamma(y), \qquad \psi_k^\star(y) = \{y_{kj}\}, \tag{2.6}$$

$$(C_\sigma^\gamma)^\star(y) = \sum_{p=1}^{\infty}(2p+1)^\gamma\sigma^{2p}\psi_{2p+1}^\star(y), \tag{2.7}$$

$$(S_\sigma^\gamma)^\star(y) = \sum_{p=1}^{\infty}(-1)^p(2p+1)^\gamma\sigma^{2p}\psi_{2p+1}^\star(y) \tag{2.8}$$

and

$$m_\gamma((C_\sigma^0)^\star(y)) = (C_\sigma^\gamma)^\star(m_\gamma(y)), \qquad m_\gamma((S_\sigma^0)^\star(y)) = (S_\sigma^\gamma)^\star(m_\gamma(y)). \tag{2.9}$$

(b) *For any* $\sigma \in (0, \sigma_\gamma)$, *the operators* $(C_\sigma^\gamma)^\star$ *and* $(S_\sigma^\gamma)^\star$ *are contractions on* ℓ^∞ *and*

$$(I + (C_\sigma^\gamma)^\star)^{-1} = [(I + C_\sigma^\gamma)^{-1}]^\star, \qquad (I + (S_\sigma^\gamma)^\star)^{-1} = [(I + S_\sigma^\gamma)^{-1}]^\star. \tag{2.10}$$

Proof. It is sufficient to prove (a). Formula (2.6) follows from

$$\langle y, \psi_k(x)\rangle = \sum_{j=1}^{\infty} y_j(\psi_k(x))_j = \sum_{n=1}^{\infty} y_{kn}x_n = \sum_{n=1}^{\infty}(\psi_k^\star(y))_n x_n,$$

and (2.7) and (2.8) follow from definitions. The first equality in (2.9) is obtained from

$$\begin{aligned} m_\gamma((C_\sigma^0)^\star(x)) &= \biggl\{\frac{1}{j^\gamma}\sum_{p=1}^{\infty}\sigma^{2p}x_{(2p+1)j}\biggr\} \\ &= \biggl\{\sum_{p=1}^{\infty}(2p+1)^\gamma\sigma^{2p}\frac{x_{(2p+1)j}}{[(2p+1)j]^\gamma}\biggr\} \\ &= (C_\sigma^\gamma)^\star(m_\gamma(x)) \end{aligned}$$

and the second is obtained similarly. □

Now, let us consider the space $L^p(\mathbb{T})$, $1 \leqslant p < \infty$, and its closed subspace

$$\mathfrak{L}^p(\mathbb{T}) = \left\{ g \in L^p(\mathbb{T}) : \int_0^{2\pi} g(\theta)\,\mathrm{d}\theta = 0 \right\}.$$

For $\sigma \in (0,1)$, let us define

$$X(g(\theta)) = \sum_{\nu=1}^{\infty} \sigma^{2\nu} g((-1)^\nu (2\nu+1)\theta). \tag{2.11}$$

Then, we have the following lemma.

Lemma 2.3. *The operator X maps $\mathfrak{L}^p(\mathbb{T})$ in $\mathfrak{L}^p(\mathbb{T})$ and, for any $\sigma \in (0, 1/\sqrt{2})$, X is a contraction on $\mathfrak{L}^p(\mathbb{T})$.*

Proof. The first part is obvious. We have

$$\begin{aligned}
\left\| \sum_{\nu=1}^{\infty} \sigma^{2\nu} g((-1)^\nu (2\nu+1)(\cdot)) \right\|_p &\leqslant \sum_{\nu=1}^{\infty} \sigma^{2\nu} \left(\int_0^{2\pi} |g((-1)^\nu (2\nu+1)\theta)|^p \,\mathrm{d}\theta \right)^{1/p} \\
&= \sum_{\nu=1}^{\infty} \sigma^{2\nu} \left(\frac{(-1)^\nu}{2\nu+1} \int_0^{2\pi(2\nu+1)(-1)^\nu} |g(\phi)|^p \,\mathrm{d}\phi \right)^{1/p} \\
&= \|g\|_p \frac{\sigma^2}{1-\sigma^2}.
\end{aligned}$$

This implies that X is bounded and that, for $\sigma \in (0, 1/\sqrt{2})$, X is a contraction on $\mathfrak{L}^p(\mathbb{T})$. □

3. Expansions in terms of Poisson kernels

Let us recall the following simple fact [**13**, p. 127].

Fact 3.1. Let $F(z)$ be a continuous function of the form $F(z) = \sum_{p=0}^{\infty} a_p z^p$ on the closed disc $|z| \leqslant R$. Then for any integer n we have

$$\frac{1}{n} \sum_{h=0}^{n-1} F(R\mathrm{e}^{2h\pi\mathrm{i}/n}) = \sum_{p=0}^{\infty} a_{np} R^{np}. \tag{3.1}$$

Now, let

$$S(r,\theta) := \frac{1 + r\mathrm{e}^{\mathrm{i}\theta}}{2(1 - r\mathrm{e}^{\mathrm{i}\theta})} = \tfrac{1}{2} + \sum_{p=1}^{\infty} r^p \mathrm{e}^{\mathrm{i}p\theta}, \quad r \in [0,1),\ \theta \in \mathbb{R}, \tag{3.2}$$

be the so-called *Schwarz kernel*. Its real part is the Poisson kernel $P(r,\theta)$.

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The following facts hold.

Lemma 3.2. *Let $0<\sigma<1$, $n\in\mathbb{N}$, $\theta\in\mathbb{R}$. Then*

$$\mathcal{S}_n(\theta) := \frac{S(\sigma,n\theta)-S(\sigma,n\theta+\pi)}{2\sigma} = \mathrm{e}^{\mathrm{i}n\theta}+\sum_{p=1}^{\infty}\sigma^{2p}\mathrm{e}^{\mathrm{i}(2p+1)n\theta} \tag{3.3}$$

is periodic with period $2\pi/n$ and

$$\mathcal{S}_n(\theta) = \frac{1}{2n\sigma}\sum_{h=0}^{n-1}\biggl[S\biggl(\sigma^{1/n},\theta+\frac{2\pi h}{n}\biggr)-S\biggl(\sigma^{1/n},\theta+\frac{\pi}{n}+\frac{2\pi h}{n}\biggr)\biggr]. \tag{3.4}$$

Moreover, $|\mathcal{S}_n(\theta)|\leqslant 1/(1-\sigma^2)$.

