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ON THE ASPHERICITY OF LENGTH-6 RELATIVE PRESENTATIONS WITH TORSION-FREE COEFFICIENTS

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Abstract An interesting result of Ivanov implies that a non-aspherical relative presentation that defines a torsion-free group would provide a potential counterexample to the Kaplansky zero-divisor conjecture. In this point of view, we prove the asphericity of the length-6 relative presentation $\langle H, x : xh_1xh_2xh_3xh_4xh_5xh_6\rangle$, provided that each coefficient is torsion free.

Keywords: asphericity; relative presentations; torsion-free groups; zero-divisors

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1. Introduction

Suppose that we are given a group H and a set of words $r(\boldsymbol{x})$, $r \in \boldsymbol{r}$, involving a set of indeterminates $x \in \boldsymbol{x}$ and elements of H. The datum $P = \langle H, \boldsymbol{x} : \boldsymbol{r}\rangle$ is called a *relative presentation* with *coefficient group* H; the group defined by these data is the quotient group $G = H * F(\boldsymbol{x})/\langle\langle \boldsymbol{r}(\boldsymbol{x})\rangle\rangle$, where $H * F(\boldsymbol{x})$ is the free product of H with the free group $F(\boldsymbol{x})$ having basis $\boldsymbol{x}$ and $\langle\langle \boldsymbol{r}(\boldsymbol{x})\rangle\rangle$ is the smallest normal subgroup of the free product containing the words $r(\boldsymbol{x})$, $r \in \boldsymbol{r}$. In this paper we focus on relative presentations involving a single indeterminate x and a single relator $r(x)$ with positive exponents. Thus, we study relative presentations of the form

$$P = \langle H, x : xh_1 \cdots xh_n\rangle, \tag{1.1}$$

so that the defining relation is a *positive equation of length* n and the elements $h_1, h_2, \ldots, h_n$ are taken from the coefficient group H. One of the early results, due to Levin [**7**], states that the natural homomorphism $H \to G$ is injective in this case. When the relator $r = xh_1xh_2 \cdots xh_n$ is not a proper power, the conjectures would predict that

(A) G is torsion free,

(B) the relative presentation P is aspherical,

(C) the group-ring $\mathbb{Z}G$ has no zero-divisors.

(C) implies (B) because, if P is not aspherical, then the Fox derivative $\partial r/\partial x$ must be a zero divisor; this is essentially what Ivanov [**6**] noted. (B) implies (A); see [**2**]. The implication (A) $\implies$ (C) (for all torsion-free G) is the Kaplansky zero-divisor conjecture [**8**]. On the other hand, the implication (A) $\implies$ (B) is open. Also, if (B) implies (C), then these groups are torsion-free groups without zero-divisors.

All of this is supposed to create a legitimate reason for examining the asphericity of P in this case. Asphericity of the relative presentations (1.1) has been addressed by several authors (see [**1**, **2**, **5**, **9**]). They studied relative presentations over any group H without mentioning the connection between asphericity and the zero-divisor conjecture. Scanning these results, one finds that, for $n \leqslant 5$, these presentations are always aspherical when H is torsion free. Also, the relative presentation $\langle H, x : (xa)^p(xb)^q(xc)^r\rangle$ is aspherical when H is torsion free, $a, b, c \in H$ and $p, q, r > 2$. Recall that a relative presentation $P = \langle H, \boldsymbol{x} : \boldsymbol{r}\rangle$ for a group G is *aspherical* if for some ordinary presentation $Q = \langle \boldsymbol{a} : \boldsymbol{s}\rangle$ for H and for some lifted presentation $\hat{P} = \langle \boldsymbol{a}, \boldsymbol{x} : \boldsymbol{s}, \hat{\boldsymbol{r}}\rangle$ for G, the second homotopy module $\pi_2(\hat{P})$ is $\mathbb{Z}G$-generated by $\pi_2(Q)$. It is then a consequence of the theory of aspherical relative presentations that those new groups defined by the relative presentations are also torsion free as long as the above relators are not proper powers in $H * F(x)$. The main result of this paper can be stated as follows.

Theorem 1.1. *A relative presentation $P = \langle H, x : xh_1xh_2 \cdots xh_n\rangle$ with $n \leqslant 6$ is aspherical if H is torsion free.*

If this is the case, we have a corollary.

Corollary 1.2. *The group defined by a relative presentation*

$$P = \langle H, x : xh_1xh_2 \cdots xh_n\rangle$$

with $n \leqslant 6$ is torsion free if H is torsion free and the relator $xh_1xh_2 \cdots xh_n$ is not a proper power.

2. Pictures and tests for asphericity

2.1. Pictures

Most of definitions of this section are taken from [**1**, **2**]. A *picture* $\boldsymbol{P}$ is defined with the following data: a finite collection of pairwise disjoint discs $\{\Delta_1, \dots, \Delta_m\}$ in the interior of an ambient disc D^2 and a finite collection of pairwise disjoint compact properly embedded one-manifolds $\{\alpha_1, \dots, \alpha_n\}$, that is with $\partial\alpha_i = \alpha_j \cap \partial(D^2 - \bigcup_{i=1}^m \mathrm{Int}(\Delta_i))$. The picture $\boldsymbol{P}$ is *non-trivial* if $m \geqslant 1$ and is *connected* if it has at most one component. The picture $\boldsymbol{P}$ is *spherical* if it is non-trivial and if none of the arcs meet the boundary of D^2.

