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ON THE CROSSING NUMBER OF THE JOIN OF THE WHEEL ON FIVE VERTICES WITH THE DISCRETE GRAPH

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Abstract

We give the crossing number of the join product $W_4 + D_n$, where W_4 is the wheel on five vertices and D_n consists of n isolated vertices. The proof is based on calculating the minimum number of crossings between two different subgraphs from the set of subgraphs which do not cross the edges of the graph W_4 and from the set of subgraphs which cross the edges of W_4 exactly once.

2010 *Mathematics subject classification*: primary 05C10; secondary 05C38.

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1. Introduction

It is well known that computing the crossing number of a graph is an NP complete problem. Nevertheless, research on the problem of reducing the number of crossings in particular classes of graphs is of interest not only in graph theory, but also in computer science. The exact value of the crossing number is known for only a few graphs and families of graphs. We use the notation and definition of the crossing number $\mathrm{cr}(G)$ of the graph G presented by Klešč in [10]. We also often use Kleitman's result [9] on the crossing numbers of the complete bipartite graphs $K_{m,n}$, that is,

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \quad \text{for all } m \leq 6.$$

Using Kleitman's result [9], the crossing numbers for the join of two paths, join of two cycles and join of a path and a cycle were studied in [10]. The exact values of the crossing numbers of $G + D_n$ and $G + P_n$ for all graphs G of order at most four are given in [14]. Here D_n denotes the discrete graph with n vertices and P_n denotes the path on n vertices. The crossing numbers of the graphs $G + D_n$ are given for a few graphs G of order five and six in [2, 3, 11–13, 15, 17–21]. In all these cases, the graph G is usually connected and contains at least one cycle. The exact values of the crossing numbers $G + P_n$ and $G + C_n$, where C_n is the cycle with n vertices, have also been investigated for some graphs G of order five and six in [6, 11, 12, 15, 16, 20, 22].

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The methods presented in the paper are based on combinatorial properties of cyclic permutations. Somewhat similar ideas were used in [8, 17]. In [2, 3, 18, 19], the properties of cyclic permutations are verified with the help of the software in [1]. In our opinion, the methods used in [11, 14, 15] do not suffice for establishing the crossing number of the join product $W_4 + D_n$.

2. Cyclic permutations and configurations

Let G be the connected graph of order five isomorphic with the wheel W_4. We consider the join product of G with the discrete graph D_n on n vertices. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \ldots, t_n$, where each vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the five edges incident with the vertex t_i. This means that the graph $T^1 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{5,n}$ and

$$G + D_n = G \cup K_{5,n} = G \cup \Big(\bigcup_{i=1}^{n} T^i \Big).$$

We use the same notation and definitions for cyclic permutations and the corresponding configurations for a good drawing D of the graph $G + D_n$ as in [18]. Let D be a good drawing of the graph $G + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave t_i, as defined by Hernández-Vélez *et al.* [8]. We use the notation (12345) if the counterclockwise order of the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4 and t_iv_5. We separate all subgraphs T^i, $i = 1, \ldots, n$, of the graph $G + D_n$ into three mutually disjoint subsets depending on how many times T^i crosses the edges of G in D. For $i = 1, \ldots, n$, let $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$. Every other subgraph T^i crosses the edges of G at least twice in D. Let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D \cup S_D$, where $i \in \{1, \ldots, n\}$. Thus, for a given subdrawing of G in D, the subgraphs F^i are exactly represented by $\mathrm{rot}_D(t_i)$.