Proof. From the definition (3.2) of the Schwarz kernel, one can directly obtain the following identity:

$$S(r,\theta)-S(r,\theta+\pi) = 2\sum_{p=0}^{\infty}r^{2p+1}\mathrm{e}^{\mathrm{i}(2p+1)\theta}, \tag{3.5}$$

and (3.3) follows from (3.5).

Now, consider formula (3.1) for $R=1$ and $F(z)=\sum_{p=1}^{\infty}r^p\mathrm{e}^{\mathrm{i}p\theta}z^p$:

$$\frac{1}{n}\sum_{h=0}^{n-1}\sum_{p=1}^{\infty}r^p\exp\biggl\{\mathrm{i}p\biggl(\theta+\frac{2\pi h}{n}\biggr)\biggr\} = \sum_{p=1}^{\infty}r^{np}\mathrm{e}^{\mathrm{i}np\theta}.$$

So, by (3.2),

$$\frac{1}{n}\sum_{h=0}^{n-1}S\biggl(r,\theta+\frac{2\pi h}{n}\biggr) = S(r^n,n\theta).$$

Hence, setting $\sigma=r^n$, (3.4) follows. The last bound is a consequence of the identity

$$\mathcal{S}_n(\theta) = \frac{\mathrm{e}^{\mathrm{i}n\theta}}{1-\sigma^2\mathrm{e}^{2\mathrm{i}n\theta}}.$$

□

Lemma 3.3. *Let $0<\sigma<1$, $n\in\mathbb{N}$, $\theta\in\mathbb{R}$. Then the functions $\mathcal{P}_n^1(\theta) := \operatorname{Re}\mathcal{S}_n(\theta)$ satisfy the following identities:*

$$\mathcal{P}_n^1(\theta) = \frac{P(\sigma,n\theta)-P(\sigma,n\theta+\pi)}{2\sigma} = \cos(n\theta)+\sum_{p=1}^{\infty}\sigma^{2p}\cos[(2p+1)n\theta], \tag{3.6}$$

$$\mathcal{P}_n^1(\theta) = \frac{1}{2n\sigma}\sum_{h=0}^{n-1}\biggl[P\biggl(\sigma^{1/n},\theta+\frac{2\pi h}{n}\biggr)-P\biggl(\sigma^{1/n},\theta+\frac{\pi}{n}+\frac{2\pi h}{n}\biggr)\biggr]. \tag{3.7}$$

Moreover, $\mathcal{P}_n^1$ is periodic of period $2\pi/n$ and $|\mathcal{P}_n^1(\theta)|\leqslant 1/(1-\sigma^2)$.

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The functions $\mathcal{P}_n^2(\theta) := \mathcal{P}_n^1(\theta - (\pi/2n))$ satisfy the identities

$$\mathcal{P}_n^2(\theta) = \frac{P(\sigma, n\theta - \frac{1}{2}\pi) - P(\sigma, n\theta + \frac{1}{2}\pi)}{2\sigma} = \sin(n\theta) + \sum_{p=1}^{\infty}(-1)^p \sigma^{2p} \sin[(2p+1)n\theta], \tag{3.8}$$

$$\mathcal{P}_n^2(\theta) = \frac{1}{2n\sigma} \sum_{h=0}^{n-1} \left[P\left(\sigma^{1/n}, \theta + \frac{2\pi h}{n} - \frac{\pi}{2n}\right) - P\left(\sigma^{1/n}, \theta + \frac{\pi}{2n} + \frac{2\pi h}{n}\right)\right]. \tag{3.9}$$

Moreover, $\mathcal{P}_n^2$ is periodic of period $2\pi/n$ and $|\mathcal{P}_n^2(\theta)| \leqslant 1/(1-\sigma^2)$.

Proof. Formulae (3.6) and (3.7) are the real parts of (3.3) and (3.4), respectively. Formulae (3.8) and (3.9) follow from (3.6) and (3.7) by replacing θ with $\theta - (\pi/2n)$. □

Remark 3.4. Notice that (3.7) and (3.9) can also be written as

$$\mathcal{P}_n^j(\theta) := \frac{1}{2n\sigma} \sum_{h=0}^{n-1} [P(\sigma^{1/n}, \theta - \arg \zeta_{2n,2h}^{(j)}) - P(\sigma^{1/n}, \theta - \arg \zeta_{2n,2h+1}^{(j)})], \tag{3.10}$$

where $0 < \sigma < 1$, $n \in \mathbb{N}$, $\theta \in \mathbb{R}$ and $\zeta_{2n,l}^{(j)}$ are the points in (1.1), (1.2), $j = 1, 2$.

For any $\theta \in \mathbb{R}$, let us denote by $c(\theta)$, $s(\theta)$ and $\mathcal{P}^j(\theta)$, $j = 1, 2$, the following elements in ℓ^∞:

$$c(\theta) = \{\cos n\theta\}, \qquad s(\theta) = \{\sin n\theta\}, \qquad \mathcal{P}^j(\theta) = \{\mathcal{P}_n^j(\theta)\}.$$

Then we have the following lemma.

Lemma 3.5. *For any $\theta \in \mathbb{R}$, $\gamma \geqslant 0$, we have*

(i) *for any $\sigma \in (0, 1)$,*

$$\mathcal{P}^1(\theta) = (I + (C_\sigma^0)^\star)(c(\theta)), \qquad \mathcal{P}^2(\theta) = (I + (S_\sigma^0)^\star)(s(\theta)), \tag{3.11}$$

$$m_\gamma(\mathcal{P}^1(\theta)) = (I + (C_\sigma^\gamma)^\star) m_\gamma(c(\theta)), \qquad m_\gamma(\mathcal{P}^2(\theta)) = (I + (S_\sigma^\gamma)^\star) m_\gamma(s(\theta)), \tag{3.12}$$

(ii) *for any $\sigma \in (0, 1/\sqrt{2})$,*

$$c(\theta) = [(I + C_\sigma^0)^{-1}]^\star(\mathcal{P}^1(\theta)), \qquad s(\theta) = [(I + S_\sigma^0)^{-1}]^\star(\mathcal{P}^2(\theta)), \tag{3.13}$$

(iii) *for any $\sigma \in (0, \sigma_\gamma)$,*

$$m_\gamma(c(\theta)) = [(I + C_\sigma^\gamma)^{-1}]^\star(m_\gamma(\mathcal{P}^1(\theta))), \qquad m_\gamma(s(\theta)) = [(I + S_\sigma^\gamma)^{-1}]^\star(m_\gamma(\mathcal{P}^2(\theta))). \tag{3.14}$$