We introduce the following labelling: each arc α_j is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an element of a group H. If κ is a corner of a disc Δ_i of $\boldsymbol{P}$, then $w(\kappa)$ is the word obtained by reading in a clockwise order the labels on the arcs and corners meeting $\partial\Delta_i$, beginning with the label on the first arc we meet as we read the clockwise corner κ. If we cross an arc labelled x in the direction of its normal orientation, we read x, otherwise we read x^{-1}.

A picture $\boldsymbol{P}$ is called a *picture over the relative presentation* $P = \langle H, \boldsymbol{x} : \boldsymbol{r} \rangle$ if the picture consists of $\boldsymbol{r}$-discs and $\boldsymbol{x}$-arcs with the following additional conditions:

(i) for each corner κ of $\boldsymbol{P}$, $w(\kappa) \in \boldsymbol{r}^*$, where $w(\kappa)$ is a word obtained by reading around a disc containing κ clockwise, and $\boldsymbol{r}^*$ is the set of all cyclic permutations of the elements of $\boldsymbol{r} \cup \boldsymbol{r}^{-1}$ which begin with an element of $\boldsymbol{x}$;

(ii) if $h_1, \dots, h_l$ is the sequence of corner labels encountered in a clockwise traversal of the boundary of an inner region F of $\boldsymbol{P}$, then the product $h_1 \cdots h_l = 1$ in H—we say that $h_1 \cdots h_l$ is the *label* of F.

A *dipole* in a labelled picture $\boldsymbol{P}$ over P consists of corners κ, κ' of $\boldsymbol{P}$ together with an arc joining the two corners such that κ and κ' belong to the same region, and such that if $w(\kappa) = Sh$, where $h \in H$ and S begins and ends with an element of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$, then $w(\kappa') = S^{-1}h^{-1}$. The picture $\boldsymbol{P}$ is *reduced* if it does not contain a dipole. An injective relative presentation P is *aspherical* whenever every connected spherical picture over P contains a dipole if no element of $\boldsymbol{r}$ is a proper power. If P is not aspherical, there is a reduced spherical picture over P (see [**1**]).

2.2. Tests of asphericity

In this section, we introduce the *n step* and see various known techniques to determine asphericity of a relative presentation. For the definition of lifting relative presentation, see [**2**, § 1.6]. Here $\pi_2(-)$ is the second homotopy module. (For more information about the second homotopy module, see [**1**, **3**].)

Lemma 2.1 ($\boldsymbol{n}$ steps). *Let the relative presentation $P = \langle H, \boldsymbol{x} : \boldsymbol{r} \rangle$ define a group G and let $Q = \langle G, \boldsymbol{t} : \boldsymbol{s} \rangle$ be another relative presentation. If Q and P are both aspherical, then the relative presentation $R = \langle H, \boldsymbol{x} \cup \boldsymbol{t} : \boldsymbol{r} \cup \tilde{\boldsymbol{s}} \rangle$ is aspherical, where $\tilde{\boldsymbol{s}}$ is an element of $H * F(\boldsymbol{x}) * F(\boldsymbol{t})$ obtained from $\boldsymbol{s}$ by lifting.*

Proof. Let K be the standard model (two-dimensional CW-complex) of a chosen ordinary presentation for H and let L and M be the standard models of presentations P and Q, respectively. Since both P and Q are aspherical, we have

$$\pi_2(L) = \mathbb{Z}\pi_1(L) \cdot \mathrm{Im}(\pi_2(K) \to \pi_2(L)),$$
$$\pi_2(M) = \mathbb{Z}\pi_1(M) \cdot \mathrm{Im}(\pi_2(L) \to \pi_2(M)),$$

where $\mathrm{Im}(\pi_2(K) \to \pi_2(L))$ and $\mathrm{Im}(\pi_2(L) \to \pi_2(M))$ are the images of the natural maps induced by embeddings. To prove the lemma, we will need to deduce that

$$\pi_2(M) = \mathbb{Z}\pi_1(M) \cdot \mathrm{Im}(\pi_2(K) \to \pi_2(M)).$$

Let ξ be an element in $\pi_2(M)$; then ξ can be expressed as $\xi = \sum_i \lambda_i \sigma_i$, where $\lambda_i \in \mathbb{Z}\pi_1(M)$ and $\sigma_i \in \mathrm{Im}(\pi_2(L) \to \pi_2(M))$ for each i. We identify the elements of $\pi_2(L)$ with their images in $\pi_2(M)$. Since P is aspherical, we can write each σ_i as $\sum_j \mu_{ij} \tau_{ij}$, where

$\mu_{ij} \in \mathbb{Z}\pi_1(L)$ and $\tau_{ij} \in \mathrm{Im}(\pi_2(K) \to \pi_2(L))$. Again, we identify the elements of $\pi_2(K)$ with their images in $\pi_2(L)$. Then, in $\pi_2(M)$,

$$\xi = \sum_i \lambda_i \sigma_i = \sum_i \lambda_i \sum_{ij} \mu_{ij} \tau_{ij},$$

so ξ is an element of $\mathbb{Z}\pi_1(M) \cdot \mathrm{Im}(\pi_2(K) \to \pi_2(M))$. This completes the proof. □

The *star-complex* P^{st} of P is a graph whose edges are labelled by elements of the coefficient group H. The vertex and edge sets are $\boldsymbol{x} \cup \boldsymbol{x}^{-1}, \boldsymbol{r}^*$, respectively. For $r \in \boldsymbol{r}^*$, write $r = Sh$, where $h \in H$ and S begins and ends with $\boldsymbol{x}$ symbols. The initial and terminal functions are given by $\iota(r)$, the first symbol of S, and $\tau(r)$, the inverse of the last symbol of S. The labelling function on the edges is defined by $\lambda(r) = h^{-1}$.