According to the arguments in the proof of our main result, Theorem 3.2, to obtain a drawing of $G + D_n$ with the smallest number of crossings, the set $R_D \cup S_D$ must be nonempty. Thus, we will only consider drawings of the graph G for which there is a possibility of the existence of a subgraph $T^i \in R_D \cup S_D$. Since the graph G consists of one dominating vertex of degree four and four vertices of degree three which form the subgraph isomorphic with the cycle C_4 (for brevity, we write $C_4(G)$), we only need to consider possibilities of crossings between subdrawings of $C_4(G)$ and four edges incident with the dominating vertex. Of course, the edges of the cycle $C_4(G)$ can cross in these subdrawings. Let us first consider a good subdrawing of G in which the edges of $C_4(G)$ do not cross each other. In this case, we obtain three drawings of G as shown in Figure 1(a), (b) and (c). If we consider a good subdrawing of G with one crossing among the edges of the cycle $C_4(G)$, then the remaining edges of G do not cross the edges of $C_4(G)$ in one case only, as shown in Figure 1(d) (and there is no possibility of obtaining a subdrawing of $G \cup T^j$ with $T^j \in R_D \cup S_D$). If the edges of $C_4(G)$ are

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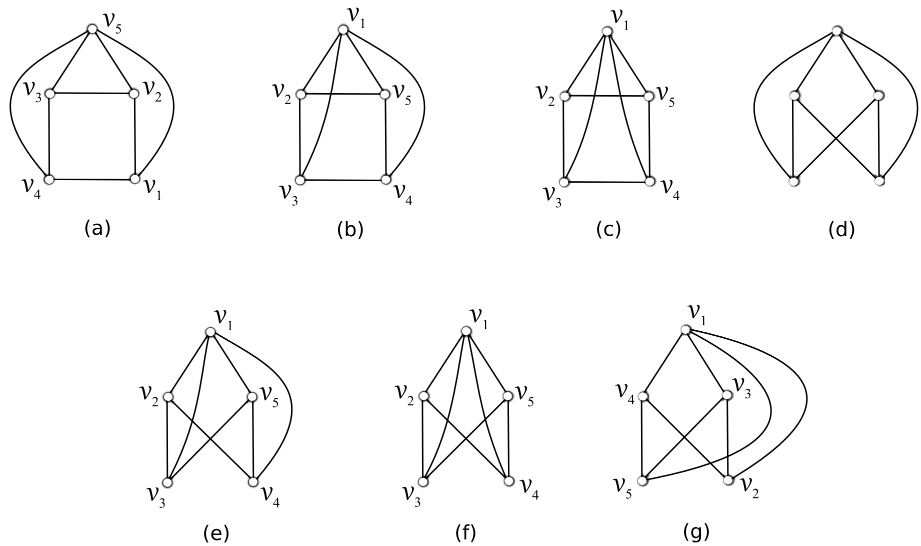

FIGURE 1. Seven possible drawings of the graph G.

crossed at least once by the remaining edges of G, then there are three possibilities, as shown in Figure 1(e), (f) and (g). The vertex notation of the graph G will be justified later.

First, let us assume a good drawing D of the graph $G + D_n$ in which the edges of G do not cross each other. In this case, without loss of generality, from the drawings in Figure 1 we can choose the vertex notation of the graph G as shown in Figure 1(a). Our aim is to list all possible rotations $\mathrm{rot}_D(t_j)$ which can appear in D if the edges of T^j cross the edges of G exactly once. Since there is only one subdrawing of $F^j \setminus \{v_5\}$ represented by the rotation (1234), there are four ways to obtain the subdrawing of F^j depending on which edge of G is crossed by the edge $t_j v_5$. We denote these four possibilities by $\mathcal{A}_k$, $k = 1, \ldots, 4$. For our purposes, it does not matter which of the regions is unbounded, so we can assume that the drawings are as shown in Figure 2. In the rest of the paper, we represent a cyclic permutation by the permutation with 1 in the first position. Thus the configurations $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ are represented by the cyclic permutations (12345), (12534), (15234) and (12354), respectively. In a fixed drawing of the graph $G + D_n$, some configurations from $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$ need not appear. So we denote by $\mathcal{M}_D$ the set of all configurations for the drawing D belonging to $\mathcal{M}$.