Proof. The identities (3.11) are simply the identities (3.6) and (3.8) written in terms of the operators $(C_\sigma^0)^\star$, $(S_\sigma^0)^\star$. Formulae (3.12) follow from (3.11) and (2.9). Equalities (3.13) and (3.14) follow from (3.11), (3.12) and Lemma 2.2 (b). □

Let us denote by $\mathbb{A}$ the space of the (complex-valued) functions which are sums of absolutely convergent Fourier series, i.e. of the functions of the form

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \tag{3.15}$$

with $a = \{a_n\}_{n\geqslant 1}$ and $b = \{b_n\}_{n\geqslant 1}$ in ℓ^1 (for more about this space, see, for example, [**1**,**15**]). For any $\gamma > 0$, also let $\mathbb{A}^{(\gamma)}$ be the space of the functions $f \in \mathbb{A}$ of the form (3.15) with $a = m_\gamma(a^1)$, $b = m_\gamma(b^1)$ and $a^1 = \{a^1_n\}_{n\geqslant 1}$, $b^1 = \{b^1_n\}_{n\geqslant 1}$ in ℓ^1. Notice that if $\gamma \in \mathbb{N}$, the condition $f \in \mathbb{A}^\gamma$ is equivalent to saying that

$$\frac{\mathrm{d}^m}{\mathrm{d}\theta^m} f \in \mathbb{A}, \quad 0 \leqslant m \leqslant \gamma.$$

We have the following result.

Theorem 3.6. *Let $\sigma \in (0, 1/\sqrt{2})$ and let $f \in \mathbb{A}$ be of the form (3.15). Then f can be written as*

$$f(\theta) = a_0 P(0,\theta) + \sum_{n=1}^{\infty}(\alpha_n \mathcal{P}^1_n(\theta) + \beta_n \mathcal{P}^2_n(\theta)), \tag{3.16}$$

with $\alpha = \{\alpha_n\}, \beta = \{\beta_n\} \in \ell^1$ given by

$$\alpha = (I + C^0_\sigma)^{-1} a, \qquad \beta = (I + S^0_\sigma)^{-1} b. \tag{3.17}$$

On the other hand, if f can be written in the form (3.16) with α and β in ℓ^1, then $f \in \mathbb{A}$ and it can be written as in (3.15), setting $a = (I + C^0_\sigma)\alpha$ and $b = (I + S^0_\sigma)\beta$. Both the series (3.15) and (3.16) satisfy the Weierstrass M-test and are absolutely and uniformly convergent.

Proof. Let $\sigma \in (0, 1/\sqrt{2})$ and $f \in \mathbb{A}$. Then (3.15) can be written as

$$f(\theta) = \tfrac{1}{2}a_0 + \langle c(\theta), a\rangle + \langle s(\theta), b\rangle, \quad \theta \in \mathbb{R}.$$

Let us write $a = (I + C^0_\sigma)\alpha, b = (I + S^0_\sigma)\beta$. Then

$$f(\theta) = a_0 P(0,\theta) + \langle (I + C^0_\sigma)^\star c(\theta), \alpha\rangle + \langle (I + S^0_\sigma)^\star s(\theta), \beta\rangle;$$

by (3.11) one gets (3.16).

In a similar way, if f is of the form

$$f(\theta) = a_0 P(0,\theta) + \langle \mathcal{P}^1(\theta), \alpha\rangle + \langle \mathcal{P}^2(\theta), \beta\rangle,$$

using (3.17) and (3.11), we obtain (3.15). □

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Remark 3.7. Let $\sigma \in (0, 1/\sqrt{2})$ and $f \in \mathbb{A}$. Using (3.7), (3.9) and Remark 3.4, (3.16) can be written as

$$f(\theta) = \alpha_0 P(0,\theta) + \sum_{n=1}^{\infty} \frac{\alpha_n}{2\sigma n} \sum_{p=0}^{n-1} [P(\sigma^{1/n}, \theta - \arg\zeta_{2n,2p}^{(1)}) - P(\sigma^{1/n}, \theta - \arg\zeta_{2n,2p+1}^{(1)})] \\ + \sum_{n=1}^{\infty} \frac{\beta_n}{2\sigma n} \sum_{p=0}^{n-1} [P(\sigma^{1/n}, \theta - \arg\zeta_{2n,2p}^{(2)}) - P(\sigma^{1/n}, \theta - \arg\zeta_{2n,2p+1}^{(2)})]. \tag{3.18}$$

In other words, f is approximated by linear combinations of delayed Poisson kernels.

Theorem 3.8. *Let $\gamma > 0$ and let σ be fixed in the interval $(0, \sigma_\gamma)$, where σ_γ is the constant in Lemma 2.1. Assume that $f \in \mathbb{A}^\gamma$ is of the form (3.15) with $a = m_\gamma(a^1)$ and $b = m_\gamma(b^1)$, $a^1 \in \ell^1$, $b^1 \in \ell^1$. Let $\alpha^1 = (I + C_\sigma^\gamma)^{-1} a^1$, $\beta^1 = (I + S_\sigma^\gamma)^{-1} b^1$. Then, f can be written in the form (3.16) with*

$$\alpha = m_\gamma(\alpha^1) = (I + C_\sigma^0)^{-1} a \tag{3.19}$$

and

$$\beta = m_\gamma(\beta^1) = (I + C_\sigma^0)^{-1} b. \tag{3.20}$$

As in the previous theorem, the converse also holds.

Proof. Using (3.14), we have

$$\begin{aligned} f(\theta) &= \tfrac{1}{2} a_0 + \langle c(\theta), m_\gamma a^1 \rangle + \langle s(\theta), m_\gamma b^1 \rangle \\ &= \tfrac{1}{2} a_0 + \langle m_\gamma c(\theta), a^1 \rangle + \langle m_\gamma s(\theta), b^1 \rangle \\ &= \tfrac{1}{2} a_0 + \langle [(I + C_\sigma^\gamma)^{-1}]^\star m_\gamma \mathcal{P}^1(\theta), a^1 \rangle + \langle [(I + S_\sigma^\gamma)^{-1}]^\star m_\gamma \mathcal{P}^2(\theta), b^1 \rangle \\ &= a_0 P(0,\theta) + \langle \mathcal{P}^1(\theta), m_\gamma (I + C_\sigma^\gamma)^{-1} a^1 \rangle + \langle \mathcal{P}^2(\theta), m_\gamma (I + S_\sigma^\gamma)^{-1} b^1 \rangle. \end{aligned}$$

Using (2.5) and defining α and β as in (3.19), (3.20), we obtain the thesis. □

Our last expansion theorem is for functions in $L^p(\mathbb{T})$. Recall that, if X is the operator in (2.11) and $\sigma \in (0, 1/\sqrt{2})$, by Lemma 2.3, X is a contraction in $\mathfrak{L}^p(\mathbb{T})$, so that for any $h \in \mathfrak{L}^p(\mathbb{T})$ the equation $g + X(g) = h$ has a unique solution $g \in \mathfrak{L}^p(\mathbb{T})$.