A non-empty cyclically reduced cycle (closed path) in P^{st} will be called *admissible* if it has trivial label in H. Each inner region of a reduced picture over P supports an admissible cycle in P^{st}.

A *weight function* θ on P^{st} is a real-valued function on the set of edges of P^{st} which satisfies $\theta(Sh) = \theta(S^{-1}h^{-1})$ with $Sh = r \in \boldsymbol{r}^*$. A weight function θ on P^{st} is *weakly aspherical* if the following two conditions are satisfied.

(i) Let $r \in \boldsymbol{r}^*$, with $r = x_1^{\varepsilon_1} h_1 \cdots x_n^{\varepsilon_n} h_n$. Then

$$\sum_{i=1}^{n} (1 - \theta(x_i^{\varepsilon_i} h_i \cdots x_n^{\varepsilon_n} h_n x_1^{\varepsilon_1} h_1 \cdots x_{i-1}^{\varepsilon_{i-1}} h_{i-1})) \geqslant 2.$$

(ii) Each admissible cycle in P^{st} has weight at least 2.

It is known [**2**] that if P^{st} admits a weakly aspherical weight function, then P is aspherical.

Let l be a positive integer. An *l-wheel* over P is a non-trivial connected picture $\boldsymbol{W}$ over P which has discs $\{\Delta_0, \Delta_1, \dots, \Delta_l\}$, and which satisfies the following conditions:

(i) each arc of $\boldsymbol{W}$ meets a disc Δ_j for some $j \in \{1, \dots, l\}$;

(ii) each arc of $\boldsymbol{W}$ meets either Δ_0 or $\partial \boldsymbol{W}$;

(iii) each disc of $\boldsymbol{W}$ has a corner which lies in a region of $\boldsymbol{W}$ that meets $\partial \boldsymbol{W}$ and the disc Δ_0 is the *hub* of the l-wheel.

Let p be a positive integer. Then P satisfies $C(p)$ if there are no reduced l-wheels over P for $l < p$. Let q be a positive integer. Then P satisfies $T(q)$ if there are no admissible cycles in P^{st} of length m for $3 \leqslant m < q$. If P satisfies $C(p), T(q)$, where $1/p + 1/q = 1/2$, then P is aspherical.

An *angle function* on a picture $\boldsymbol{P}$ is a real-valued function ϕ on the set of corners of $\boldsymbol{P}$. Associated with ϕ is a curvature function c defined on the discs Δ of $\boldsymbol{P}$ by

$$c(\Delta) = 2\pi - \sum_{\kappa \subseteq \partial \Delta} \phi(\kappa)$$

and on the regions F of $\boldsymbol{P}$ by

$$c(F) = 2\pi - \biggl(\sum_{\kappa \subseteq \partial F} (\pi - \phi(\kappa)) \biggr).$$

If $\boldsymbol{P}$ is a connected spherical picture, then there is the fundamental *curvature formula*

$$\sum_{\Delta} c(\Delta) + \sum_{F} c(F) = 2\pi\chi(S^2) = 4\pi.$$

The sums are taken over all discs and regions of $\boldsymbol{P}$. It follows immediately that, for any angle function on any connected spherical picture, some disc or region has positive curvature.

An application of the curvature formula we use in this paper is due to Edjvet [**4**] and is called *curvature distribution*. More specifically, let ϕ be an angle function with associated curvature function c. Suppose also that every disc Δ of $\boldsymbol{P}$ is *flat* in the sense that $c(\Delta) = 0$. Then the curvature formula implies the existence of at least one region F such that $c(F) > 0$. If F' is a region of $\boldsymbol{P}$ that neighbours F across an arc α in the boundary of F, then we can subtract any real number η from one of the corners of F that touches α and then add η to the adjacent corner in F'. This results in a new angle function on $\boldsymbol{P}$ with associated curvature function c^*. Obviously, $c^*(\Delta) = c(\Delta)$ for each disc Δ of $\boldsymbol{P}$, $c^*(F) = c(F) - \eta$ and $c^*(F') = c(F') + \eta$. Other regions are unaffected. More generally, if $\mathcal{F}$ is the set of regions of $\boldsymbol{P}$, we can define a *distribution scheme* on $\boldsymbol{P}$ as the function $\eta : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$.

3. Reduction to special cases

First, we start with the following lemma.