We can extend the idea of the minimum number of crossings between two different subgraphs T^i and T^j from the set R_D onto the set S_D. Let $\mathcal{X}$, $\mathcal{Y}$ be configurations from $\mathcal{M}_D$. We denote by $\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in S_D$ such that F^i, F^j have configurations $\mathcal{X}$, $\mathcal{Y}$, respectively. Let $\mathrm{cr}(\mathcal{X}, \mathcal{Y}) = \min\{\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})\}$ over all good drawings of the graph $G + D_n$ with $\mathcal{X}, \mathcal{Y} \in \mathcal{M}_D$. Our aim is to calculate $\mathrm{cr}(\mathcal{X}, \mathcal{Y})$ for all pairs $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$. In particular,

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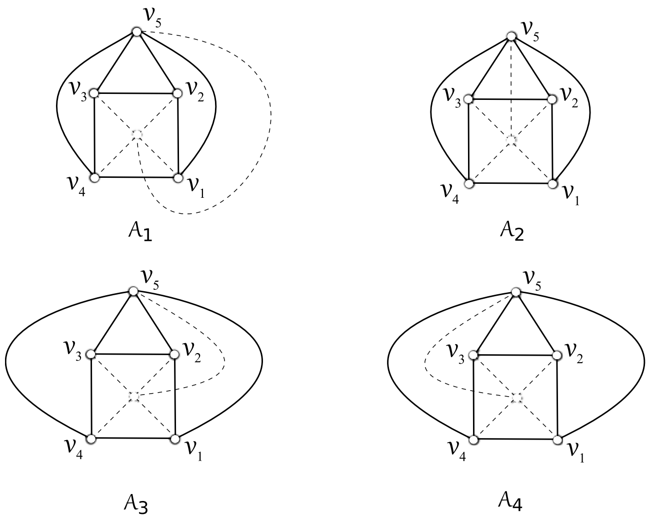

FIGURE 2. Drawings of four possible configurations from $\mathcal{M}$ of the subgraph F^j.

TABLE 1. The necessary number of crossings between T^i and T^j for the configurations $\mathcal{A}_k$, $\mathcal{A}_l$.

—	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_1$	4	2	3	3
$\mathcal{A}_2$	2	4	3	3
$\mathcal{A}_3$	3	3	4	2
$\mathcal{A}_4$	3	3	2	4

the configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ are represented by the cyclic permutations (12345) and (12534), respectively. Since the minimum number of interchanges of adjacent elements of (12345) required to produce the cyclic permutation $\overline{(12534)} = (14352)$ is two, any subgraph T^j with the configuration $\mathcal{A}_2$ of F^j crosses the edges of T^i with the configuration $\mathcal{A}_1$ of F^i at least twice, that is, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_2) \geq 2$. Details have been worked out by Woodall [23]. The same reasoning gives $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_3) \geq 3$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_4) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_3) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_4) \geq 3$ and $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_4) \geq 2$, Clearly, also $\mathrm{cr}(\mathcal{A}_i, \mathcal{A}_i) \geq 4$ for any $i = 1, 2, 3, 4$. The resulting lower bounds for the number of crossings of configurations from $\mathcal{M}$ are summarised in the symmetric Table 1. (Here, $\mathcal{A}_k$ and $\mathcal{A}_l$ are configurations of the subgraphs F^i and F^j, where $k, l \in \{1, 2, 3, 4\}$.)

3. The crossing number of $G + D_n$

Recall that two vertices t_i and t_j of $G + D_n$ are *antipodal* in a drawing of $G + D_n$ if the subgraphs T^i and T^j do not cross. A drawing is *antipodal-free* if it has no antipodal vertices. Now we are able to prove the main result of the paper. We compute the exact values of crossing numbers of the small graphs in this paper using the algorithm

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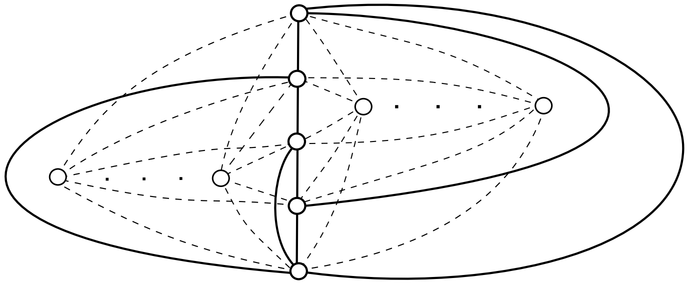

FIGURE 3. The good drawing of $G + D_n$ with $4\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor + n + \lfloor n/2\rfloor$ crossings.

located on the website http://crossings.uos.de/. This algorithm can find the crossing numbers of small undirected graphs. It uses integer linear programming, based on Kuratowski subgraphs and solves it via branch-and-cut-and-price. The system also generates verifiable formal proofs, as described in [4]. Unfortunately, the capacity of this system is restricted.