Theorem 3.9. *Let $p \in (1, \infty)$ and $f \in L^p(\mathbb{T})$,*

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

and let $g \in \mathfrak{L}^p(\mathbb{T})$ be the solution to

$$g + X(g) = f - \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\,\mathrm{d}\theta \in \mathfrak{L}^p(\mathbb{T}).$$

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Let

$$g(\theta) \sim \sum_{n=1}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta). \tag{3.21}$$

be the Fourier expansion of g. Then

$$\lim_{N\to\infty} \left\| f - \frac{a_0}{2} - \sum_{n=1}^{N} (\alpha_n \mathcal{P}_n^1 + \beta_n \mathcal{P}_n^2) \right\|_p = 0. \tag{3.22}$$

On the other hand, if $g \in \mathfrak{L}^p(\mathbb{T})$ is of the form (3.21) and f is defined as

$$f = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\,\mathrm{d}\theta + g + X(g),$$

then (3.22) holds.

Proof. By (2.11), (3.6) and (3.8), we have that $(I+X)(\cos n\theta) = \mathcal{P}_n^1(\theta)$ and $(I+X)(\sin n\theta) = \mathcal{P}_n^2(\theta)$. From this and by Lemma 2.3 we obtain that

$$\begin{aligned}
&\left\| f - \frac{a_0}{2} - \sum_{n=1}^{N} (\alpha_n \mathcal{P}_n^1 + \beta_n \mathcal{P}_n^2) \right\|_p \\
&\quad = \left\| g + X(g) - \sum_{n=1}^{N} (\alpha_n (I+X)(\cos n(\cdot)) + \beta_n (I+X)(\sin n(\cdot))) \right\|_p \\
&\quad = \left\| (I+X)\left(g - \sum_{n=1}^{N} (\alpha_n \cos n(\cdot) + \beta_n \sin n(\cdot)) \right) \right\|_p \\
&\quad \leqslant \frac{1}{1-\sigma^2} \left\| g - \sum_{n=1}^{N} (\alpha_n \cos n(\cdot) + \beta_n \sin n(\cdot)) \right\|_p
\end{aligned}$$

tends to zero when $N \to \infty$. □

Remark 3.10. Let $f \in L^1(\mathbb{T})$,

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta);$$

then g, defined by

$$g + X(g) = f - \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\,\mathrm{d}\theta,$$

satisfies $g \in L^1(\mathbb{T})$ and can be written as in (3.21). In this case, however, one cannot have a convergence as in (3.22). Using the notation of the previous theorem, a weaker convergence such as

$$\lim_{\rho\to 1} \left\| f - \frac{a_0}{2} - \sum_{n=1}^{\infty} \rho^n (\alpha_n \mathcal{P}_n^1 + \beta_n \mathcal{P}_n^2) \right\|_1 = 0$$

holds.

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As a first application of Theorem 3.6 we give an expansion as a sum of Poisson kernels for harmonic functions u in D such that $u|_{\partial D} \in \mathbb{A}$. We need the following lemma.

Lemma 3.11. *Let $0 \leqslant r_1, r_2 < 1$, $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$. Then*

$$\frac{1}{\pi}\int_0^{2\pi} P(r_1, \theta - t)P(r_2^n, nt)\,\mathrm{d}t = P((r_1 r_2)^n, n\theta). \tag{3.23}$$

Proof. Notice that, for any $n \in \mathbb{N}$, the function

$$v(r\mathrm{e}^{\mathrm{i}\theta}) = P((rr_2)^n, n\theta) = \tfrac{1}{2} + \sum_{\nu=1}^{\infty}(r_2 r)^{n\nu}\cos n\nu\theta$$

is harmonic in D and $v(\mathrm{e}^{\mathrm{i}\theta}) = P(r_2^n, n\theta)$. Then, representing v by Poisson's formula, we obtain (3.23). □

Theorem 3.12. *Let $\sigma \in (0, 1/\sqrt{2})$ and $f \in \mathbb{A}$ be of the form in (3.15). Then, the solution u to the Dirichlet problem*

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D,\\ u &= f \quad \text{on } \partial D \end{aligned}$$

can be written as

$$\begin{aligned} u(r\mathrm{e}^{\mathrm{i}\theta}) = a_0 P(0,\theta) &+ \sum_{n=1}^{\infty}\frac{\alpha_n}{2\sigma}[P(r^n\sigma, n\theta) - P(r^n\sigma, n\theta+\pi)]\\ &+ \sum_{n=1}^{\infty}\frac{\beta_n}{2\sigma}[P(r^n\sigma, n\theta - \tfrac{1}{2}\pi) - P(r^n\sigma, n\theta + \tfrac{1}{2}\pi)], \end{aligned} \tag{3.24}$$

where α_n, β_n, given by (3.17), are the coefficients in the expansion (3.16) of f. Moreover, if $\zeta_{2n,l}^{(j)}$, $n \in \mathbb{N}$, $l = 0, \dots, 2n-1$, $j = 1, 2$, are the points in (1.1), (1.2), let

$$A^{(j)} := \left\{\frac{1}{n}\sum_{h=0}^{n-1}[u(\zeta_{2n,2h}^{(j)}) - u(\zeta_{2n,2h+1}^{(j)})]\right\}_{n\in\mathbb{N}}, \quad j = 1, 2.$$

Then $A^{(j)} \in \ell^1$, $j = 1, 2$, and

$$A^{(1)} = 2\sigma(I + (C_\sigma^0)^\star)(a), \tag{3.25}$$

$$A^{(2)} = 2\sigma(I + (S_\sigma^0)^\star)(b), \tag{3.26}$$

where a, b are the sequences of the Fourier coefficients of f.