Lemma 3.1. *The relative presentation $P = \langle H, x : x^m g x^n h \rangle$, where g and h are non-trivial elements of H, is aspherical for any positive integers m and n if H is torsion free.*

Proof. The star-complex of P has just two vertices with labels x and x^{-1}, two edges with labels g^{-1} and h^{-1}, and has other edges with label the identity. We assign a weight function in such a manner that each edge with label the identity has a weight 1 and g^{-1}, h^{-1} have a weight 0. It is sufficient to consider admissible cycles with weights up to 2. Also, there are no admissible cycles of weight 0 since H is torsion free. Therefore, possible admissible cycles are $(gh^{-1})^i g^{\pm 1}$ for non-negative integer i or its conjugates. But this means that g is in $\langle gh^{-1} \rangle$. Then, since the subgroup generated by $\langle g, h \rangle$ equals the subgroup generated by $\langle g, gh^{-1} \rangle$, the group generated by coefficients is infinite cyclic. This implies asphericity by [**1**, Lemma 3]. □

This result also holds if m and n have opposite signs. One can show this by using a weight test.

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For convenience, we will write a length-6 relative presentation with a torsion-free coefficient group H as

$$P = \langle H, x : xaxbxcxdxexf \rangle,$$

where each coefficient is in H.

If the relator is a proper power, then it is aspherical by applying Lemma 2.1. So we assume that the relator $xaxbxcxdxexf$ is not a proper power and assume that the group H is torsion free. Our aim in this section is to classify the relative presentation $P = \langle H, x : xaxbxcxexexf \rangle$ in all the possible cases to show that P is aspherical.

First of all, if the coefficients a, b, c, d, e and f are all distinct, then each disc is connected to at least four more discs and so there are no reduced l-wheels for $f < 4$. Therefore, the presentation satisfies $C(4)$. On the other hand, the star complex P^{st} of P consists of five edges oriented from x^{-1} to x with labels a, b, c, d, e, f. Hence, the smallest admissible cycle in P is of length 4 and therefore P satisfies $T(4)$. This implies that P satisfies the small cancellation condition $C(4)$–$T(4)$. So P is aspherical. In fact, one can observe that if the unknown variable x has all positive exponents, then the number of discs of any region in a picture over P is even and so P automatically satisfies $T(4)$. Thus, we may assume that some coefficients are the same. Then, use the substitution $t = xg$, where g is the coefficient with the most consecutive appearances. For example, if the relator is $x^3gxgxgxh$, then, taking $t = xg$, we have $t^4g^{-1}htg^{-1}tg^{-1}$ up to cyclic permutation. It is sufficient to show that $\langle H, x : x^4hxgxg \rangle$ is aspherical with torsion-free coefficients. If there are no consecutive repeats of coefficients, then we choose $t = xa$ so that the relation starts with t^2. Without loss of generality, we can assume that the relation begins with the largest power of x at least two. But the relative presentations $\langle H, x : x^6g \rangle$, where $g \neq 1$, are clearly aspherical and $\langle H, x : x^5gxh \rangle$ is aspherical by Lemma 3.1. So we have three special cases:

$$P_1 = \langle H, x : x^4gxhxk \rangle,$$
$$P_2 = \langle H, x : x^3gxhxkxl \rangle,$$
$$P_3 = \langle H, x : x^2gxhxkxlxm \rangle,$$

where $1 \notin \{g, h, k\}$ in the relative presentation P_1, $1 \notin \{g, l\}$, neither $g = h = k$ nor $h = k = l$ in P_2, and $1 \notin \{g, m\}$ and any two consecutive coefficients are not the same in P_3. Next, we shall consider further special cases which come from the relative presentations P_1, P_2 and P_3.

In presentation P_1, if we suppose that all the coefficients are different, then the presentation satisfies the condition $C(4)$–$T(4)$ and so P_1 is aspherical. If all the coefficients are the same, then the coefficient group is cyclic and again it is aspherical. Thus, we can consider three special cases obtained from P_1, that is, $g = h$, $g = k$ or $h = k$. But the case $h = k$ is equivalent to the case $g = h$, because taking the inverse of the relation gives us $x^{-4}h^{-1}x^{-1}h^{-1}x^{-1}g^{-1}$, and then we use the substitutions $t = x^{-1}$, $h^{-1} = g_1$ and $g^{-1} = g_2$. Therefore, we have two more relative presentations to consider:

$$Q_1 = \langle H, x : x^4gxgxk \rangle,$$
$$Q_2 = \langle H, x : x^4gxhxg \rangle,$$

where all the coefficients in each relation are different and not equal to 1.

In presentation P_2, if all the coefficients are different and not equal to 1, then P_2 satisfies $C(4)$–$T(4)$ and so it is aspherical. If the coefficients are all the same, then P_2 is also aspherical by [**1**, Lemma 3]. Thus, we can consider three special cases, i.e. $h = 1$ and $k \neq 1$, $h \neq 1$ and $k = 1$, and $h \neq 1$ and $k \neq 1$. We can see that the case $h \neq 1$ and $k = 1$ is equivalent to the case $h = 1$ and $k \neq 1$ by using substitutions and taking the inverse. Thus, two special cases come from the relative presentation P_2. Firstly, consider the case $h = 1$ and $k \neq 1$, i.e. x^3gx^2kxl. If all coefficients are distinct, then P_2 satisfies the condition $C(4)$–$T(4)$ so that P_2 is aspherical. Thus, we have three subcases, $g = k$, $g = l$ and $k = l$, and so we have three relative presentations to consider:

$$Q_3 = \langle H, x : x^3gx^2gxl\rangle,$$
$$Q_4 = \langle H, x : x^3gx^2kxg\rangle,$$
$$Q_5 = \langle H, x : x^3gx^2kxk\rangle,$$

where all coefficients in each relation are different and not equal to 1. Secondly, in the case $h \neq 1$ and $k \neq 1$ in P_2, if all coefficients are distinct then P_2 also satisfies $C(4)$–$T(4)$ so that P_2 is aspherical. Thus, we obtain three subcases, $g = h$ (equivalently) $k = l$, $g \neq h$. Look at the case $g = h$. If $k \neq l$, then P_2 satisfies $C(4)$–$T(4)$ and so P_2 is aspherical. Thus, we have one relative presentation from the case $g = h$:

$$Q_6 = \langle H, x : x^3gxgxkxk\rangle,$$

where all the coefficients are different and not equal to 1. In the case $g \neq h$, we again have two subcases, $g = k$ and $g \neq k$. From the case $g \neq h$, $g = k$, we have three more subcases: g, h and l all different, $g = l$ and $h = l$. But the case $g = l$ is equivalent to Q_5 by using the substitution $t = xg$ followed by taking the inverse. Thus, we have two relative presentations to consider:

$$Q_7 = \langle H, x : x^3gxhxgxl\rangle,$$
$$Q_8 = \langle H, x : x^3gxhxgxh\rangle,$$

where all coefficients in each relation are different and not equal to 1. Consider the case $g \neq h$, $g \neq k$. We may assume that $g = l$ in this case, for otherwise P_2 satisfies $C(4)$–$T(4)$ and so is aspherical. If $h \neq k$, then $x^3gxhxkxg$ satisfies $C(4)$–$T(4)$ so that it is aspherical. Therefore, we have just one more case to consider:

$$Q_9 = \langle H, x : x^3gxhxhxg\rangle,$$

where all coefficients are different and not equal to 1 in Q_9. All remaining cases with $h \neq 1$ and $k \neq 1$, that is, $h = k$, $h = l$ and $k = l$, are equivalent to the above cases.

For presentation P_3, we can see by similar arguments that we have the following relative presentations to consider:

$$Q_{10} = \langle H, x : x^2gx^2kx^2m\rangle,$$
$$Q_{11} = \langle H, x : x^2gx^2gx^2m\rangle,$$
$$Q_{12} = \langle H, x : x^2gx^2gxlxm\rangle,$$

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$$Q_{13} = \langle H, x : x^2gx^2kxgxm\rangle,$$
$$Q_{14} = \langle H, x : x^2gx^2kxgxk\rangle,$$
$$Q_{15} = \langle H, x : x^2gxhx^2gxm\rangle,$$

where all coefficients in each relation are different and not equal to 1.

Thus, there is a total of 15 relative presentations Q_j $(1 \leqslant j \leqslant 15)$ to be considered and this is done in the next section.

4. Proof of Theorem 1.1

We prove Theorem 1.1 by the following sequence of lemmas.

Lemma 4.1. *The relative presentations Q_4, Q_{10}, Q_{11} and Q_{15} are aspherical.*

Proof. We use the n-steps technique for the relative presentation $Q_4 = \langle H, x : x^3gx^2kxg\rangle$. Consider the relative presentation $\langle H, y : y^2k\rangle$. This is aspherical, since the length of the equation is less than 6. Also, this presentation defines a torsion-free group F by [**1**, Theorem 1]. Consider the presentation $\langle F, x : xgx^2y^{-1}\rangle$. Then the relation xgx^2y^{-1} is not a proper power and has length 3, so the presentation is aspherical. Therefore, the presentation $\langle H, x, y : y^2k, xgx^2y^{-1}\rangle$ is aspherical by n steps. By solving the second relation for y and substituting into the first, we conclude that Q_4 is aspherical. The other cases are almost the same. □

Lemma 4.2. *The relative group presentations Q_2, Q_3 and Q_8 are aspherical.*

Proof. Consider the relative presentation

$$\langle H, x : x^2(xgxh)a_1(xgxh)a_2\rangle,$$

which, by using $s = xgxh$, can be transformed to

$$R_1 = \langle H, x, s : xgxhs^{-1}, x^2sa_1sa_2\rangle.$$

Figure 1a shows the star complex over R_1.

For Q_2, with $h = a_2 = 1$ and $a_1 \neq 1$, we assign a weight function θ in such a way that $\theta(g) = \theta(a_1) = 0$, one edge with label 1 from x^{-1} to x in the star complex has weight 1 and the other edges have weight $\frac{1}{2}$. Thus, there are no admissible cycles with weight less than 2 to avoid a non-cyclic coefficient group, so the star complex over Q_2 admits an aspherical weight function. Thus, Q_2 is aspherical.

For Q_3, with $h = a_1 = 1$ and $a_2 \neq 1$, we assign a weight function θ in such a way that $\theta(g) = \theta(a_2) = 0$, one edge with label 1 from x^{-1} to s in the star complex has weight 1 and the others have weight $\frac{1}{2}$. Then the star complex over Q_3 admits an aspherical weight function, so Q_3 is aspherical.