LEMMA 3.1. $\mathrm{cr}(G + D_1) = 1$ *and* $\mathrm{cr}(G + D_2) = 3$.

THEOREM 3.2. *If* $n \geq 1$, *then* $\mathrm{cr}(G + D_n) = 4\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor + n + \lfloor n/2\rfloor$.

PROOF. Figure 3 exhibits a drawing of $G + D_n$ with $4\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor + n + \lfloor n/2\rfloor$ crossings. Thus, $\mathrm{cr}(G + D_n) \leq 4\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor + n + \lfloor n/2\rfloor$. We prove the reverse inequality by induction on n. By Lemma 3.1, the result is true for $n = 1$ and $n = 2$. Now suppose that, for some $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G + D_n) < 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor \tag{3.1}$$

and that

$$\mathrm{cr}(G + D_m) \geq 4\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor + m + \left\lfloor\frac{m}{2}\right\rfloor \quad \text{for any } m < n. \tag{3.2}$$

We claim that the drawing D must be antipodal-free. For a contradiction, suppose, without loss of generality, that $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Using the positive values in Table 1, it is easy to verify that the subgraphs T^n and T^{n-1} are not both from the set S_D, and if $T^n \in R_D$, then $\mathrm{cr}_D(G, T^{n-1}) \geq 3$ from the possible subdrawings in Figure 1, that is, $\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 3$. Since $\mathrm{cr}(K_{5,3}) = 4$, any T^k, for $k = 1, 2, \ldots, n-2$, crosses the edges of the subgraph $T^{n-1} \cup T^n$ at least four times. So the number of crossings $\mathrm{cr}_D(G + D_n)$ of $G + D_n$ in D is given by

$$\begin{aligned}
&\mathrm{cr}_D(G + D_{n-2}) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{5,n-2}, T^{n-1} \cup T^n) + \mathrm{cr}_D(G, T^{n-1} \cup T^n)\\
&\quad \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + n - 2 + \left\lfloor\frac{n-2}{2}\right\rfloor + 4(n-2) + 3\\
&\quad = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor.
\end{aligned}$$

This contradicts the assumption (3.1) and confirms that D is antipodal-free.

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If $r = |R_D|$ and $s = |S_D|$, the assumption (3.2) together with the well-known fact $\mathrm{cr}(K_{5,n}) = 4\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor$ imply that, in D, if $r = 0$, then there are at least $\lceil n/2\rceil + 1$ subgraphs T^i which cross the edges of G exactly once. More precisely,

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{5,n}) \le \mathrm{cr}_D(G) + 0r + 1s + 2(n - r - s) < n + \left\lfloor \frac{n}{2} \right\rfloor,$$

that is,

$$s + 2(n - r - s) < n + \left\lfloor \frac{n}{2} \right\rfloor. \tag{3.3}$$

This readily forces $2r + s \ge n - \lfloor n/2\rfloor + 1$, and if $r = 0$, then $s \ge n - \lfloor n/2\rfloor + 1 = \lceil n/2\rceil + 1$.

Now, for $T^i \in R_D \cup S_D$, we consider the possible configurations of $F^i = G \cup T^i$ in the drawing D in four cases.

Case 1. $\mathrm{cr}_D(G) = 0$. We can choose the drawing with the vertex notation of G as shown in Figure 1(a). Clearly the set R_D is empty, that is, $r = 0$. Thus, we now have two possibilities for the nonempty set of configurations $\mathcal{M}_D$.