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Proof. Let us write f in the expansion (3.16). By Poisson's formula, as the series in (3.16) is uniformly convergent, we have

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{\pi}\int_0^{2\pi} P(r,\theta-t)f(t)\,dt \\ &= \frac{a_0}{\pi}\int_0^{2\pi} P(r,\theta-t)P(0,t)\,dt \\ &\quad + \sum_{n=1}^{\infty}\frac{\alpha_n}{2\sigma}\frac{1}{\pi}\int_0^{2\pi}(P(\sigma,nt)-P(\sigma,nt+\pi))P(r,\theta-t)\,dt \\ &\quad + \sum_{n=1}^{\infty}\frac{\beta_n}{2\sigma}\frac{1}{\pi}\int_0^{2\pi}(P(\sigma,nt-\tfrac{1}{2}\pi)-P(\sigma,nt+\tfrac{1}{2}\pi))P(r,\theta-t)\,dt. \end{aligned}$$

From this, applying Lemma 3.11, we obtain (3.24).

Moreover, again by Poisson's formula, we have

$$A^{(1)} = \biggl\{\frac{1}{n\pi}\int_0^{2\pi} f(\theta)\sum_{h=0}^{n-1}\biggl[P\biggl(\sigma^{1/n},\theta+\frac{2\pi h}{n}\biggr)-P\biggl(\sigma^{1/n},\theta+\frac{2\pi h}{n}+\frac{\pi}{n}\biggr)\biggr]\,d\theta\biggr\}_{n\in\mathbb{N}}.$$

Therefore, by (3.7) and (3.6),

$$\begin{aligned} A^{(1)} &= \biggl\{\frac{2\sigma}{\pi}\int_0^{2\pi} f(\theta)\biggl(\cos n\theta+\sum_{p=1}^{\infty}\sigma^{2p}\cos[(2p+1)n\theta]\biggr)\,d\theta\biggr\}_{n\in\mathbb{N}} \\ &= 2\sigma\biggl\{a_n+\sum_{p=1}^{\infty}\sigma^{2p}a_{(2p+1)n}\biggr\}_{n\in\mathbb{N}} \\ &= 2\sigma(I+(C_\sigma^0)^\star)(a), \end{aligned}$$

i.e. (3.25) holds. Formula (3.26) has a similar proof. □

4. Comparison with previous results

Let us recall (see, for example, [**5**]) that a subset E of D is called non-tangentially dense for ∂D if almost every point of ∂D is the non-tangential limit of some sequence in E. More precisely, for $w\in\partial D$, $\psi\in(0,\pi/2)$ and $\epsilon>0$, let us denote by $\Delta_{w,\psi,\epsilon}$ the symmetric Stolz angle with vertex w and of opening 2ψ, i.e.

$$\Delta_{w,\psi,\epsilon} = \{z\in D : |\arg(1-\bar{w}z)|<\psi,\ |z-w|<\epsilon\}.$$

Then, E is non-tangentially dense for ∂D if, for almost all $w\in\partial D$, there exists $\psi\in(0,\pi/2)$ such that $E\cap\Delta_{w,\psi,\epsilon}\neq\emptyset$ for all $\epsilon>0$.

Let us recall the following results.

Fact 4.1 (Bonsall [2]). Let $\mathcal{M}=\{\mathfrak{b}_j\}$ be a subset of D which is non-tangentially dense for ∂D. Then, $L^1(\partial D)$ is the set of all functions f of the form

$$f = \sum_{\mu=1}^{\infty}\lambda_\mu P(|\mathfrak{b}_k|,(\cdot)-\arg\mathfrak{b}_k) \tag{4.1}$$

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with $\sum_{\mu=1}^{\infty} |\lambda_\mu| < \infty$. Also

$$\|f\|_{L^1(\partial D)} = \inf \sum_{\mu=1}^{\infty} |\lambda_\mu|,$$

with the infimum taken over all decompositions (4.1).

Fact 4.2 (Bonsall [3]). Let $\mathcal{M} = \{\mathfrak{b}_j\}$ be a subset of D which is non-tangentially dense for ∂D and let $BH(D)$ be the family of bounded complex valued harmonic functions in D. Then, for all $u \in BH(D)$,

$$\sup_{z \in D} |u(z)| = \sup_{n \in \mathbb{N}} |u(\mathfrak{b}_n)|.$$

Fact 4.3 (Bonsall and Walsh [5]). The map T of ℓ^1 into $L^1(\partial D)$ given by (4.1) is onto (Fact 4.1) and $\ker T \neq \{0\}$.

Remark 4.4. Let $\mathcal{N}$ be the set of points defined in § 1. Then $\mathcal{N}$ is non-tangentially dense for ∂D. To prove this, let ψ be such that

$$\frac{2}{\pi}(\tan\psi)\log\frac{1}{\sigma} > 1.$$

We will prove that, for every $w \in \partial D$, n sufficiently large and $j = 1, 2$, one has $\zeta_{2n,l}^{(j)} \in \Delta_{w,\psi,\epsilon}$ for some $l = 0, \dots, 2n-1$. This is certainly true if, for n sufficiently large, the arc $\Delta_{w,\psi,\epsilon} \cap \{|z| = \sigma^{1/n}\}$ bounds a sector, centred at 0, with opening $2\varphi_n > \pi/n$. As

$$\frac{\sigma^{1/n}}{\sin\psi} = \frac{1}{\sin(\varphi_n + \psi)},$$

it follows that

$$\lim_{n\to\infty} \frac{2\varphi_n}{\pi/n} = \lim_{n\to\infty} \frac{2(\arcsin((\sin\psi)/\sigma^{1/n}) - \psi)}{\pi/n} = \frac{2}{\pi}\log\frac{1}{\sigma}\tan\psi.$$

From this fact, the thesis follows.

Remark 4.5. Let $f \in \mathbb{A}$ and let $\mathcal{M} = \{\mathfrak{b}_\nu\} \subset D$ be a set which is non-tangentially dense for ∂D. As $\mathbb{A} \subset L^1(\partial D)$, there are infinitely many $\lambda \in \ell^1$ such that f can be written in the form (4.1). The series in (4.1) converges in $L^1(\partial D)$, but nothing can be said about the continuous dependence of the ℓ^1 norm of λ upon the Fourier coefficients of f.