For Q_8, with $a_1 = a_2 = 1$ and $h \neq 1$, the weight function is given in the same way as for Q_3. This completes the proof. □

Lemma 4.3. *The relative group presentations Q_5, Q_9, Q_{13} and Q_{14} are aspherical.*

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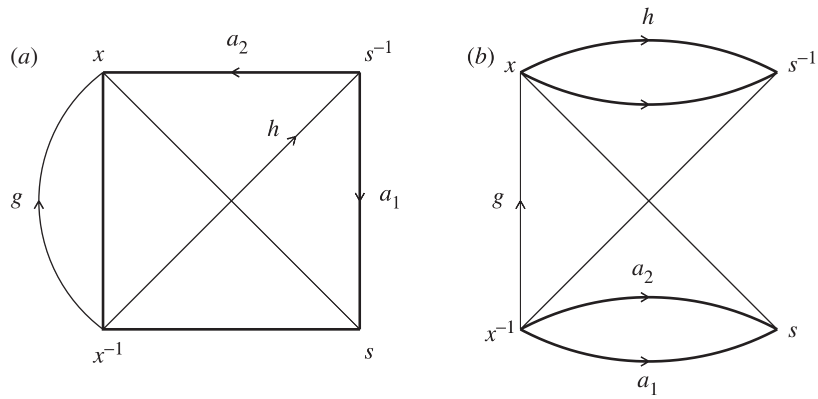

Figure 1. Star complexes.

Proof. Consider the relative presentation

$$\langle H, x : (xgx)xa_1(xgx)hxa_2 \rangle,$$

which, by using $s = xgx$, can be transformed to

$$R_2 = \langle H, x, s : xgxs^{-1}, sxa_1shxa_2 \rangle.$$

Then the star complex is shown in Figure 1*b*.

If $g = 1$ and $h = a_2$, then the relative group presentation is exactly Q_5. We assign the weight function θ as follows: the edge with label h from x to s^{-1} has weight 0, edges with label 1 from x^{-1} and s to x have weight 0, and the others have weight 1. The cases Q_9, with $a_1 = 1$ and $h = a_2$, Q_{13}, with $a_2 = 1$ and $h \neq a_1$, and Q_{14}, with $a_2 = 1$ and $h = a_1$, have the same weight function, which is given as follows: $\theta(g) = 0$ and the others have weight $\frac{1}{2}$. One can show that all the star complexes admit aspherical weight functions. This completes the proof. □

There remain just four cases Q_1, Q_6, Q_7, and Q_{12}: we now apply the curvature distribution to these relative presentations.

One can observe that the four remaining relative presentations satisfy the condition $C(3)$. This means that if we assume that there is a picture over Q_1, Q_6, Q_7 or Q_{12}, then there is no 2-wheel on the picture. Thus, each disc on the picture is an l-wheel, where $l \geqslant 3$.

Define an angle function called the *standard angle function* on a picture over relative group presentations Q_1, Q_6, Q_7 or Q_{12} such that each corner within a double bond has angle 0 and every other corner has angle $2\pi/\deg(\Delta)$, where $\deg(\Delta)$ is the degree of the disc Δ. With this angle function we have the curvature $c(\Delta) = 0$ for each disc Δ and any double bond is flat. Since each disc is an l-wheel, where $l \geqslant 3$, any n-region has the maximum curvature

$$c(\Delta) = 2\pi - n\left(\frac{\pi - 2\pi}{3}\right) = \frac{(6-n)\pi}{3}.$$

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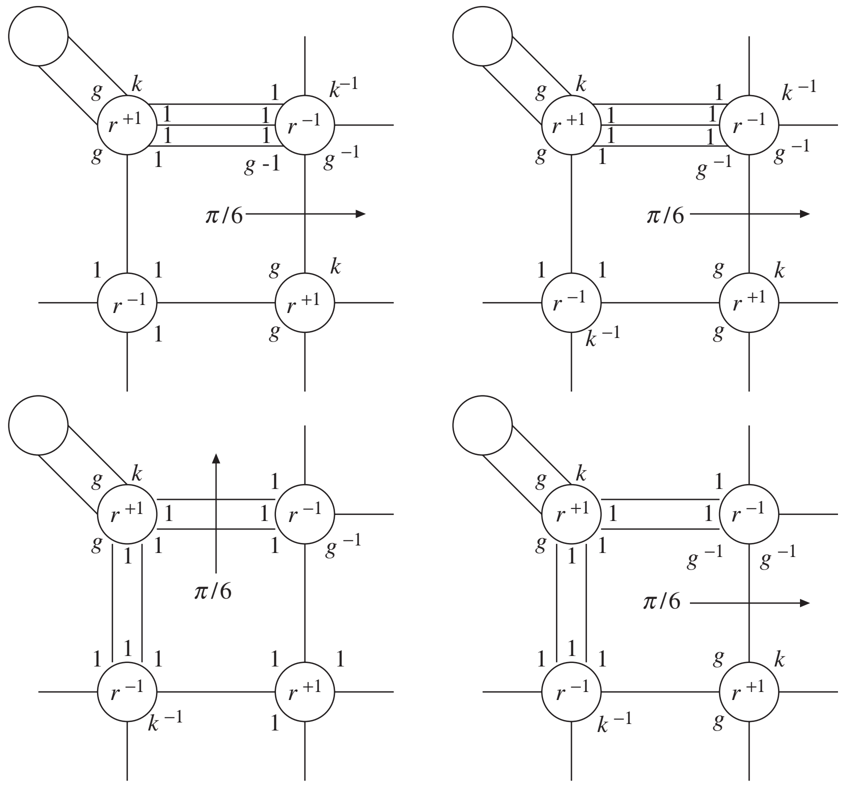

Figure 2. Positive curved regions and curvature distributions.