(a) $\{\mathcal{A}_i, \mathcal{A}_{i+1}\} \subseteq \mathcal{M}_D$ *for some* $i \in \{1, 3\}$. Without loss of generality, fix $T^{n-1}, T^n \in S_D$ such that F^{n-1} and F^n have different configurations from $\{\mathcal{A}_1, \mathcal{A}_2\}$. Then we have $\mathrm{cr}_D(T^{n-1} \cup T^n, T^j) \ge 6$ for any $T^j \in S_D$ with $j \ne n-1, n$ by summing the values in all columns in the first two rows of Table 1. Moreover, $\mathrm{cr}_D(G \cup T^{n-1} \cup T^n, T^j) \ge 2 + 2 = 4$ trivially for any subgraph $T^j \notin S_D$. As $\mathrm{cr}_D(G \cup T^{n-1} \cup T^n) \ge 2 + 2$, by fixing the graph $G \cup T^{n-1} \cup T^n$,

$$\begin{aligned}
&\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(K_{5,n-2}, G \cup T^{n-1} \cup T^n) + \mathrm{cr}_D(G \cup T^{n-1} \cup T^n)\\
&\ge 4\left\lfloor \frac{n-2}{2} \right\rfloor\left\lfloor \frac{n-3}{2} \right\rfloor + 7(s-2) + 4(n-s) + 2 + 2 = 4\left\lfloor \frac{n-2}{2} \right\rfloor\left\lfloor \frac{n-3}{2} \right\rfloor + 4n + 3s - 10\\
&\ge 4\left\lfloor \frac{n-2}{2} \right\rfloor\left\lfloor \frac{n-3}{2} \right\rfloor + 4n + 3\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) - 10 > 4\left\lfloor \frac{n}{2} \right\rfloor\left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

(b) $\{\mathcal{A}_i, \mathcal{A}_{i+1}\} \not\subseteq \mathcal{M}_D$ *for any* $i = 1, 3$. Without loss of generality, we can assume that $T^n \in S_D$ with the configuration $\mathcal{A}_j \in \mathcal{M}_D$ of the subgraph F^n for some $j \in \{1, \dots, 4\}$. Then $\mathrm{cr}_D(T^n, T^k) \ge 3$ for any $T^k \in S_D$ with $k \ne n$, by the remaining values in Table 1. Hence, by fixing the subgraph $G \cup T^n$,

$$\begin{aligned}
&\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n)\\
&\quad\ge 4\left\lfloor \frac{n-1}{2} \right\rfloor\left\lfloor \frac{n-2}{2} \right\rfloor + 4(s-1) + 3(n-s) + 1 = 4\left\lfloor \frac{n-1}{2} \right\rfloor\left\lfloor \frac{n-2}{2} \right\rfloor + 3n + s - 3\\
&\quad\ge 4\left\lfloor \frac{n-1}{2} \right\rfloor\left\lfloor \frac{n-2}{2} \right\rfloor + 3n + \left\lceil \frac{n}{2} \right\rceil + 1 - 3 > 4\left\lfloor \frac{n}{2} \right\rfloor\left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

Case 2. $\mathrm{cr}_D(G) = 1$. Without loss of generality, we can choose the vertex notation of G as shown in Figure 1(b). In this case, the set R_D is also empty. So we will consider only the subgraphs T^j whose edges cross the edges of G exactly once. Since the edge

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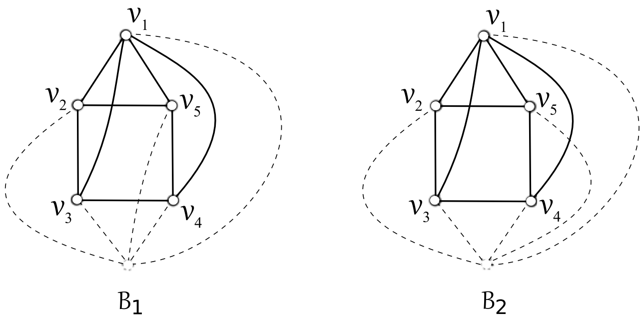

FIGURE 4. Drawings of two possible configurations from $\mathcal{N}$ of the subgraph F^i.

t_jv_5 crosses either v_3v_4 or v_1v_4 in G, there are two possibilities, which we denote by $\mathcal{B}_1$ and $\mathcal{B}_2$. It does not matter which of the regions is unbounded, so we can assume that the drawings are as shown in Figure 4.