If we choose $\mathcal{M} = \mathcal{N}$, our results imply that in (4.1) (or, more explicitly, in (3.18)) one can make a choice for λ in order to obtain more precise results:

(i) there is a one-to-one mapping between the Fourier coefficients of f and the coefficients of the expansion in Poisson kernels (now written as (3.16));

(ii) if we start with $f \in L^1(\partial D)$, and f is the sum of a series of the form (3.16) with coefficients in ℓ^1, then $f \in \mathbb{A}$.

5. A Cauchy-type problem

Let $\mathcal{M} = \{\mathfrak{b}_\nu\} \subset D$ be a set which is non-tangentially dense for ∂D, without limit points in D. Let us consider a class of functions of the form

$$u(z) = h(z) + \sum_{\nu=1}^{\infty} \lambda_\nu G(z, \mathfrak{b}_\nu), \tag{5.1}$$

where h is harmonic in D, of the form

$$h(\rho e^{i\theta}) = h_0 + \sum_{n=1}^{\infty} \frac{h'_n \cos n\theta + h''_n \sin n\theta}{n} \rho^n$$

with

$$h_{\boldsymbol{n}} \sim \sum_{n=1}^{\infty} (h'_n \cos n\theta + h''_n \sin n\theta) \in L^1(\partial D);$$

$G(z, \zeta)$ is the Green function in D for the Laplacian

$$G(z, \zeta) = -\frac{1}{2\pi} \ln \left| \frac{z - \zeta}{1 - z\bar{\zeta}} \right|, \quad z \neq \zeta,$$

and $\lambda = \{\lambda_\nu\}$ is a sequence in ℓ^1. The function $u - h$ belongs to $W^{1,p}(D)$, $1 \leqslant p < 2$, u is smooth in $D \setminus \mathcal{M}$ and has a distributional Laplacian which is a complex measure μ supported on $\mathcal{M}$:

$$\mu := \Delta u = -\sum_{\nu=1}^{\infty} \lambda_\nu \delta_{\mathfrak{b}_\nu} \tag{5.2}$$

(δ_ζ denotes the Dirac function with singularity ζ). A generalized (exterior) normal derivative $\partial_{\boldsymbol{n}} u$ on ∂D can be defined as

$$\partial_{\boldsymbol{n}} u = h_{\boldsymbol{n}} - \frac{1}{\pi} \sum_{\nu=1}^{\infty} \lambda_\nu P(|\mathfrak{b}_\nu|, \theta - \arg \mathfrak{b}_\nu). \tag{5.3}$$

By Fact 4.1 and the properties of h, we have that $u|_{\partial D}$ is in $L^p(\partial D)$ for $1 \leqslant p < \infty$ and $\partial_{\boldsymbol{n}} u$ is in $L^1(\partial D)$. The next lemma shows that it satisfies natural boundary integral formulae.

Lemma 5.1. *Let u be of the form (5.1). Then, for any $v \in C^1(\bar{D})$, v harmonic in D,*

$$\int_D v \, d\mu = \int_{\partial D} \left(v \partial_{\boldsymbol{n}} u - u \frac{\partial v}{\partial n} \right) ds. \tag{5.4}$$

Proof. Since

$$\int_{\partial D} \left(v h_{\boldsymbol{n}} - h \frac{\partial v}{\partial n} \right) ds = 0,$$

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we have

$$\int_{\partial D}\left(v\partial_{\boldsymbol{n}}u - u\frac{\partial v}{\partial n}\right)\mathrm{d}s = -\frac{1}{\pi}\sum_{\nu=1}^{\infty}\lambda_\nu\int_0^{2\pi}v(\theta)P(|\mathfrak{b}_\nu|,\theta-\arg\mathfrak{b}_\nu)\,\mathrm{d}\theta$$
$$= -\sum_{\nu=1}^{\infty}\lambda_\nu v(\mathfrak{b}_\nu)$$
$$= \int_D v\,\mathrm{d}\mu.$$

□

By using Fact 4.1, we get the following theorem.

Theorem 5.2. *Let*

(i) $f^{(0)}$ *be defined on* ∂D *of the form*

$$f^{(0)}(\theta) \sim f_0^{(0)} + \sum_{n=1}^{\infty}\frac{f_n^{(0)'}\cos n\theta + f_n^{(0)''}\sin n\theta}{n}$$

with

$$L^1(\partial D) \ni g^{(0)} \sim \sum_{n=1}^{\infty}(f_n^{(0)'}\cos n\theta + f_n^{(0)''}\sin n\theta),$$

(ii) $f^{(1)} \in L^1(\partial D)$.

Then, there exist μ *of the form (5.2) and* u *of the form (5.1) satisfying*

$$\left.\begin{aligned}\Delta u &= \mu \quad \text{in } D,\\ u|_{\partial D} &= f^{(0)},\\ \partial_{\boldsymbol{n}}u|_{\partial D} &= f^{(1)}.\end{aligned}\right\} \tag{5.5}$$

The boundary data are assumed according to (5.4).

Proof. Let

$$v(\rho\mathrm{e}^{\mathrm{i}\theta}) := f_0^{(0)} + \sum_{n=1}^{\infty}\frac{f_n^{(0)'}\cos n\theta + f_n^{(0)''}\sin n\theta}{n}\rho^n.$$

By Fact 4.1, there exists $\tilde{\lambda} = \{\tilde{\lambda}_\nu\} \in \ell^1$ such that

$$f^{(1)} - g^{(0)} = \sum_{\nu=1}^{\infty}\tilde{\lambda}_\nu P(|\mathfrak{b}_\nu|,(\cdot)-\arg\mathfrak{b}_\nu).$$

Then

$$\mu = \pi\sum_{\nu=1}^{\infty}\tilde{\lambda}_\nu\delta_{\mathfrak{b}_\nu} \quad \text{and} \quad u = v - \pi\sum_{\nu=1}^{\infty}\tilde{\lambda}_\nu G(\cdot\,,\mathfrak{b}_\nu)$$

satisfy (5.5). □

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The Bonsall–Walsh result gives us that any $f \in L^1(\partial D)$ can be written in the form (4.1), but it does not say anything about the dependence of λ upon f. Our set of points $\mathcal{N}$ and a suitable choice of λ give a more precise result.