If $n \geqslant 6$, then it has non-positive curvature. By the fundamental curvature formula, we know there is at least one region with positive curvature, so the picture has at least one 4-region. If such a 4-region contains discs Δ with $\deg(\Delta) \geqslant 4$, then the maximum curvature

$$c(\Delta) \geqslant 2\pi - 4\left(\frac{\pi - 2\pi}{4}\right) = 0.$$

By the fundamental curvature formula, this leads to a contradiction. We therefore have at least one 4-region with at least one disc containing a 3-wheel. If such a 4-region does not exist, then we conclude that there is no picture over Q_1, Q_6, Q_7 and Q_{12} and so we can say that Q_1, Q_6, Q_7 and Q_{12} are aspherical. Recall that if we read off corner labels counterclockwise in each inner region on a picture, then the word obtained in such a manner is the identity in H and if such a word gives us that H is cyclic, then the relative presentation is aspherical. Therefore, we will focus on finding each possible 4-region satisfying the condition that it has no dipole and corner labels of such a 4-region does not give that H is cyclic.

Lemma 4.4. *The relative group presentation $Q_6 = \langle H, x : x^3gxgxkxk \rangle$ is aspherical.*

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Proof. For Q_6, we observe that there does not exist a 4-region without a dipole. This proves that Q_6 is aspherical. $\square$

Lemma 4.5. *The relative group presentation $Q_1 = \langle H, x : x^4gxgxk\rangle$ is aspherical.*

Proof. Suppose that Q_1 is not aspherical. First, we need to find 4-regions with at least one 3-wheel. One can find there are four such regions without a dipole up to inversion with corner labels $1 \cdot 1 \cdot g \cdot g^{-1}$ or $1 \cdot 1 \cdot 1 \cdot 1$ (see Figure 2). The maximum curvature of these regions is $\pi/6$. Define a distribution scheme as follows:

$$\eta(F, F') = \begin{cases} \pi/6 & \text{if } c(F) > 0 \text{ and } F \text{ is separated from } F' \text{ by a single bond} \\ & \text{with corner labels } (g^{-1}k)^{\pm 1} \text{ or if } c(F) > 0 \text{ and } F \text{ is separated} \\ & \text{from } F' \text{ by a double bond with corner labels } (k1)^{\pm 1}, \\ 0 & \text{otherwise.} \end{cases}$$

Since 4-regions do not have corners with labels $k^{\pm 1}$, F' has at least six corners. One can easily show that two inward bonds in F' are not adjacent. Now we compute the curvature of F'. Let n be the number of total discs in F', let m be the number of inwardly oriented bonds in F' and let p be the number of other discs in F'. Therefore, we have $n = 2m + p$, and

$$\begin{aligned} c^*(F') &\leqslant 2\pi - m\left(\frac{\pi - 2\pi}{3}\right) - m\left(\frac{\pi - 2\pi}{4}\right) - p\left(\frac{\pi - 2\pi}{3}\right) + \frac{m\pi}{6} \\ &= \frac{2\pi - (m + m + p)\pi}{3} \\ &= \frac{(6 - n)\pi}{3}. \end{aligned}$$

Since $n \geqslant 6$ we deduce that $c^*(F')$ is non-positive. This is a contradiction. $\square$

Lemma 4.6. *The relative presentation $Q_{12} = \langle H, x : x^2gx^2gxlxm\rangle$ is aspherical.*

Proof. If we suppose that Q_{12} is not aspherical, then there is a picture with at least one 4-region with at least one 3-wheel up to inversion. We assign the standard angle function to the picture. Let F be such a 4-region. Then the maximum curvature is $c(F) = \pi/6$ if it has a triple bond and $c(F) = \pi/15$ otherwise. Now apply the curvature distribution scheme to adjacent regions with at least six discs as follows. Define

$$\eta(F, F') = \begin{cases} \pi/6 & \text{if } c(F) > 0 \text{ and } F \text{ has a triple bond,} \\ \pi/15 & \text{if } c(F) > 0 \text{ and } F \text{ has a double bond,} \\ 0 & \text{otherwise.} \end{cases}$$

Let m be the number of triple bonds in F', let n be the number of total vertices and let p be the number of remaining vertices. So we have $n = 2m + p$, where $p = p_1 + p_2$. We