Consequently, the configurations $\mathcal{B}_1$ and $\mathcal{B}_2$ are represented by the cyclic permutations (14532) and (15432), respectively. In a fixed drawing of the graph $G + D_n$, some configurations from $\mathcal{N} = \{\mathcal{B}_1, \mathcal{B}_2\}$ need not appear. Thus we denote by $\mathcal{N}_D$ the subset of $\mathcal{N}$ consisting of all configurations that exist in the drawing D. From the properties of cyclic rotations, we can easily verify that $\mathrm{cr}(\mathcal{B}_1, \mathcal{B}_2) \geq 3$. (This idea was also used to establish the values in Table 1.) Choosing $T^j \in S_D$ to fix the graph $G \cup T^j$, we find that

$$\begin{aligned}\mathrm{cr}_D(G + D_n) &\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(s-1) + 3(n-s) + 1 + 1\\ &= 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + s - 2\\ &\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + \left\lceil\frac{n}{2}\right\rceil + 1 - 2 > 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

Case 3. $\mathrm{cr}_D(G) = 2$. Without loss of generality, we can consider the drawing with the vertex notation of the graph G as shown in Figure 1(c). In this case, there is no possibility of obtaining a subdrawing of $G \cup T^j$ for a $T^j \in S_D$, that is, the set S_D must be empty. This fact, with property (3.3), confirms that $r \geq 2$. So we only need to consider the subgraphs T^j whose edges do not cross the edges of G. For a $T^j \in R_D$, it is easy to verify that the subgraph $F^j = G \cup T^j$ is uniquely represented by $\mathrm{rot}_D(t_j) = (15432)$, and $\mathrm{cr}_D(T^j, T^i) \geq 4$ for any $T^i \in R_D$ with $i \neq j$ provided that $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$ (for more details, see [23]). Moreover, it is not difficult to show over all possible drawings that $\mathrm{cr}_D(G \cup T^j, T^k) \geq 4$ for any subgraph $T^k \notin R_D$. By fixing the subgraph $G \cup T^j$,

$$\begin{aligned}\mathrm{cr}_D(G + D_n) &\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(r-1) + 4(n-r) + 2\\ &= 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4n - 2 > 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

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If we choose the vertex notation of the graph G as shown in Figure 1(e), then we can apply the same process as for the drawing of G in Figure 1(b).

Case 4. $\mathrm{cr}_D(G) = 3$. If we assume the vertex notation of the graph as shown in Figure 1(f), then we can apply the same process as for the drawing of G in Figure 1(c). Finally, without loss of generality, we can consider the vertex notation of the graph G as shown in Figure 1(g). In this case, the set R_D is again empty. Further, any subgraph $F^j = G \cup T^j$, for $T^j \in S_D$, is uniquely represented by $\mathrm{rot}_D(t_j) = (12543)$. Since $\mathrm{cr}_D(T^j, T^i) \geq 4$ trivially for any $T^i \in S_D$ with $i \neq j$ provided that $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$, we can repeat the same idea as for the drawing of G in Figure 1(b).

We have shown, in all cases, that there is no good drawing D of the graph $G + D_n$ with fewer than $4\lfloor n/2\rfloor\lfloor (n-1)/2\rfloor + n + \lfloor n/2\rfloor$ crossings. This completes the proof of the theorem. □

Let W_n and S_n denote the wheel and the star on $n + 1$ vertices, respectively. In general, the graph $S_n + C_m$ is isomorphic with the graph $W_m + D_n$ for all integers $n \geq 1$ and $m \geq 3$. In [13], the crossing numbers of the graphs $W_m + D_n$ for $n \leq 5$ and $m \geq 3$ were established. Theorem 3.2 extends this result for the graphs $W_4 + D_n$ for any $n \geq 1$. The result in Theorem 3.2 has already been claimed in [7] (see [5]). The paper [7] does not seem to be available in English and we have not been able to verify the results.

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