Let $\gamma > 0$, $0 < \sigma < \sigma_\gamma$, and let $\mathcal{N}$ be the set of the points defined in § 1 (which is non-tangentially dense for ∂D by Remark 4.4). Let also

$$u = h - \pi a_0 G((\cdot), 0) - \pi \sum_{n=1}^{\infty} \frac{\alpha_n}{2\sigma n} \sum_{p=0}^{n-1} [G((\cdot), \zeta^{(1)}_{2n,2p}) - G((\cdot), \zeta^{(1)}_{2n,2p+1})] - \pi \sum_{n=1}^{\infty} \frac{\beta_n}{2\sigma n} \sum_{p=0}^{n-1} [G((\cdot), \zeta^{(2)}_{2n,2p}) - G((\cdot), \zeta^{(2)}_{2n,2p+1})], \tag{5.6}$$

where $h \in C^1(\bar{D})$ is a harmonic function such that $\partial h/\partial n \in \mathbb{A}^\gamma$ and α and β are sequences in ℓ^1 such that $\alpha = m_\gamma(\alpha^1)$, $\beta = m_\gamma(\beta^1)$, with $\alpha^1, \beta^1 \in \ell^1$.

We have that $u \in W^{1,p}(D)$, $1 \leqslant p < 2$, $u|_{\partial D} = h|_{\partial D}$ and

$$\mu = \Delta u = \pi a_0 \delta_0 + \pi \sum_{n=1}^{\infty} \frac{\alpha_n}{2\sigma n} \sum_{p=0}^{n-1} \Big[\delta_{\zeta^{(1)}_{2n,2p}} - \delta_{\zeta^{(1)}_{2n,2p+1}}\Big] + \pi \sum_{n=1}^{\infty} \frac{\beta_n}{2\sigma n} \sum_{p=0}^{n-1} \Big[\delta_{\zeta^{(2)}_{2n,2p}} - \delta_{\zeta^{(2)}_{2n,2p+1}}\Big]. \tag{5.7}$$

Moreover,

$$\partial_{\boldsymbol{n}} u = \frac{\partial h}{\partial n} + a_0 P(0, (\cdot)) + \sum_{n=1}^{\infty} (\alpha_n \mathcal{P}^1_n + \beta_n \mathcal{P}^2_n).$$

Therefore, using Theorem 3.8 instead of Fact 4.1, we obtain the following result.

Theorem 5.3. *Let $\gamma > 0$ and let $f^{(0)}$, $f^{(1)}$ be such that $\mathrm{d}f^{(0)}/\mathrm{d}\theta, f^{(1)} \in \mathbb{A}^\gamma$. Denote also by h the solution to the Dirichlet problem $\Delta h = 0$ in D, $h = f^{(0)}$ on ∂D. Then we have the following.*

(i) $f := f^{(1)} - \dfrac{\partial h}{\partial n} \in \mathbb{A}^\gamma$.

(ii) *Let*

$$f(\theta) = a_0 P(0, \theta) + \sum_{n=1}^{\infty} (\alpha_n \mathcal{P}^1_n(\theta) + \beta_n \mathcal{P}^2_n(\theta)).$$

Then the function u given by (5.6) solves the Cauchy problem (5.5) with μ given by (5.7). The sequences α, β are of the form $\alpha = m_\gamma(\alpha_1)$, $\beta = m_\gamma(\beta_1)$ with $\alpha_1, \beta_1 \in \ell^1$ and they are in a one-to-one correspondence with the Fourier coefficients a, b of f. Moreover, u is the unique solution to (5.5) of the form (5.6).

Remark 5.4. Under the hypotheses of Theorem 5.3, the function (5.6) can also be considered as a solution to the Cauchy problem (5.5) in the following sense. For any $N \in \mathbb{N}$,

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let

$$u^N = h - \pi a_0 G((\cdot),0) - \pi \sum_{n=1}^{N} \frac{\alpha_n}{2\sigma n} \sum_{p=0}^{n-1} [G((\cdot),\zeta^{(1)}_{2n,2p}) - G((\cdot),\zeta^{(1)}_{2n,2p+1})] \\ - \pi \sum_{n=1}^{N} \frac{\beta_n}{2\sigma n} \sum_{p=0}^{n-1} [G((\cdot),\zeta^{(2)}_{2n,2p}) - G((\cdot),\zeta^{(2)}_{2n,2p+1})].$$

Then the following hold.

(i) u^N converges to u uniformly on any compact subset of $D \setminus \mathcal{N}$.

(ii) u^N is harmonic in $\{\sigma^{1/N} < |z| < 1\}$; indeed,

$$\Delta u^N = \pi a_0 \delta_0 + \pi \sum_{n=1}^{N} \frac{\alpha_n}{2\sigma n} \sum_{p=0}^{n-1} \left[\delta_{\zeta^{(1)}_{2n,2p}} - \delta_{\zeta^{(1)}_{2n,2p+1}}\right] \\ + \pi \sum_{n=1}^{N} \frac{\beta_n}{2\sigma n} \sum_{p=0}^{n-1} \left[\delta_{\zeta^{(2)}_{2n,2p}} - \delta_{\zeta^{(2)}_{2n,2p+1}}\right].$$

(iii) u^N is of class C^2 in a neighbourhood of ∂D, $u^N|_{\partial D} = f^{(0)}$ and $\partial u^N/\partial n|_{\partial D} \in \mathbb{A}^\gamma$.

(iv) We have

$$\left\| \frac{\partial u^N}{\partial n}\Big|_{\partial D} - f^1 \right\|_{\mathbb{A}^\gamma} \to 0,$$

where $\|\cdot\|_{\mathbb{A}_\gamma}$ denotes the norm defined by

$$\|f\|_{\mathbb{A}^\gamma} = \|m_\gamma^{-1}(a)\|_{\ell^1} + \|m_\gamma^{-1}(b)\|_{\ell^1}$$

for any

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \in \mathbb{A}^\gamma.$$

6. An interpolation-type result

Let $\zeta^{(j)}_{2n,l}$ be the points in (1.1), (1.2) and let $A_0, A^{(j)}_{2n,l} \in \mathbb{C}$, $j = 1,2$, $n \in \mathbb{N}$, $l = 0,\dots,2n-1$. We now investigate whether there exists a function u, harmonic in D, satisfying

$$u(0) = A_0, \quad u(\zeta^{(j)}_{2n,l}) = A^{(j)}_{2n,l}, \quad j = 1,2,\ n \in \mathbb{N},\ l = 0,\dots,2n-1, \tag{6.1}$$

and, moreover, if it exists, whether it is unique.

This is a special case (with fixed points in D) of a more general problem in harmonic analysis called the 'interpolation problem' (see, for example, [**7**, **8**, **11**, **14**]).