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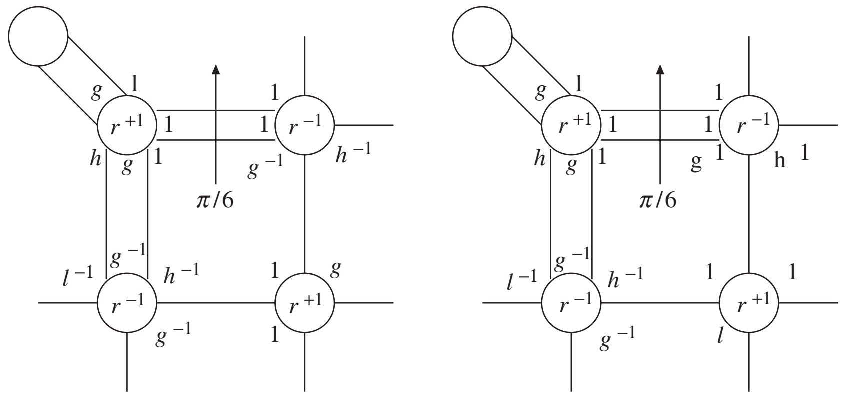

Figure 3. Corner labels $hg^{\pm 1}$.

note that triple bonds are not adjacent in this case. Then

$$\begin{aligned} c^*(F') &\leqslant 2\pi - m\left(\frac{\pi - 2\pi}{3}\right) - m\left(\frac{\pi - 2\pi}{4}\right) - p_2\left(\frac{\pi - 2\pi}{3}\right) \\ &\qquad - p_1\left(\frac{\pi - 2\pi}{5}\right) + \frac{m\pi}{6} + \frac{p_1\pi}{15} \\ &= 2\pi - \frac{(m + m + p_2 + p_1)\pi}{3} - \frac{p_1 4\pi}{14} + \frac{p_1\pi}{15} \\ &\leqslant \frac{(6 - n)\pi}{3}. \end{aligned}$$

Since $n \geqslant 6$, it is non-positive. This leads us to a contradiction. $\square$

We will show the last relative presentation Q_7 is aspherical. In this case, the argument is a little more complicated.

Lemma 4.7. *The relative presentation $Q_7 = \langle H, x : x^3gxhxgxl\rangle$ is aspherical.*

Proof. As before, we find possible 4-regions with at least one 3-wheel up to inversion. One can see that there are nine possible 4-regions, and they give us the following relations in H:

$$(hg)^{\pm 1}, \quad (hgl^{-1})^{\pm 1}, \quad (h^{-1}lg)^{\pm 1}, \quad (h^{-1}l^2)^{\pm 1}, \quad (l^{-1}h^2).$$

Recall that each coefficient is not the identity and any two of coefficients are not the same. Then, it may be shown easily that two such relations are not satisfied simultaneously, thus avoiding an infinite cyclic subgroup. We therefore observe that 4-regions with different relations cannot be present in the same picture. Consider a spherical picture containing the relation $(hg)^{\pm 1}$ (see Figure 3). Since any 4-region in it does not have corner labels $(l1)^{\pm 1}$, if we apply the curvature distribution as shown in Figure 3, then the adjacent

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region F' has at least six corners because two bonds with corner labels 11 are not adjacent:

$$\eta(F,F') = \begin{cases} \pi/6 & \text{if } c(F) > 0 \text{ and } F \text{ is separated from } F' \text{ by} \\ & \text{a double bond with corner labels 11,} \\ 0 & \text{otherwise.} \end{cases}$$

Let m be the number of discs of inwardly oriented double bonds in F' and let $n = m + p$ be the total number of discs. Then

$$\begin{aligned} c^*(F') &\leqslant 2\pi - \frac{m}{2}\left(\frac{\pi - 2\pi}{3}\right) - \frac{m}{2}\left(\frac{\pi - 2\pi}{4}\right) - p\left(\frac{\pi - 2\pi}{3}\right) + \frac{m}{2}\frac{\pi}{6} \\ &= 2\pi - (m+p)\frac{\pi}{3} \\ &= \frac{(6-n)\pi}{3}. \end{aligned}$$

Since $n \geqslant 6$, it has a non-positive curvature. This is a contradiction. In pictures with corner labels $(l^{-1}hg)$, F' has at least six corners because each 4-region does not have $(h^{-1}1)^{\pm 1}, (1l)^{\pm 1}$. Also, two such bonds are not adjacent. In this case,

$$\begin{aligned} c^*(F') &\leqslant 2\pi - m\left(\frac{\pi - 2\pi}{4}\right) - p\left(\frac{\pi - 2\pi}{3}\right) + \frac{m}{2}\frac{\pi}{3} \\ &= \frac{(6-n)\pi}{3}. \end{aligned}$$

For the reason given above, this leads us to a contradiction.

In a picture with corner labels $(h^{-1}lg)^{\pm 1}$, F' has at least six corners and two such bonds are not adjacent. Then

$$\begin{aligned} c^*(F') &\leqslant 2\pi - m\left(\pi - \frac{2\pi}{4}\right) - p\left(\pi - \frac{2\pi}{3}\right) + \frac{m}{2}\frac{\pi}{6} \\ &\leqslant \frac{(6-n)\pi}{3}. \end{aligned}$$

For the same reason, this leads us to a contradiction. It is shown in the same way, i.e. in pictures with corner labels $(l^{-1}h^2)^{\pm 1}$. This completes the proof. □

We have now proved that all 15 relative group presentations are aspherical. This completes the proof of the theorem.

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