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Let us recall that a sequence of points $z_n \in D$ is called an interpolating sequence for the Hardy space H^∞ if, for each bounded complex sequence A_n, there exists $f \in H^\infty$ satisfying $f(z_n) = A_n$ (for interpolating sequences in other spaces of functions see [**14**] and the bibliography therein).

Concerning our set $\mathcal{N}$, we point out that, as it is non-tangentially dense, *its elements cannot be an interpolating sequence* [**5**].

One can nevertheless ask if there are conditions that characterize the sequence of values that are assumed on $\mathcal{N}$ by the harmonic functions. In what follows we give the only positive results that we have been able to determine in this regard.

We first prove a uniqueness theorem in a suitable class of complex harmonic functions.

Theorem 6.1. *Let $\mathbb{A}'$ be the dual space of $\mathbb{A}$ and let $T \in \mathbb{A}'$. Let us assume that the harmonic function*

$$u(z) := \frac{1}{\pi}\langle T, P(|z|, (\cdot) - \arg z)\rangle, \quad z \in D, \tag{6.2}$$

satisfies the conditions

$$u(0) = 0 \quad \textit{and} \quad \sum_{p=0}^{n-1}[u(\zeta^{(j)}_{2n,2p+1}) - u(\zeta^{(j)}_{2n,2p})] = 0, \tag{6.3}$$

$j = 1, 2$, $n \in \mathbb{N}$. *Then* $u \equiv 0$.

Proof. Let $T \in \mathbb{A}'$ and let

$$\frac{t_0}{2} + \sum_{n=1}^{\infty}(t_n \cos n\theta + \tau_n \sin n\theta), \quad \{t_n\}, \{\tau_n\} \in \ell^\infty,$$

be the Fourier expansion of T. Then, for

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \quad \text{in } \mathbb{A}$$

(hence, with $\{a_n\}, \{b_n\} \in \ell^1$), we have

$$\langle T, f\rangle = \pi\left\{\frac{a_0 t_0}{2} + \sum_{n=1}^{\infty}(a_n t_n + b_n \tau_n)\right\}.$$

Notice that, as $P(\rho, (\cdot) - \phi) \in \mathbb{A}$, $0 \leqslant \rho < 1$, $\phi \in \mathbb{T}$, (6.2) makes sense and u can also be written as

$$u(\rho e^{i\phi}) = \frac{t_0}{2} + \sum_{n=1}^{\infty}\rho^n(t_n \cos n\phi + \tau_n \sin n\phi).$$

Let us write $f \in \mathbb{A}$, by using the representation formula (3.18) and applying T to both members of (3.18). Using (6.2) we have

$$\langle T, f\rangle = \alpha_0 u(0) + \sum_{n=1}^{\infty} \frac{\alpha_n}{2\sigma n} \sum_{p=0}^{n-1} [u(\zeta^{(1)}_{2n,2p}) - u(\zeta^{(1)}_{2n,2p+1})] + \sum_{n=1}^{\infty} \frac{\beta_n}{2\sigma n} \sum_{p=0}^{n-1} [u(\zeta^{(2)}_{2n,2p}) - u(\zeta^{(2)}_{2n,2p+1})].$$

Then, (6.3) implies that, for every $f \in \mathbb{A}$, $\langle T, f\rangle = 0$. Thus, $T = 0$ and $u \equiv 0$ in D. □

Let us prove now an existence theorem. For this, we need compatibility conditions for the As.

Theorem 6.2. *Let $A_0, A^{(j)}_{2n,l} \in \mathbb{C}$, $j = 1, 2$, $n \in \mathbb{N}$, $l = 0, \dots, 2n-1$, which satisfy the following conditions:*

(i)

$$A^{(j)} := \left\{ \frac{1}{n} \sum_{p=0}^{n-1} [A^{(j)}_{2n,2p} - A^{(j)}_{2n,2p+1}] \right\}_{n\in\mathbb{N}} \in \ell^1, \quad j = 1, 2;$$

(ii)

$$A^{(j)}_{2n,l} = A_0 + \sum_{\nu=1}^{\infty} \frac{[(I + (C^0_\sigma)^\star)^{-1} A^{(1)}]_\nu}{2\sigma} \sigma^{\nu/n} \cos(\nu \arg \zeta^{(j)}_{2n,l}) + \sum_{\nu=1}^{\infty} \frac{[(I + (S^0_\sigma)^\star)^{-1} A^{(2)}]_\nu}{2\sigma} \sigma^{\nu/n} \sin(\nu \arg \zeta^{(j)}_{2n,l}),$$

$$j = 1, 2, \quad n \in \mathbb{N}, \quad l = 0, \dots, 2n-1.$$

Then, there exists u harmonic in D, continuous in $\bar{D}$, with $u|_{\partial D} \in \mathbb{A}$, satisfying (6.1).

On the other hand, if u is harmonic in D, continuous in $\bar{D}$, with $u|_{\partial D} \in \mathbb{A}$, then $A_0 = u(0)$, $A^{(j)}_{2n,l} = u(\zeta^{(j)}_{2n,l})$, $j = 1, 2$, $n \in \mathbb{N}$, $l = 0, \dots, 2n-1$, satisfy (i) and (ii).

Proof. Let us define

$$a = (I + (C^0_\sigma)^\star)^{-1} \frac{A^{(1)}}{2\sigma} \quad \text{and} \quad b = (I + (S^0_\sigma)^\star)^{-1} \frac{A^{(2)}}{2\sigma}.$$

Then, $f(\theta) = A_0 + \langle c(\theta), a\rangle + \langle s(\theta), b\rangle \in \mathbb{A}$ and the solution u to the Dirichlet problem

$$\Delta u = 0 \text{ in } D, \quad u = f \text{ on } \partial D,$$

i.e. the function $u(re^{i\theta}) = A_0 + \langle c(\theta), r^n a\rangle + \langle s(\theta), r^n b\rangle$ satisfies (i) and (ii).

On the other hand, let u be harmonic in D, continuous in $\bar{D}$, with $u|_{\partial D}(\theta) = A_0 + \langle c(\theta), a\rangle + \langle s(\theta), b\rangle \in \mathbb{A}$. If $A^{(j)}_{2n,l} = u(\zeta^{(j)}_{2n,l})$, $j = 1, 2$, $n \in \mathbb{N}$, $l = 0, \dots, 2n-1$, then (i) (by Theorem 3.12) and (ii) hold. □

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